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The Stability of Basis Properties of Multiple Systems in a Banach Space With Respect to Certain Transformations

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Abstract. In this paper a method for constructing a basis of a Banach space based on the bases of subspaces is proposed. The completeness, minimality, uniform minimality and basicity with the parentheses of the corresponding systems are also studied. The obtained abstract results are applied to the study of the basis properties of the eigenfunctions of a discontinuous differential operator of second order.

Key Words and Phrases: basis, completeness, minimality, uniformly minimality, discontinuous differential operator

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1. Introduction

The study of the spectral properties of some discrete differential operators leads to the development of new methods for constructing bases. In this regard, many mathematicians have paid attention to the study of basis properties (completeness, minimality, basicity) of systems of functions of special types, often being eigen and associated functions of differential operators. At the same time, various methods for studying these properties were proposed. Among such works are the works of the authors [1-6]. In the case of discontinuous differential operators, from eigenfunctions arise systems that for the study of the basicity the previously known methods are not applicable.

In this work is considered an abstract approach to the problem described above. The stability of the basis properties of multiple systems in a Banach space with respect to certain transformations is studied, a method for constructing a basis for the whole space is proposed, based on the bases of subspaces, which has wide application in the spectral theory of discontinuous differential operators.

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2. Necessary information

Recall the definitions of some notions from the theory of basis in a Banach space. Let X be a Banach space.

Definition 1. *The system $\{x_n\}_{n\in N} \subset X$ is called uniformly minimal in X, if*

$$\exists \delta > 0 : \inf_{\forall u\in L\left[\{x_n\}_{n\neq k}\right]} \|x_k - u\| \geq \delta \|x_k\| \ , \ \forall k \in N.$$

Definition 2. *If there exists a sequence of indexes, such that $\{n_k\}_{k\in N} \subset N : n_k < n_{k+1}$, $\forall k \in N$ and any element $x \in X$ is uniquely represented in the form*

$$x = \sum_{k=0}^{\infty} \sum_{j=n_k+1}^{n_{k+1}} c_j x_j \quad (n_0 = 0),$$

then the system $\{x_n\}_{n\epsilon N} \subset X$ is called a basis with parentheses in X.

For $n_k = k$ the system $\{x_n\}_{n\in N}$ forms a usual basis for X.
We need the following easily proved statements.

Statement 1. *Let the system $\{x_n\}_{n\in N}$ form a basis with parentheses for X. If the system $\{x_n\}_{n\in N}$ is uniformly minimal and the sequence $\{n_{k+1} - n_k\}_{k\in N}$ is bounded, then this system forms a usual basis for X.*

Statement 2. *Let the system $\{x_n\}_{n\in N}$ form a Riesz basis with parentheses for a Hilbert space X. If the sequence $\{n_{k+1} - n_k\}_{n\in N}$ is bounded and the following condition*

$$\sup_n \{\|x_n\| : \|v_n\|\} < \infty$$

holds, where $\{v_n\}_{n\in N}$ is a biorthogonal system, then $\{x_n\}_{n\in N}$ forms a usual Riesz basis for X.

Definition 3. *The basis $\{u_n\}_{n\in N}$ of Banach space X is called a p-basis, if for any $x \in X$ the condition*

$$\left(\sum_{n=1}^{\infty} |\langle x, \vartheta_n\rangle|^p\right)^{\frac{1}{p}} \leq M \|x\| ,$$

holds, where $\{\vartheta_n\}_{n\in N}$ - is a biorthogonal system to $\{u_n\}_{n\in N}$.

Definition 4. *The sequences $\{u_n\}_{n\in N}$ and $\{\varphi_n\}_{n\in N}$ of Banach space X are called a p-close, if the condition*

$$\sum_{n=1}^{\infty} \|u_n - \varphi_n\|^p < \infty,$$

holds.

We will also use the following results from [3,5] (see, also [6-8]).

Theorem 1. *[3] Let $\{x_n\}_{n\in N}$ form a q-basis for a Banach space X, and the system $\{y_n\}_{n\in N}$ is p-close to $\{x_n\}_{n\in N}$, where $\frac{1}{p}+\frac{1}{q}=1$. Then the following properties are equivalent:*

i) $\{y_n\}_{n\in N}$-is complete in X;

ii) $\{y_n\}_{n\in N}$-is minimal in X;

iii) $\{y_n\}_{n\in N}$-forms an isomorphic basis to $\{x_n\}_{n\in N}$ for X.

Let $X_1 = X \oplus C^m$ and $\{\hat{u}_n\}_{n\in N} \subset X_1$ be some minimal system and $\left\{\hat{\vartheta}_n\right\}_{n\in N} \subset X_1^* = X^* \oplus C^m$ be its biorthogonal system:

$$\hat{u}_n = (u_n; \alpha_{n1}, ..., \alpha_{nm}); \quad \hat{\vartheta}_n = (\vartheta_n; \beta_{n1}, ..., \beta_{nm}).$$

Let $J = \{n_1, ..., n_m\}$ be some set of m natural numbers. Suppose

$$\delta = \det \|\beta_{n_i j}\|_{i,j=\overline{1,m}}.$$

The following theorem is true.

Theorem 2. *[5] Let the system $\{\hat{u}_n\}_{n\in N}$ form a basis for X_1. In order to the system $\{u_n\}_{n\in N_J}$, where $N_J = N \backslash J$ form a basis for X it is necessary and sufficient that the condition $\delta \neq 0$ be satisfied. In this case the biorthogonal system to $\{u_n\}_{n\in N_J}$ is defined by*

$$\vartheta_n^* = \frac{1}{\delta}\begin{vmatrix} \vartheta_n & \vartheta_{n1} & \dots & \vartheta_{nm} \\ \beta_{n1} & \beta_{n_1 1} & \dots & \beta_{n_m 1} \\ \dots & \dots & \dots & \dots \\ \beta_{nm} & \beta_{n_1 m} & \dots & \beta_{n_m m} \end{vmatrix}.$$

In particular, if $X$is a Hilbert space and the system $\{u_n\}_{n\in N}$ forms a Riesz basis for X_1,then under the condition $\delta \neq 0$, the system $\{u_n\}_{n\in N_J}$ also forms a Riesz basis for X. For $\delta = 0$ the system $\{u_n\}_{n\in N_J}$is not complete and is not minimal in X.

3. Stability of the basis properties of systems

Suppose that the direct decomposition $X = X_1 \oplus \ldots \oplus X_m$ holds, where X_i,$i=\overline{1,m}$ are Banach spaces. For convenience, the elements of X are identified with vectors: $x \in X \Leftrightarrow x = (x_1; ...; x_m)$, where $x_k \in X_k$, $k=\overline{1,m}$. The norm in X is defined by the formula $\|x\|_X = \sqrt{\sum_{i=1}^m \|x_i\|_{X_i}^2}$. It is clear that $X^* = X_1^* \oplus \ldots \oplus X_m^*$ and for $f \in X^*$ and $x \in X$ it holds $< x; f >= \sum_{i=1}^m < x_i; f_i >$ ($< \cdot\, ; \,\cdot >$ –is the value of the functional), where $f = (f_1, ..., f_m)$, $f_k \in X_k^*$, $k=\overline{1,m}$. For $x_k \in X_k$ let us denote by $\tilde{x}_k$ the element from X, which is defined by the formula $\tilde{x}_k = \left(\underbrace{0, ..., x_k}_{k}, ..., 0\right)$.

Suppose that a system $\{u_{in}\}_{n\in N}$ is given in each space X_i, $i=\overline{1,m}$, Consider the following system in X:

$$\hat{u}_{in}=(a_{i1}^{(n)}u_{1n},...,a_{im}^{(n)}u_{mn}), i=\overline{1,m}, n\in N, \tag{1}$$

where $a_{ij}^{(n)}$−are some numbers. Let $A_n=\left(a_{ij}^{(n)}\right)_{i,\,j=\overline{1,\,m}}$; $\Delta_n=\det A_n$.

The following theorem is proved.

Theorem 3. *Let the system $\{u_{in}\}_{n\in N}$ be complete (minimal) in $X_i, i=\overline{1,m}$. If $\Delta_n\neq 0$, $\forall n\in N$, then the system $\{\hat{u}_{in}\}_{i=\overline{1,m};n\in N}$ is also complete (minimal) in X.*

Proof. Let the system $\{u_{in}\}_{n\in N}$ be complete (minimal) in X_i, $i=\overline{1,m}$. If for any $\vartheta\in X^*$

$$<\hat{u}_{in},\,\vartheta>=0\,,\ \ i=\overline{1,\,m}, n\in N,$$

then from the representation $X^*=X_1^*\dot{\oplus}\ldots\oplus X_m^*$ and $\vartheta=(\vartheta_1,...,\vartheta_m)^t$, $\vartheta_i\in X_i^*$, $i=\overline{1,m}$, implies

$$\sum_{j=1}^{m}a_{ij}<u_{jn},\,\vartheta_j>=0\,,\quad i=\overline{1,\,m}. \tag{2}$$

Since $\Delta_n=\det\left(a_{ij}^{(n)}\right)\neq 0$, $n\in N$, then (2) has only trivial solution for each $n\in N$:

$$<u_{jn},\,\vartheta_j>=0,\ \ j=\overline{1,\,m}\,,\ \ n\in N.$$

Then from the completeness of the system $\{u_{jn}\}_{n\in N}$ in X_j implies that $\vartheta_j=0$, $j=\overline{1,\,m}$, i.e. $\vartheta=0$.

Now let the system $\{u_{in}\}_{n\in N}$ be minimal in X_i, and $\{\vartheta_{in}\}_{n\in N}\subset X_i^*$ be conjugate-biorthogonal system. Consider the following system in X^*

$$\hat{\vartheta}_{in}=\left(b_{1i}^{(n)}\vartheta_{1n};b_{2i}^{(n)}\vartheta_{2n};...;b_{mi}^{(n)}\vartheta_{mn}\right)=\sum_{s=1}^{m}b_{si}^{(n)}\tilde{\vartheta}_{sn},\quad i=\overline{1,\,m}\,,\ \ n\in N,$$

where the numbers $b_{ji}^{(n)}$− are the elements of the inverse matrix A_n^{-1}. We obtain

$$<\hat{u}_{in},\,\hat{\vartheta}_{lk}>=\sum_{j=1}^{m}\sum_{s=1}^{m}a_{ij}^{(n)}b_{sl}^{(k)}<\tilde{u}_{jn},\,\tilde{\vartheta}_{sk}>=$$

$$=\sum_{j=1}^{m}a_{ij}^{(n)}b_{jl}^{(k)}<u_{jn},\vartheta_{jk}>=\sum_{j=1}^{m}a_{ij}^{(n)}b_{jl}^{(k)}\delta_{nk}=\sum_{j=1}^{m}a_{ij}^{(n)}b_{jl}^{(n)}\delta_{nk}=\delta_{il}\delta_{nk}\,,\ i;l=\overline{1,m};n;k\in N.$$

The last expressions mean that the system $\left\{\hat{\vartheta}_{in}\right\}_{i=\overline{1,m};n\in N}$ is conjugated to the system $\{\hat{u}_{in}\}_{i=\overline{1,m};n\in N}$, i.e. the system $\{\hat{u}_{in}\}_{i=\overline{1,m};n\in N}$ is minimal in X.

Theorem is proved.

Theorem 4. *Let the system* $\{u_{in}\}_{n\in N}$ *be minimal in* X_i, $i=\overline{1,m}$. *If* $\exists n_0 \in N$, $\Delta_{n_0}=0$ *then the system* $\{\hat{u}_{in}\}_{i=\overline{1,m};n\in N}$ *is not minimal in* X.

Proof. Let for any $n_0 \in N$, $\Delta_{n_0}=0$. We will show that the system $\{\hat{u}_{in_0}\}_{i=\overline{1,m}}$ is linear dependent. From the condition $\det\left(a_{ij}^{(n_0)}\right)=0$ implies that, there are numbers c_i, $i=\overline{1,\,m}$, which not all equal to zero and such that

$$\sum_{i=1}^{m} a_{ij}^{(n_0)} c_i = 0\,,\; j=\overline{1,m}.$$

Then

$$\sum_{i=1}^{m} c_i \hat{u}_{in_0} = \sum_{i=1}^{m} c_i \sum_{j=1}^{m} a_{ij}^{(n_0)} \tilde{u}_{jn_0} =$$

$$= \sum_{j=1}^{m}\left(\sum_{i=1}^{m} a_{ij}^{(n_0)} c_i\right)\tilde{u}_{jn_0} = 0\,.$$

Thus, the system $\{\hat{u}_{in_0}\}_{i=\overline{1,m}}$ is linear dependent, consequently, all of the systems $\{\hat{u}_{in}\}_{i=\overline{1,m};n\in N}$ are linear dependent and especially are not minimal. Theorem is proved.

Theorem 5. *Let the system* $\{u_{in}\}_{n\in N}$ *be complete and minimal in* X_i , *for each* $i\in 1:m$. *If* $\exists n_0\in N$, $\Delta_{n_0}=0$, *then the system* $\{\hat{u}_{in}\}_{i=\overline{1,m};n\in N}$ *is not complete and is not minimal in* X.

Proof. Non-minimality of the system $\{\hat{u}_{in}\}_{i=\overline{1,m};n\in N}$ in X implies from the previous theorem. We will show that, it is not complete in X. From the condition $\Delta_{n_0} = \det\left(a_{ij}^{(n_0)}\right)=0$ implies that, there are numbers c_j, $j=\overline{1,\,m}$, which not all are equal to zero such that

$$\sum_{j=1}^{m} a_{ij}^{(n_0)} c_j = 0\,,\; j=\overline{1,m}.$$

Suppose

$$\tilde{u}_{jn} = \left(\underbrace{0,\,\ldots,\,u_{jn}}_{j},\,\ldots,\,0\right)\in X,\;\; j=\overline{1,\,m}.$$

Then the system $\{\tilde{u}_{jn}\}_{j=\overline{1,m};\;\, n\in N}$ is complete and minimal in X, and its conjugated system is in the following form

$$\tilde{\vartheta}_{jn} = \left(\underbrace{0,\,\ldots,\,\vartheta_{jn}}_{j},\,\ldots,\,0\right),\quad j=\overline{1,m};\;\; n\in N,$$

where $\left\{\vartheta_{jn}\right\}_{n\in N}\subset X_j^*$–is conjugate system to $\left\{u_{jn}\right\}_{n\in N}$. Consider the following functional

$$\vartheta_0=\sum_{s=1}^{m}c_s\tilde{\vartheta}_{sn_0}.$$

It is clear that $\vartheta_0\in X^{\,*}$ and $\vartheta_0\neq 0$. We will show that the functional , ϑ_0 annuls the system $\{\hat{u}_{in}\}$. Indeed, for $n=n_0$ we obtain

$$<\hat{u}_{in_0},\vartheta_0>=\sum_{j=1}^{m}a_{ij}^{(n_0)}<\tilde{u}_{jn_0},\,\vartheta_0>=\sum_{j=1}^{m}a_{ij}^{(n_0)}\sum_{s=1}^{m}c_s<\tilde{u}_{jn_0},\,\tilde{\vartheta}_{sn_0}>=$$

$$=\sum_{j=1}^{m}a_{ij}^{(n_0)}\sum_{s=1}^{m}c_s\delta_{js}=\sum_{j=1}^{m}a_{ij}^{(n_0)}c_j=0.$$

For $n\neq n_0$ we have

$$<\tilde{u}_{in},\,\vartheta_0>=\sum_{j=1}^{m}a_{ij}^{(n)}\sum_{s=1}^{m}c_s<\tilde{u}_{jn},\,\tilde{\vartheta}_{sn_0}>=0.$$

Thus, the system $\{\hat{u}_{in}\}_{i=\overline{1,m};\;n\in N}$ is not complete in X. Theorem is proved.

Theorem 6. *If all $\Delta_n=\det\left(a_{ij}^{(n)}\right)\neq 0$, $n\in N$, and for each $i\in 1:m$ the system $\{u_{in}\}_{n\in N}$ forms a basis in X_i , then the system $\{\hat{u}_{in}\}_{i=\overline{1,m};n\in N}$ forms a basis with parentheses in X. If, the conditions*

$$\sup_n\{\|u_{in}\|\,;\|\vartheta_{in}\|\}<+\infty,i=\overline{1,m},\sup_n\left\{\|A_n\|\,,\left\|A_n^{-1}\right\|\right\}<+\infty,\tag{3}$$

also hold, then the system $\{\hat{u}_{in}\}_{i=\overline{1,m};n\in N}$ forms a usual basis in X.

Proof. Let us present the system $\{\hat{u}_{in}\}$ in the following form

$$\hat{u}_{in}=\sum_{j=1}^{m}a_{ij}^{(n)}\tilde{u}_{jn}\,,\,i=\overline{1,m}\,;\,n\in N.\tag{4}$$

As shown above, the conjugated system is in the following form

$$\hat{\vartheta}_{in}=\sum_{j=1}^{m}b_{li}^{(n)}\tilde{\vartheta}_{l\,n},l=\overline{1,m};n\in N,\tag{5}$$

where the numbers b_{ji} are the elements of the inverse matrix A^{-1}. Hence we get (for $x\in X$)

$$\sum_{i=1}^{m}<x,\,\hat{\vartheta}_{in}>\hat{u}_{in}=\sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{l=1}^{m}a_{ij}^{(n)}b_{li}^{(n)}<x,\,\tilde{\vartheta}_{l\,n}>\tilde{u}_{jn}=$$

$$= \sum_{j=1}^{m} \sum_{l=1}^{m} \left(\sum_{i=1}^{m} b_{li}^{(n)} a_{ij}^{(n)} \right) < x,\, \tilde{\vartheta}_{l\,n} > \tilde{u}_{jn} =$$

$$= \sum_{j=1}^{m} \sum_{l=1}^{m} \delta_{lj} < x,\, \tilde{\vartheta}_{l\,n} > \tilde{u}_{jn} = \sum_{j=1}^{m} < x,\, \tilde{\vartheta}_{jn} > \tilde{u}_{jn}.$$

Consequently

$$S_N(x) = \sum_{n=1}^{N} \sum_{i=1}^{m} < x,\, \hat{\vartheta}_{in} > \hat{u}_{in} = \sum_{n=1}^{N} \sum_{j=1}^{m} < x,\, \tilde{\vartheta}_{jn} > \tilde{u}_{jn} =$$

$$= \sum_{j=1}^{m} \sum_{n=1}^{N} < x,\, \tilde{\vartheta}_{jn} > \tilde{u}_{jn} \to x, \quad \text{as } N \to \infty.$$

Thus, the system $\{\hat{u}_{in}\}_{i=\overline{1,m}\,;\,n\in N}$ forms a basis with parentheses in X.

Now let us assume that the condition (3) be fulfilled. Then

$$\sup_{i,n} \left\{ \|\tilde{u}_{in}\| \,;\, \left\|\tilde{\vartheta}_{in}\right\| \right\} < +\infty, i = \overline{1,m},$$

And from the representations (4) and (5) we obtain

$$\sup_{i,n} \left\{ \|\hat{u}_{in}\| \,;\, \left\|\hat{\vartheta}_{in}\right\| \right\} < +\infty.$$

Consequently, the system $\{\hat{u}_{in}\}$ is uniformly minimal and by Statement 1 it forms a usual bases in X.

Theorem 7. *If $X_i-$are Hilbert spaces, and $\{u_{in}\}_{n\in N}$ is a Riesz basis in X_i, $i = \overline{1,m}$, then for $\Delta_n \neq 0$, $n \in N$, the system $\{\hat{u}_{in}\}_{i=\overline{1,m}\,;\,n\in N}$ forms Riesz basis with parentheses in X, and under the condition* (3) *it forms a usual Riesz basis in X.*

Proof of the theorem implies from the Theorem 6 and Statement 2. Note that, in particular, when the matrixes A_n do not depend on n: $A_n = A$, $n \in N$, the similar results were obtained in [9,10].

4. Application to discontinuous differential operators

Consider the following model spectral problem for a second-order discontinuous differential operator

$$-y''(x) + q(x)\,y = \lambda y(x)\,,\; x \in (-1,\, 0) \bigcup (0,\, 1)\,, \tag{6}$$

with boundary conditions

$$y(-1) = y(1) = 0,$$

$$y(-0) = y(+0), \tag{7}$$
$$y'(-0) - y'(+0) = \lambda m y(0).$$

where $m \neq 0-$ is any complex number, $q(x)$ - summable complex-valued function. Such spectral problems arise when the problem of vibrations of a loaded in the middle of the string with fixed ends is solved by applying the Fourier method [11,12]. The justification of the Fourier method requires the study of the basis properties of the eigenfunctions of the spectral problem in the appropriate spaces of functions (as a rule, in Lebesgue or Sobolev spaces). Such questions for the problem (6),(7) studied by another method in [13,14]. Following two theorems are proved in [13].

Theorem 8. *[13] Let*

$$d = 4 + (mq_2(0))^2 + (mq_1(0))^2 + 8mq_2(0) - 2m^2 q_2(0) q_1(0) \neq 0,$$

where

$$q_1(0) = \frac{1}{2}\int_{-1}^{0} q(t)\,dt$$

and

$$q_1(0) = \frac{1}{2}\int_{-1}^{0} q(t)\,dt.$$

Then the spectral problem (6), (7) *has two series asymptotically simple eigenvalues* $\lambda_{1,n} = \rho_{1,n}^2$, $n = 1, 2, \ldots$ *and* $\lambda_{2,n} = \rho_{2,n}^2$, $n = 1, 2, \ldots$, *where* $\rho_{1,n}$ *and* $\rho_{2,n}$ *have asymptotics*

$$\rho_{1,n} = \pi n + \frac{\alpha_1}{n} + o\left(\frac{1}{n}\right)$$

and

$$\rho_{2,n} = \pi n + \frac{\alpha_2}{n} + o\left(\frac{1}{n}\right)$$

respectively, and the numbers α_1 *and* α_2 *are different complex numbers and are defined as follows:*

$$\alpha_1 = \frac{-\left(2mq_2(0) + mq_1(0)\right) + \sqrt{d}}{-2m\pi},$$
$$\alpha_2 = \frac{-\left(2mq_2(0) + mq_1(0)\right) - \sqrt{d}}{-2m\pi},$$

where $0 \leq \arg\sqrt{d} < \pi$.

Theorem 9. *[13] Let the function* $q(x)$ *satisfy the condition of the Theorem 8. Then the eigen functions* $y_{1,n}(x)$ *of the problem* (6),(7), *corresponding to eigen values* $\lambda_{1,n} = (\rho_{1,n})^2$ *and the eigen functions* $y_{2,n}(x)$, *which correspond to eigen values* $\lambda_{2,n} = (\rho_{2,n})^2$ *have the following asymptotics:*

$$y_{1,n}(x) = \begin{cases} \sin \pi n x + O\left(\frac{1}{n}\right), & x \in [-1, 0], \\ \gamma_{1,n} \sin \pi n x + O\left(\frac{1}{n}\right), & x \in [0, 1], \end{cases} \tag{8}$$

$$y_{2,\,n}\left(x\right)=\left\{\begin{array}{ll}\sin\pi nx+\mathrm{O}\left(\frac{1}{n}\right), & x\in\left[-1,\,0\right],\\ \gamma_{2,\,n}\sin\pi nx+\mathrm{O}\left(\frac{1}{n}\right), & x\in\left[0,\,1\right],\end{array}\right. \tag{9}$$

where the numbers $\gamma_{1,\,n}$ $\gamma_{2,\,n}$ are defined by the formula

$$\gamma_{1,\,n}=1+mq_1\left(0\right)-m\alpha_1\pi+O\left(\frac{1}{n}\right),$$

$$\gamma_{2,\,n}=1+mq_1\left(0\right)-m\alpha_2\pi+O\left(\frac{1}{n}\right).$$

By $W_p^k\left(-1,\,0\right)\oplus\left(0,\,1\right)$ we denote a space of functions whose constrictions on segments $\left[-1,\,0\right]$ and $\left[0,\,1\right]$ belong to Sobolev spaces $W_p^k\left(-1,\,0\right)$ and $W_p^k\left(0,\,1\right)$, respectively. Let's define the operator L in $L_p\left(-1,\,1\right)\oplus C$ as follows :

$$\begin{array}{l}D\left(L\right)=\left\{\hat{u}\in L_p\left(-1,\,1\right)\oplus C:\hat{u}=\left(u;\,mu\left(0\right)\right),\,u\in W_p^2\left(-1,\,0\right)\bigcup\left(0,\,1\right),\right.\\ \qquad\left.u\left(-1\right)=u\left(1\right)=0,\;\;u\left(-0\right)=u\left(+0\right)\right\}\end{array} \tag{10}$$

and for $\hat{u}\in D\left(L\right)$

$$L\hat{u}=\left(-u''+q\left(x\right)u;\;\;u'\left(-0\right)-u'\left(+0\right)\right). \tag{11}$$

Lemma 1. *Operator L, defined by the formulas* (10), (11) *is a linear closed operator with dense definitional domain in $L_p\left(-1,\,1\right)\oplus C$. Eigenvalues of the operator L and of the problem* (6), (7) *coincide, and $\left\{\hat{y}_k\right\}_{k=0}^{\infty}$ are eigenvectors of the operator L, where $\hat{y}_{2n-1}=\left(y_{2n-1}\left(x\right);\,my_{2n-1}\left(0\right)\right)$, $\hat{y}_{2n}=\left(y_{2n}\left(x\right);\,my_{2n}\left(0\right)\right)$.*

Proof. To prove the first part of the lemma we take $\hat{y}=\left(y;\,\alpha\right)\in L_p\left(-1,\,1\right)\oplus C$ and we define the functional $F\left(\hat{y}\right)$ as follows:

$$F\left(\hat{y}\right)=my\left(+0\right)-\alpha.$$

Let us assume

$$U_\nu\left(\hat{y}\right)=U_\nu\left(y\right),\nu=1,2,3,$$

where

$$U_1\left(y\right)=y\left(-1\right),\quad U_2\left(y\right)=y\left(1\right),\quad U_3\left(y\right)=y\left(-0\right)-y\left(+0\right).$$

Then F, $U_v, v=1,2,3,$ are bounded linear functionals on $W_p^2\left(-1,\,0\right)\bigcup\left(0,\,1\right)\oplus C$, but unbounded on $L_p\left(-1,\,1\right)\oplus C$. Therefore , (see, e.g. [15, pp. 27-29]) the set

$$D\left(L\right)=\left\{\hat{y}=\left(y;\,\alpha\right),\,y\in W_p^2\left(-1,\,0\right)\bigcup\left(0,\,1\right),\;\mathrm{F}\left(\hat{y}\right)=U_\nu\left(\hat{y}\right)=0,\;\;\nu=1,\,2,\,3\right\}$$

is dense everywhere in $L_p\left(-1,\,1\right)\oplus C$, and L is a closed operator as constriction of corresponding closed maximal operator.

The second part of the lemma is verified directly.

The lemma is proved.

Theorem 10. *In conditions of the Theorem 8 eigenvectors and conjugate vectors of the operator L, linearized problem* (6), (7) *form basis in* $L_p(-1,\,1)\oplus C$,*and for* $p=2$ *this basis is a Riesz basis.*

Proof. From the Lemma 1 implies that , L is a dense defined closed operator with compact resolvent. Then the system $\{\hat{y}_n\}_{n=0}^{\infty}$ of eigenvectors of the operator L is minimal in $L_p(-1,\,1)\oplus C$, and its conjugate system $\left\{\hat{\vartheta}_n\right\}_{n=0}^{\infty}$ is the system of eigenvectors of the conjugate operator L^* and is in the form

$$\hat{\vartheta}_n=(\vartheta_n,\bar{m}\vartheta_n(0))\ ,\ n=0,1,...,$$

here $\vartheta_n(x)\ ,\ n=0,1,...,$ are eigenfunctions of the conjugate spectral problem

$$-\vartheta''+\overline{q(x)}\vartheta=\lambda\vartheta, \tag{12}$$

$$\vartheta(-1)=\vartheta(1)=0\,;\ \vartheta(-0)=\vartheta(+0)\ ;\ \vartheta'(-0)-\vartheta'(+0)=\lambda\bar{m}\vartheta(0)\ . \tag{13}$$

By the similar way, for the problem (12), (13) we obtain, that for $\vartheta_n(x)$ hold following formulas:

$$\vartheta_{1,\,n}(x)=\begin{cases}\sin\pi nx+\mathrm{O}\left(\frac{1}{n}\right), & x\in[-1,\,0]\,,\\ \mu_{1,\,n}\sin\pi nx+\mathrm{O}\left(\frac{1}{n}\right), & x\in[0,\,1]\,,\end{cases} \tag{14}$$

$$\vartheta_{2,\,n}(x)=\begin{cases}\sin\pi nx+\mathrm{O}\left(\frac{1}{n}\right), & x\in[-1,\,0]\,,\\ \mu_{2,\,n}\sin\pi nx+\mathrm{O}\left(\frac{1}{n}\right), & x\in[0,\,1]\,,\end{cases} \tag{15}$$

where $\mu_{1,n},\mu_{2,n}$ are the normalization numbers and for which holds

$$\mu_{1,n}=a_1+O\left(\frac{1}{n}\right),\quad \mu_{2,n}=a_2+O\left(\frac{1}{n}\right),$$

and $a_1a_2\neq 0$. Denote

$$e_{1,\,n}(x)=\begin{cases}\sin\pi nx, & x\in[-1,\,0]\,,\\ \gamma_{1,\,n}\sin\pi nx, & x\in[0,\,1]\,,\end{cases} \tag{16}$$

$$e_{2,\,n}(x)=\begin{cases}\sin\pi nx, & x\in[-1,\,0]\,,\\ \gamma_{2,\,n}\sin\pi nx, & x\in[0,\,1]\,,\end{cases} \tag{17}$$

and consider the system $\{\hat{e}_n\}_{n=0}^{\infty}$, where

$$\hat{e}_0=(0;1)\,,\hat{e}_{2n}=(e_{2,n};0)\,,\hat{e}_{2n-1}=(e_{1,n};0)\,,n\in N.$$

Then $\{\hat{e}_n\}_{n=0}^{\infty}$ is basis in $L_p(-1,\,1)\oplus C$, besides for $1<p\leq 2$, from the formulas (16),(17) implies, that according to inequality Hausdorf-Young for trigonometric system (see., for example, [16]) for each $\hat{f}\in L_p(-1,1)\oplus C$ the inequality

$$\left(\sum_{B=0}^{\infty}\left|\left\langle\hat{f},\hat{e}_n\right\rangle\right|^q\right)^{\frac{1}{q}}\leq c\left\|\hat{f}\right\|_{L_q\oplus C},$$

is fulfilled and from the formulas (8),(9) implies that

$$\sum_n \|\hat{y}_n - \hat{e}_n\|^p_{L_p \oplus C} < \infty.$$

Then by Theorem 1 the system $\{\hat{y}_n\}_{n=0}^{\infty}$ also forms a basis in $L_p(-1, 1) \oplus C$ isomorphic to $\{\hat{e}_n\}_{n=0}^{\infty}$. If $p > 2$ $(1 < q < 2)$, then in this case from the formulas (14),(15) implies that, the system $\left\{\hat{\vartheta}_n\right\}_{n=0}^{\infty}$ is q- close to $\{\hat{e}_n\}_{n=0}^{\infty}$:

$$\sum_n \left\|\hat{\vartheta}_n - \hat{e}_n\right\|^q_{L_q \oplus C} < \infty,$$

and for each $\hat{g} \in L_q(-1, 1) \oplus C$

$$\left(\sum_{B=0}^{\infty} |\langle g, \hat{e}_n\rangle|^p\right)^{\frac{1}{p}} \leq c \|\hat{g}\|_{L_q \oplus C},$$

and by Theorem 1 the system $\left\{\hat{\vartheta}_n\right\}_{n=0}^{\infty}$ forms a basis in $L_q(-1, 1) \oplus C$ and consequently, the system $\{\hat{y}_n\}_{n=0}^{\infty}$ forms a basis in $L_p(-1, 1) \oplus C$ isomorphic to $\{\hat{e}_n\}_{n=0}^{\infty}$.

As noted in the Theorem 8, $\alpha_1 \neq \alpha_2$, because, although one of these numbers does not equal zero. With this in mind and applying the Theorem 2 and 7, we obtain, that right is next

Theorem 11. *If $\alpha_1 \neq 0$, then for sufficiently great values of n_0 we eliminate $y_{1, n_0}(x)$, and if $\alpha_2 \neq 0$, then for sufficiently great values of n_0 we eliminate $y_{2, n_0}(x)$ from the system of the eigen and conjugate functions of the problem* (6), (7) *we obtain a basis in $L_p(-1, 1)$, and for $p = 2$ we obtain a Riesz basis in $L_2(-1, 1)$.*

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