Caspian Journal of Applied Mathematics, Ecology and Economics
V. 3, No 2, 2015, December
ISSN 1560-4055

Absence of Positive Solutions of a Semi-linear Parabolic Equation with Lowest Derivatives and Time Periodic Coefficients in External Domains

S.H. Bagirov*, T.Z. Garayev

Abstract. In the exterior of a compact we consider a semi-linear parabolic equation with lowest derivatives and with time periodic coefficients. Depending on degree of nonlinearity and the coefficients of the equation, we find exact estimations on non-existence of positive solutions.

Key Words and Phrases: semi-linear parabolic equation, time periodic global positive solutions, inequality Harnack.

2010 Mathematics Subject Classifications: 35A01

Let Ω be the exterior of some compact in R_x^n, containing the origin of coordinates. In the cylinder $Q = \Omega \times (-\infty, +\infty)$ consider the following equation

$$\frac{\partial u}{\partial t} = Lu + C\,|x|^{\sigma}\,|u|^{p-1}\,u, \tag{1}$$

where

$$L \equiv \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^{n} b_i(x,t)\frac{\partial u}{\partial x_i} \equiv$$

$$\equiv div(A\nabla u)(x,t) + B(x,t)\nabla u(x,t),$$

$$A = A(x,t) = (a_{ij}(x,t))_{i,j=1}^{n}, \;\; B(x,t) = (b_1(x,t), \ldots, b_n(x,t)).$$

Suppose that $n \geqslant 3$, $p > 1$, $\sigma > -2$, the coefficients $a_{ij}(x,t)$, $b_i(x,t)$ are measurable, T periodic with respect to t functions in $R^n \times (-\infty; +\infty)$ and $a_{ij}(x,t)$ satisfy the following conditions:

I) $A(x,t)$ is a symmetric matrix with the Holder continuous functions $a_{ij}(x,t)$ in $R^n \times (-\infty; +\infty)$ and there exists $\lambda > 1$ such that $\lambda^{-1}I \leqslant A(x,t) \leqslant \lambda I$ for all $(x,t) \in R^n \times [0,T]$.

We will study the existence of the positive solution of equation (1). Note that the case when $B(x,t) \equiv 0$ was considered in the paper [1], the case when the coefficients are time-independent, in the paper [2].

*Corresponding author.

A great number of works were devoted to nonlinear elliptic equations of type (1) (see for example [3, 4, 7]). Such equations may occur in geometry [6].

We will determine the conditions on $B(x,t)$ later.

At first we introduce some denotation.

Denote $Q_T = \Omega \times (0,T)$, $S_R = \{x; |\,x\,| = R\} \times \{-\infty; +\infty\}$, $B_R = \{x; |\,x\,| < R\}$, $B_R^C = \{x; |\,x\,| > R\}$; $B_{\rho_1,\rho_2} = \{x; \rho_1 < x < \rho_2\}$, $Q_T^R = B_R \times (0,T)$, $Q_T^{R,C} = B_R^C \times (0,T)$, $Q_T^{\rho_1,\rho_2} = B_{\rho_1,\rho_2} \times (0,T)$.

We assume that $u(x,t) \in W_2^{1,1/2}(Q_T)$, if $u(x,t+T) \in u(x,t)$, $u(x,t) \in \ \in W_2^{1,0}(Q_T)$ and $\|u\|^2 = \sum\limits_{k=-\infty}^{+\infty} |k| \cdot \int\limits_{\Omega} |u_k(x)|^2 \, dx < \infty$, where $u_k(x) = \frac{1}{T} \int\limits_0^T u(x,t) e^{-ik\frac{2\pi}{T}t} dt$.

The space $W_2^{1,1/2}(Q_T)$ will be a Hilbert space if we define the norm in it by the equality

$$\|u\|_{W_2^{1,1/2}(Q_T)} = \left[\|u\|_{2,Q_T}^2 + \|u_x\|_{2,Q_T}^2 + \|u\|^2 \right]^{1/2}, \tag{2}$$

where

$$u_x \equiv \nabla u \equiv (u_{x_1}, \ldots, u_{x_n}), \quad \|u\|_{2,Q_T}^2 = \int\limits_{Q_T} |u(x,t)|^2 \, dxdt.$$

Denote by $\overset{\circ}{W}{}_2^{1,1/2}(Q_T)$ the completion of $C^{0,\infty}(Q_T)$ by the norm (2), where $C^{0,\infty}(Q_T)$ is the set of infinitely smooth, T-periodic with respect to t functions equal to zero in the vicinity of $\partial\Omega$ and infinity.

We call the function $u(x,t)$ a generalized solution of the equation (1), if $u(x,t) \in W_{2,loc}^{1,1/2}(Q_T) \cap L_{\infty,loc}(Q_T)$, $B\nabla u \in L_{1,loc}(Q_T)$ and for any $\varphi(x,t) \in \ \in \overset{\circ}{W}{}_2^{1,1/2}(Q_T)$ the following integral identity is fulfilled:

$$2\pi \sum_{k=-\infty}^{+\infty} (ik) \int\limits_{\Omega} u_k(x) \varphi_{-k}(x) dx + \int\limits_{Q_T} \sum_{i,j=1}^{n} a_{ij}(x,t) \frac{\partial u}{\partial x_j} \frac{\partial \varphi}{\partial x_i} dxdt -$$

$$- \int\limits_{Q_T} \sum_{i=1}^{n} b_i(x,t) \frac{\partial u}{\partial x_i} \varphi \, dxdt = C \cdot \int\limits_{Q_T} |x|^{\sigma} |u|^{p-1} u\varphi \, dxdt.$$

Denote (see [7])

$$N_h^{\alpha}(b) \equiv \sup_{x,t} \int\limits_{t-h}^{t} \int\limits_{R^n} |B(y,s)| \frac{1}{(t-s)^{\frac{n+1}{2}}} \exp\left(-\alpha \frac{|x-y|^2}{t-s} \right) dyds +$$

$$+ \sup_{y,s} \int\limits_{s}^{s+h} \int\limits_{R^n} |B(x,t)| \frac{1}{(t-s)^{\frac{n+1}{2}}} \exp\left(-\alpha \frac{|x-y|^2}{t-s} \right) dxdt,$$

where α is a fixed positive constant.

Definition 1. It is said that the vector B satisfies the condition K if

$$\lim_{h\to 0} N_h^\alpha(B)=0$$

for all $\alpha>0$.

Denote $N_\infty^\alpha(B)\equiv \lim\limits_{h\to\infty} N_h^\alpha(B)$.

Definition 2. Let $H(x)\in L_{1,loc}(R^n)$. It is said the $H(x)$ belongs to the class $\widehat{K}_{n+1,\infty}$ if

$$M_{n+1}(H)\equiv \sup_{x\in R^n}\int\limits_{R^n}\frac{|H(y)|}{|x-y|^{n-1}}dy<\infty.$$

It is easy to show that (see [7]) if $H(x)\in\widehat{K}_{n+1,\infty}$, then for any $\alpha>0$ $\quad N_\infty^\alpha(H)<\infty$.

Denote by $\Gamma(x,t;y,s)$ the weak fundamental solution of the equation

$$Lu-\frac{\partial u}{\partial t}=0. \tag{3}$$

If follows from the results of the papers [7, 8] that if for some α $N_\infty^\alpha(B)$ is rather small, then equation (2)has a unique fundamental solution $\Gamma(x,t;y,s)$ and there exist the constants C_1, $C_2>0$, such that

$$\frac{1}{C_1(t-s)^{n/2}}\exp\left(-\frac{|x-y|^2}{C_2(t-s)}\right)\leqslant\Gamma(x,t;y,s)\leqslant$$

$$\leqslant\frac{C_1}{(t-s)^{n/2}}\exp\left(-C_2\frac{|x-y|^2}{t-s}\right), \tag{4}$$

$$|\nabla_x G(x,t;y,s)|\leqslant\frac{C_1}{(t-s)^{(n+1)/2}}\exp\left(-C_2\frac{|x-y|^2}{t-s}\right) \tag{5}$$

for all $x,y\in R^n,\ t>s$

$$\Gamma(x,t;y,s)=0\quad\text{for}\quad t<s.$$

Let $B(x,t)$ satisfy the following conditions:

II) $|B(x,t)|\leqslant C_3\,|V(x)|$, where $V(x)\in\widehat{K}_{n+1,\infty}$ and there exists $\varepsilon>0$ such that $M_{n+1}(V)<\varepsilon$.

III) There exist the constants $C_4>0$, $\beta\in(0,1)$ such that

$$\int\limits_{Q_T^{R,C}}\left|\bar{B}\right|^2\varphi^2dxdt\leqslant C_4\int\limits_{Q_T^{R,C}}|\nabla\varphi|^2\,dxdt,$$

$$(1-\beta)\int\limits_{Q_T^{R,C}}\sum_{i,j=1}^n a_{ij}(x,t)\frac{\partial\varphi}{\partial x_j}\frac{\partial\varphi}{\partial x_i}dxdt-\int\limits_{Q_T^{R,C}}\sum_{i,j=1}^n\bar{b}_i(x,t)\frac{\partial\varphi}{\partial x_i}\varphi\,dxdt\geqslant 0$$

for any $\varphi(x,t) \in \overset{\circ}{W}{}_2^{1,1/2}(Q_T^{R,C})$, where $\bar{B} := B \cdot \chi_{Q_T^{R,C}}$, $\chi_{Q_T^{R,C}}$ is a characteristic function $Q_T^{R,C}$.

Before we pass to the main result, we prove some auxiliary lemmas.

Lemma 1. Let $A(x,t)$, $B(x,t)$ satisfy the conditions **I**), **II**), **III**) and $u(x,t)$ be a nonnegative solution the inequality $Lu - \frac{\partial u}{\partial t} \leqslant 0$ be such that $u|_{|x|=R_0} > 0$. Then $u(x,t) \geqslant C_0 |x|^{2-n}$.

Proof. Let $\Gamma(x,t)$ be a fundamental solution of equation (3) with a singularity in the origin of coordinates.

Consider the function

$$\Gamma'(x,t) = \sum_q \Gamma(x, t+Tq), \tag{6}$$

where the summation is taken over all integer q. If series (6) converges, then it is a periodic solution of equation (3). According to estimation (4) for $\Gamma(x,t+Tq)$ we get

$$\Gamma'(x,t) \geqslant \sum_q \frac{1}{C_1}(t+Tq)^{-n/2} e^{-\frac{|x|^2}{C_2(t+Tq)}} \geqslant$$

$$\geqslant C_5 \int\limits_{-t/T}^{\infty} (t+Ts)^{-n/2} e^{-\frac{|x|^2}{C_2(t+Ts)}} ds - C_6 |x|^{-n} \geqslant C_7 |x|^{2-n}.$$

By the results of the paper [7], $u(x,t)$ satisfies the Harnack inequality. Then there exists $C_0 = const > 0$ such that $u(x,t) - C_0\Gamma'(x,t) > 0$ for $|x| = R_0$.

Consider the function

$$v(x,t) = u(x,t) - C_0\Gamma'(x,t) + \sigma,$$

where $0 < \sigma < \inf\limits_{|x|=R_0} (u - C_0\Gamma')$.

Since, $\Gamma'(x,t) \to 0$ as $|x| \to \infty$, then for large R $\quad v|_{|x|=R} > 0$.

Thus in $Q_T^{R_0,C}$

$$Lv - \frac{\partial v}{\partial t} \leqslant 0 \quad \text{and} \quad v|_{|x|=R_0} > 0, \quad v|_{|x|=R} > 0.$$

Prove that $v > 0$ in $Q_T^{R_0,R}$. In the definition of the solution we take the test function $\varphi(x,t) = \max(-v, 0) \equiv v_-$. Then we get:

$$-\int\limits_{Q_T^{R_0,R}} v\frac{\partial v}{\partial t} dxdt - \int\limits_{Q_T^{R_0,R}} \sum_{i,j=1}^n a_{ij}(x,t) \frac{\partial v}{\partial x_j}\frac{\partial v_-}{\partial x_i} dxdt +$$

$$+ \int\limits_{Q_T^{R_0,R}} \sum_{i=1}^n b_i(x,t) \frac{\partial v}{\partial x_i} v_- \, dxdt \leqslant 0.$$

Hence

$$\beta \int\limits_{Q_T^{R_0,R}} \sum_{i,j=1}^{n} a_{ij}(x,t) \frac{\partial v_-}{\partial x_j} \frac{\partial v_-}{\partial x_i} dxdt + (1-\beta) \int\limits_{Q_T^{R_0,R}} \sum_{i,j=1}^{n} a_{ij}(x,t) \frac{\partial v_-}{\partial x_j} \frac{\partial v_-}{\partial x_i} dxdt -$$

$$- \int\limits_{Q_T^{R_0,R}} \sum_{i=1}^{n} b_i(x,t) \frac{\partial v_-}{\partial x_i} v_- \, dxdt \leqslant 0.$$

Using condition **III**), we have

$$\beta \int\limits_{Q_T^{R_0,R}} \sum_{i,j=1}^{n} a_{ij}(x,t) \frac{\partial v_-}{\partial x_j} \frac{\partial v_-}{\partial x_i} dxdt \leqslant 0.$$

So $v_- \equiv 0$. Then $v > 0$ in $Q_T^{R_0,C}$. Tending δ zero, we get

$$u \geqslant C_0 \Gamma' \geqslant C_0 \left|x\right|^{2-n} \quad \text{for} \quad |x| \geqslant R_0.$$

This proves Lemma 1.

Lemma 2. Let the conditions of lemma 1 be fulfilled, $0 \leqslant W(x,t) \in L^{\infty}_{loc}(Q_T^{R_0,C})$, $W(x,t+T) = W(x,t)$ and $|x|^2 W(x,t) \to \infty$ as $x \to \infty$. Then in the cylinder $Q_T^{R_0,C}$ there is no positive solution of the inequality

$$Lu + W(x,t)u - \frac{\partial u}{\partial t} \leqslant 0.$$

Proof. Let it be not so, i.e. there exists the positive solution $u(x,t)$. Then in definition of the solution we take the test function in the form φ^2/u, where $\varphi \in C_0^{\infty}(B_{\rho,2\rho})$, $0 \leqslant \varphi \leqslant 1$, $\varphi = 1$ for $\frac{5}{4}\rho < |x| < \frac{7}{4}\rho$ and $|\nabla \varphi| < \frac{5}{\rho}$. Then we get

$$\inf_{Q_T^{\rho,2\rho}} W(x,t) \cdot \int\limits_{B_{\rho,2\rho}} \varphi^2 dx \leqslant \frac{1}{T} \int\limits_{Q_T^{R_0,C}} W(x,t)\varphi^2 dxdt \leqslant$$

$$\leqslant \frac{1}{T}\mu_1 \sum_{j=1}^{n} \int\limits_{Q_T^{R_0,C}} \left(\sum_{i=1}^{n} a_{ij} \frac{\partial \varphi}{\partial x_i} - \frac{1}{2} b_j \varphi \right)^2 dxdt \leqslant$$

$$\leqslant \frac{1}{T}\mu_1 \int\limits_{Q_T^{R_0,C}} \left[\sum_{j=1}^{n} \left(\sum_{i=1}^{n} a_{ij} \frac{\partial \varphi}{\partial x_i} \right)^2 + |B|^2 \cdot \varphi^2 \right] dxdt \leqslant C_8 \int\limits_{B_{\rho,2\rho}} |\nabla \varphi|^2 \, dx.$$

Hence, it follows that $\inf\limits_{Q_T^{\rho,2\rho}} W(x,t) \left|x\right|^2$ is bounded for large ρ. This contradicts the condition of lemma 2.

This proves lemma 2.

In $Q_T^{R_0,C}$ consider the equation

$$-\frac{\partial v}{\partial t}+L^{*}v=0, \tag{7}$$

where

$$L^{*}v\equiv\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial v}{\partial x_j}\right)-\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(b_i(x,t)v\right).$$

Lemma 3. Let the conditions **I**), **II**), **III**) be fulfilled. There exists $\varepsilon>0$ such that condition **II**) imply that the equation (7) has in $Q_T^{R_0,C}$ the solution $v(x,t)$, and $C_9\leqslant v(x,t)\leqslant C_{10}$, $0<C_9<C_{10}<\infty$.

Proof. Let $R<R_0$. Consider the following problem:

$$-\frac{\partial \omega}{\partial t}+\sum_{i,j=1}^{n}\frac{\partial}{\partial x_j}\left(a_{ij}(x,t)\frac{\partial \omega}{\partial x_i}\right)=\sum_{i=1}^{n}\frac{\partial}{\partial x_i}(b_i^m(x,t)(\omega+1)), \tag{8}$$

$$\omega|_{\partial B_{R_0,R}}=0,\quad \omega(x,t+T)=\omega(x,t), \tag{9}$$

where

$$b_i^{(m)}(x,t)=\begin{cases} m & \text{if} \quad b>m,\\ b_i(x,t) & \text{if} \quad b\leqslant m.\end{cases}$$

Problem (8), (9) has the solution (see [9]) from the class $\overset{\circ}{W}{}_2^{1,1/2}(Q_T^{R_0,R})\cap\ L_\infty(Q_T^{R_0,R})$.

Let $G_R(x,t;y,s)$ be the Green function of the equation

$$\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial u}{\partial x_j}\right)=\frac{\partial u}{\partial t}\quad \text{in}\quad Q_T^{R_0,R}.$$

Then we can write the solution of problem (8), (9) in the form

$$\omega_{m,R}(x,t)=\int\limits_{-\infty}^{t}\int\limits_{B_{R_0,R}}G_R(x,t;y,s)\nabla_y(B^m(y,s))(\omega_{m,R}(y,s)+1)dyds=$$

$$=-\int\limits_{-\infty}^{t}\int\limits_{B_{R_0,R}}\nabla_y G_R(x,t;y,s)B^m(y,s)(\omega_{m,R}(y,s)+1)dyds.$$

Using estimation (5), for the derivatives of the fundamental solution we get

$$\|\omega_{m,R}\|_\infty\leqslant\|\omega_{m,R}+1\|_\infty\cdot\int\limits_{-\infty}^{t}\int\limits_{R^n}C_1(t-s)^{-\frac{n+1}{2}}e^{-C_2\frac{|x-y|^2}{t-s}}\left|B(y,s)\right|dyds\leqslant$$

$$\leqslant C_1 \cdot \|\omega_{m,R}+1\|_\infty N_\infty^{C_2}(|\,b\,|) \leqslant C_1 \cdot \varepsilon \, \|\omega_{m,R}+1\|_\infty .$$

If we take $\varepsilon < \frac{1}{2C_1}$, we get $|\omega_{m,R}| < 1$. So, there exist the constants R and m independent of $C_9, C_{10} > 0$ such that $C_9 < \omega_{m,R}+1 < C_{10}$. The function $v_{m,R} = \omega_{m,R}+1$ is the solution of equation (7), and $C_9 < v_{m,R} < C_{10}$. Then $v_{m,R}$ is weakly compact in $W_{loc}^{1,1/2}(Q_T^{R_0,C})$. Passing to limit as $m, R \to \infty$, we get the statement of the lemma.

In $Q_T^{R_0,C}$ we consider the linear equation

$$-\frac{\partial v}{\partial t} + Lv + \alpha_1 \cdot |x|^{-2}\, v = 0, \tag{10}$$

where $R > 1$, $\quad \alpha_1$ is a rather small positive number.

Lemma 4. Let the conditions of lemma 3 be fulfilled, and $v(x,t)$ be a nonnegative solution of equation (10) such that $v|_{|x|=R} > 0$. Then there exist $C_{11} > 0$, $R_1 > R$, such that $v(x,t) \geqslant C_{11}\, |x|^{2-n} \log|x|$ for $|x| \geqslant R_1$.

Proof. Let $\varphi(x) \in C_0^1(B_R^C)$, be such that $0 \leqslant \varphi \leqslant 1$, $\varphi = 1$ for $2R \leqslant |x| \leqslant \rho$, $\varphi = 0$ for $|x| \leqslant \frac{3R}{2}$, $x \geqslant 2\rho$, $|\nabla\varphi| \leqslant \frac{C}{\rho}$ for $\rho \leqslant |x| \leqslant 2\rho$.

For $\rho > 2R$ denote $m_\rho := \inf\limits_{|x|=\rho} v(x,t)$. In the definition of the solution take the test function in the form $v_1\varphi$, where $v_1(x,t)$ is the solution of equation (7) such that $0 < C_9 \leqslant v_1(x,t) \leqslant C_{10}$, $C_9, C_{10} = const$.

Then we get:

$$-\sum_{k=-\infty}^{+\infty} (ik) \int\limits_{B_R^C} (v_1)_{-k} \cdot (v\varphi)_{-k} dx - \int\limits_{Q_T^{R_0,C}} \sum_{i,j=1}^{n} a_{ij}(x,t) \frac{\partial (v\varphi)}{\partial x_j} \frac{\partial v_1}{\partial x_i}\, dxdt +$$

$$+ \int\limits_{Q_T^{R_0,C}} \sum_{i=1}^{n} b_i(x,t) v_1 \cdot \frac{\partial (v\varphi)}{\partial x_i}\, dxdt + \int\limits_{Q_T^{R_0,C}} \sum_{i,j=1}^{n} a_{ij}(x,t) \frac{\partial v_1}{\partial x_i}\, v \frac{\partial \varphi}{\partial x_j} dxdt -$$

$$- \int\limits_{Q_T^{R_0,C}} \sum_{i,j=1}^{n} a_{ij}(x,t) \frac{\partial v}{\partial x_j}\, v_i \frac{\partial \varphi}{\partial x_i} dxdt - \int\limits_{Q_T^{R_0,C}} \sum_{i=1}^{n} b_i(x,t) v_1 \cdot v \frac{\partial \varphi}{\partial x_i}\, dxdt +$$

$$+\alpha_1 \cdot \int\limits_{Q_T^{R_0,C}} \frac{1}{|x|^2} v \cdot v_1 \cdot \varphi \, dxdt = 0.$$

Taking into account that $v_1(x,t)$ is the solution of equation (7), hence we get,

$$\alpha_1 \cdot \int\limits_{Q_T^{R_0,C}} \frac{1}{|x|^2} v \cdot v_1 \cdot \varphi \, dxdt = - \int\limits_{Q_T^{R_0,C}} \sum_{i,j=1}^{n} a_{ij}(x,t) \frac{\partial v_1}{\partial x_i}\, v \cdot \frac{\partial \varphi}{\partial x_j} dxdt +$$

$$+ \int\limits_{Q_T^{R_0,C}} \sum_{i,j=1}^{n} a_{ij}(x,t) \frac{\partial v}{\partial x_j}\, v_1 \cdot \frac{\partial \varphi}{\partial x_i} dxdt + \int\limits_{Q_T^{R_0,C}} \sum_{i=1}^{n} b_i(x,t) v_1 \cdot v \frac{\partial \varphi}{\partial x_i}\, dxdt =$$

$$= \int\limits_{\frac{3R}{2}\leqslant|x|\leqslant 2R} \sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial v_1}{\partial x_j}\, v\cdot\frac{\partial\varphi}{\partial x_j}dxdt + \int\limits_{\frac{3R}{2}\leqslant|x|\leqslant 2R} \sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial v}{\partial x_j}\, v_1\cdot\frac{\partial\varphi}{\partial x_i}dxdt+$$

$$+ \int\limits_{\frac{3R}{2}\leqslant|x|\leqslant 2R} \sum_{i=1}^{n} b_i(x,t)v_1 v\cdot\frac{\partial\varphi}{\partial x_i}dxdt + \int\limits_{\rho\leqslant|x|\leqslant 2\rho} \sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial v_1}{\partial x_i}v\cdot\frac{\partial\varphi}{\partial x_i}dxdt+$$

$$+ \int\limits_{Q_T^{\rho,2\rho}} \sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial v}{\partial x_i}v_1\cdot\frac{\partial\varphi}{\partial x_i}dxdt + \int\limits_{\rho\leqslant|x|\leqslant 2\rho} \sum_{i=1}^{n} b_i(x,t)v_1\cdot v\cdot\frac{\partial\varphi}{\partial x_i}dxdt.$$

Using condition **III**), the Cacciopoli and Harnack inequality, we get

$$\alpha_1\cdot\int\limits_{Q_T^{R_0,C}} |x|^{-2}\, v\cdot v_1\cdot\varphi\, dxdt \leqslant C_{12} + C_9 m_\rho\cdot\rho^{\frac{n}{2}-1}\left\|\psi\nabla v_1\right\| +$$

$$+C_{10}\rho^{\frac{n}{2}-1}\left\|\psi\nabla v\right\| + C_{13}\left\|\,|B|\cdot\psi v\,\right\| \leqslant C_{12} + C_{14}m_\rho\cdot\rho^{n-2}, \tag{11}$$

where $\psi\in C_0^1\left(B_{\frac{3\rho}{4},\frac{2\rho}{4}}\right)$, $0\leqslant\psi\leqslant 1$ and $|\nabla\psi|\leqslant\frac{C}{\rho}$.

Estimating the left hand side of (11):

$$\alpha_1\cdot\int\limits_{Q_T^{R_0,C}} |x|^{-2}\, v\cdot v_1\cdot\varphi\, dxdt \geqslant \alpha_2 C_3 \int\limits_{Q_T^{R,C}} |x|^{-2}\, v dxdt \geqslant$$

$$\geqslant \alpha_3\cdot\int |x|^{-4}\, dx \geqslant \alpha_4\cdot\ln\rho. \tag{12}$$

Combining (11) and (12), we arrive at the inequality $\alpha_4\cdot\ln\rho\leqslant C + C_6\cdot m_\rho\cdot\rho^{n-2}$. Hence, using the Harnack inequality, we get the proof of the lemma.

Theorem 1. Let the conditions **I**), **II**), **III**) be fulfilled, and $n\geqslant 3$, $p>1$, $\sigma>-2$. There exists $\varepsilon>0$, such that conditions **II**) imply that the equation (1) for $2+\sigma+(2-n)(p-1)\geqslant 0$ has no positive solutions in Q_T.

Proof.a) Let at first $2+\sigma+(2-n)(p-1)>0$. Denote $W(x,t)=a_0(x,t)\,|u|^{p-1}$. If $u(x,t)$ is a positive solution of equation (1), then $u|_{|x|=R_0}>0$ and $Lu-\frac{\partial u}{\partial t}\leqslant 0$, where R_0 is such that $B_{R_0}^C\subset\Omega$. By lemma 1, $|x|^2\,W(x,t)\geqslant\quad C_{15}\,|x|^{\sigma}\,|x|^{(2-n)(p-1)}=C_{15}\,|x|^{2+\sigma+(2-n)(p-1)}$. Then $|x|^2\,W(x,t)\to\infty$ as $|x|\to\infty$. So, by lemma 2 a positive solution does not exist.

b) Let now $2+\sigma+(2-n)(p-1)=0$. If there exists a positive solution of equation (1), then by lemma 1

$$-\frac{\partial u}{\partial t} + Lu + \alpha_2\cdot|x|^{-2}\, u\leqslant 0.$$

In $Q_T^{R_0,\infty}$ consider the equation

$$-\frac{\partial v}{\partial t} + Lv + \alpha_2 \cdot |x|^{-2} v = 0. \tag{13}$$

If $v(x,t)$ is a positive solution of equation (13), then by lemma 4

$$v(x,t) \geqslant C_{11} |x|^{2-n} \cdot \lg |x| .$$

Now we show that indeed the equation (13) has a positive solution in $Q_T^{R_0,\infty}$. For simplicity of notation, we take $R_0 = 1$. Consider the following problem:

$$-\frac{\partial v_R}{\partial t} + Lv_R + \alpha_2 |x|^{-2} v_R = 0, \tag{14}$$

$$v_R\big|_{|x|=1} = 1, \quad v\big|_{|x|=R} = 0, \quad v_R(x,t+\tau) = v_R(x,t). \tag{15}$$

It is known that problem (14), (15) has the solution $v_R(x,t)$ (see [9]). Prove that $0 \leqslant v_R \leqslant 1$ in $Q_T^{1,R}$. At first show that $v_R \leqslant 1$. Let it be not so. Denote $\varphi(x,t) = (v_R - 1)^+$ and in the definition of the solution we take the test function $\varphi(x,t)$. Then we get:

$$2\pi \sum_{i=-\infty}^{+\infty} (ik) \int\limits_{1<|x|<R} v_{R_k}(x)\varphi_{-k}(x)dx + \int\limits_{Q'} \sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial \varphi}{\partial x_j}\frac{\partial \varphi}{\partial x_i} dxdt -$$

$$- \int\limits_{Q'} \sum_{i=1}^{n} b_i(x,t)\frac{\partial \varphi}{\partial x_i}\varphi \, dxdt - \alpha_2 \cdot \int\limits_{Q'} \frac{1}{|x|^2} v_R(v_R - 1)dxdt = 0,$$

where $Q' = \operatorname{supp} \varphi \subset Q_T^{1,R}$.

Taking the test function in the form

$$\varphi_h(x,t) = h^{-1} \int\limits_{t}^{t+h} \psi(x,\tau)d\tau,$$

is easy to show that the first addend equals zero. Using condition **II**) and the Hardy inequality, we get

$$\alpha\lambda^{-1} \int\limits_{Q'} |\nabla v_R|^2 \, dxdt + \alpha_2 \int\limits_{Q'} \frac{1}{|x|^2} v_R \, dxdt =$$

$$= \alpha_2 \int\limits_{Q'} \frac{1}{|x|^2} v_R^2 \, dxdt \leqslant \alpha_2 C_{16} \int\limits_{Q'} |\nabla v_R|^2 \, dxdt.$$

Then

$$(\alpha\lambda^{-1} - \alpha_2 C_{16}) \cdot \int\limits_{Q'} |\nabla v_R|^2 \, dxdt + \alpha_2 \int\limits_{Q'} \frac{1}{|x|^2} v_R \, dxdt \leqslant 0.$$

Since α_2 is rather small, then $\alpha \cdot \lambda^{-1} - \alpha_2 C_{16} > 0$.

Hence $v_R \equiv 0 \;\; Q'$. So $v_R \leqslant 1$.

Similarly we can show that $v_R \geqslant 0$. For any R, the functions $v_R(x,t)$ are uniformly bounded. Then by the compactness theorem, as $R \to \infty$ the functions $v_R(x,t)$ converge to some function $v(x,t)$, that will be a sought - for solution of equation (13). Assume

$$H_R(x,t) = u(x,t) - C_{17} v_R(x,t),$$

where $C_{17} = \frac{1}{2} \min\limits_{|x|=1} u(x,t)$.

Then

$$-\frac{\partial H_R}{\partial t} + LH_R + \alpha_2 \cdot |x|^{-2} H_R \leqslant 0,$$

$$H_R|_{|x|=1} > 0, \quad H_R|_{|x|=R} > 0, \quad H_R(x,t+T) = H_R(x,t).$$

As above we can show that $H_R \geqslant 0$ in $Q_T^{1,R}$ for any R. Passing to limit as $R \to +\infty$, we get $u(x,t) \geqslant v(x,t)$ in $Q_T^{R,C}$. Hence

$$u(x,t) \geqslant C_{11} |x|^{2-n} \lg |x| \,.$$

Then as in a), by lemma 2 we get that there is no positive solution in $Q_T^{1,C}$. This proves the theorem.

References

[1] Sh.G. Bagyrov, *On the existence of positive solution of a second order linear parabolic equation with periodic time coefficients*, Diff. Uravn., **43(4)**, 2007, 562–565. (in Russian)

[2] V. Kondratiev, V. Liskevich, Z. Sobol, O. Us, *Heat kernel estimates with applications to an exterior problem for a class of semi-linear equation*, J. London Math. Soc., **2(69)**, 2004, 107–127.

[3] E. Mitidieri, S.Z. Pohozhayev, *A priori estimations and no solutions of nonlinear partial equations and inequalities*, Proc. of V.A. Steklov Mathematics Institute of NAS, **234**, 2001, 9-234.

[4] B. Gidas, J. Spruck, *Global and local behavior of positive solutions of nonlinear elliptic equations*, Comm. Pure. Appl. Math., **34(4)**, 1981, 525–598.

[5] V. Kondratiev, V. Liskevich, Z. Sobol, *Second-order semilinear elliptic inequalities in exterior domains*, J. Diff. Eq., **187**, 2003, 429–455.

[6] W.M. Ni, *On the elliptic equation, its generalizations, and applications*, Indiana Univ. Math. J., **31(4)**, 1982, 493–529.

[7] Zhang, S. Qi, *A Harnack inequality for the equation* $\nabla(a\nabla u) + b\nabla u = 0$, *when* $|b| \in Kn + 1$, Manuscript Math., **89**, 1996, 61–77.

[8] Zhang, S. Qi, *Gaussian bounds for the fundamental solutions of* $\nabla(A\nabla u) + B\nabla u - ut = 0$, Manuscript Math., **93**, 1997, 381–390.

[9] O.A. Ladyzhenskaya, V.A. Solonnikov, N.N. Uralceva, *Linear and Quasilinear Equations of Parabolic Type*, Providence, AMS, 1968.

Shirmayil H. Bagirov
Baku State University
23, Z. Khalilov Str., AZ1148, Baku, Azerbaijan
Institute of Mathematics and Mechanics, Azerbaijan National Academy of Sciences
9, B.Vahabzadeh Str., AZ1141, Baku, Azerbaijan
E-mail: sh_bagirov@yahoo.com

Tarlan Z. Garayev
Institute of Mathematics and Mechanics, Azerbaijan National Academy of Sciences
9, B.Vahabzadeh Str., AZ1141, Baku, Azerbaijan
E-mail: qarayevtarlan@gmail.com

Received 18 November 2015
Accepted 01 December 2015