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On Asymptotics of Eigenvalues for Second Order Differential Operator Equation

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Abstract. We investigate the boundary value problem with an eigenvalue dependent boundary condition for a second order differential equation with an operator coefficient. The symmetry and the self-adjointness of the operator associated with this problem are established. Also, the asymptotic formula for the eigenvalues is derived.

Key Words and Phrases: Hilbert space, self-adjoint operator, positive definite operator, eigenvalues, eigenvectors, discrete spectrum.

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1. Introduction and Preliminaries

Spectral theory of operators plays a major role in mathematics and applied sciences. Boundary value problems with a spectral parameter in the boundary conditions are one of the most important fields in the spectral theory of operators. It is well known that Sturm-Liouville type problems often arise in the solution of problems in mathematical physics. That is why the Sturmian theory is one of the most actual and extensively developed fields in theoretical and applied mathematics. Many researches have been dedicated to studying the spectral properties of the boundary value problems with eigenparameter dependent boundary conditions (see, e.g., Fulton [1], Walter [2]). Various physical applications of such problems can be found in [1].

Note that many problems of mechanics, mathematical physics, theory of partial differential equations, etc are reduced to the study of boundary value problems for operator-differential equations in different spaces. The asymptotic distribution of eigenvalues for boundary-value problems with operator coefficients was first considered by A.G. Kostyuchenko and B.M. Levitan [3] There followed a lot of papers dedicated to the investigation of spectrum of differential operators with operator coefficients. The asymptotic distribution of the eigenvalues of operators defined on the whole space and having a discrete spectrum can be interesting for those who specialize in quantum mechanics.

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In [4], M.A. Rybak studied the asymptotic behavior of eigenvalues of a boundary value problem with a spectral parameter in the boundary conditions for a second order elliptic operator-differential equation. The symmetry and self-adjointness of the operator associated with this problem were established. It was shown that if the considered operator has a discrete spectrum, then the operator associated with this problem has a discrete spectrum, too. The asymptotic formula for the eigenvalues of this problem was derived.

Asymptotics of eigenvalues for boundary value problems with unbounded operator coefficients and eigenvalue dependent boundary conditions were studied, for example, in [5-11].

In [6], where both boundary conditions depend on λ, B.A. Aliev showed that the operator defined in the space $L_2(H, (0, 1)) \oplus H \oplus H$ (where H is a separable Hilbert space) is symmetric and positive definite. The asymptotics of eigenvalues were also obtained.

In [8], M.Bayramoglu and N.M. Aslanova considered the spectral problem

$$-y''(x) + Ay(x) + q(x)y(x) = \lambda y(x),$$

$$y'(0) = 0, y(1) - hy'(1) = \lambda y(1),$$

in the space $L_2(H, (0, 1))$. The asymptotic formulas for eigenvalues and trace formula were derived.

Let $L_2 = L_2(H, (0, 1)) \oplus H$, where H is a separable Hilbert space. Denote by $(\cdot, \cdot)$ and $|\cdot|$ the scalar product and the norm in H, respectively. Define the scalar product in L_2 as

$$(Y, Z)_{L_2} = \int_0^1 (y(t), z(t))\, dt + \frac{1}{\rho}(y_1, z_1), \tag{1}$$

where $Y = \{y(t), y_1\} \in L_2$, $Z = \{z(t), z_1\} \in L_2$, $y(t), z(t) \in L_2(H, (0, 1))$, $y_1, z_1 \in H$, for which $L_2(H, (0, 1))$ is a space of vector functions $y(t)$ such that $\int_0^1 \|y(t)\|^2 dt < \infty$,

$$\rho = \begin{vmatrix} c & -a \\ d & b \end{vmatrix} = bc + ad > 0, \quad ad < 0, \quad ac > 0, \quad bd < 0.$$

Consider the spectral problem

$$l[y] \equiv -y''(t) + Ay(t) + q(t)y(t) = \lambda y(t), \tag{2}$$

$$y'(0) = 0, \tag{3}$$

$$ay(1) + by'(1) = \lambda\left(cy(1) - dy'(1)\right), \tag{4}$$

in $L_2(H, (0,1))$, where A is a self-adjoint positive definite operator in H, $A > E$, E is the identity operator in H, $A^{-1} \in \sigma_\infty$. In this paper, unlike [8], the boundary condition (4) is given in general form.

Suppose that the operator-valued function $q(t)$ is weakly measurable, and $\| q(t)\|$ is bounded on [0, 1].

For $q(t) \equiv 0$, we can associate the problem (2)-(4) in the space L_2 with the operator L_0 defined as

$$D(L_0) = \{Y : Y = \{y(t), y_1\} / -y''(t) + Ay(t) \in L_2(H, (0,1)),$$
$$y'(0) = 0\,,\ y_1 = cy(1) - dy'(1)\},$$

$$L_0 Y = \{-y''(t) + Ay(t),\ ay(1) + by'(1)\}. \tag{5}$$

The operator corresponding to the case $q(t) \not\equiv 0$ is denoted by $L = L_0 + Q$, where $Q : Q\{y(t), cy(1) - dy'(1)\} = \{q(t)\,y(t), 0\}$ is a bounded self-adjoint operator in L_2.

In this paper, we study the asymptotics of eigenvalue distribution of the operator L. Because of the appearance of eigenvalue parameter in the boundary condition at the end point, the operator associated with the problem (2), (3), (4) in the space $L_2(H, (0, 1))$ is not self-adjoint. That is why we introduce the space $L_2(H, (0, 1)) \oplus H$ with the scalar product defined by the formula (1). In this space, the operator L_0 becomes self-adjoint.

2. Proofs of Some Lemmas and Main Result

In this section, we prove the symmetry and positive definiteness of the operator L_0. We also establish the discreteness of the spectrum of L_0 and obtain the asymptotic formulas for the eigenvalues of the operators L_0 and L.

Lemma 1. The operator L is symmetric in L_2.

Proof. Let $Y \in D(L)$. Then

$$(LY, Y)_{L_2} = \int_0^1 (l[y], y)_H\, dt + \frac{1}{\rho}\left(ay(1) + by'(1), cy(1) - dy'(1)\right) =$$
$$= -\int_0^1 \left(y'', y\right)_H dt + \int_0^1 \left((Ay(t), y(t)) + (q(t)\,y(t), y(t))\right)_H dt +$$
$$+\frac{1}{\rho}\left(ay(1) + by'(1), cy(1) - dy'(1)\right) = -\left(y'(1), y(1)\right) +$$
$$+\int_0^1 \left(y', y'\right)_H dt + \int_0^1 \left((Ay(t), y(t)) + (q(t)\,y(t), y(t))\right)_H dt +$$

$$+\frac{1}{\rho}\left(ay(1) + by'(1), cy(1) - dy'(1)\right) = -\left(y'(1), y(1)\right) + \left(y(1), y'(1)\right) -$$
$$-\int_0^1 \left(y, y''\right)_H dt + \int_0^1 \left((y(t), Ay(t)) + (y(t), q(t)\,y(t))\right)_H dt +$$
$$+\frac{ac}{\rho}\left(y(1), y(1)\right) - \frac{ad}{\rho}\left(y(1), y'(1)\right) +$$

$$+\frac{bc}{\rho}\left(y'(1)\,,\,y(1)\right)-\frac{bd}{\rho}\left(y'(1)\,,\,y'(1)\right)=\int_0^1(y(t)\,,\,l[y])_H\,dt+\frac{ac}{\rho}(y(1)\,,\,y(1))+$$

$$+\frac{bc}{\rho}\left(y(1)\,,\,y'(1)\right)-\frac{ad}{\rho}\left(y'(1)\,,\,y(1)\right)-\frac{bd}{\rho}\left(y'(1)\,,\,y'(1)\right)=\int_0^1(y(t)\,,\,l[y])_H\,dt+$$

$$+\frac{1}{\rho}\left(cy(1)-dy'(1)\,,\,ay(1)+by'(1)\right)=(Y,\,LY)_{L_2}\,,$$

that completes the proof.

Lemma 2. If $A>E$, then the operator L_0 is positive definite in L_2.

Proof. Taking into consideration (1) and (5), we have

$$(L_0Y,Y)_{L_2}=\int_0^1(l[y]\,,\,y(t))_H\,dt+\frac{1}{\rho}\left(ay(1)+by'(1)\,,\,cy(1)-dy'(1)\right)=-\left(y'(1)\,,\,y(1)\right)+$$

$$+\int_0^1\left\|y'(t)\right\|_H^2dt+\int_0^1(Ay(t)\,,\,y(t))_H\,dt+\frac{ac}{\rho}\left\|y(1)\right\|_H^2-\frac{ad}{\rho}\left(y(1)\,,\,y'(1)\right)+\frac{bc}{\rho}\left(y'(1)\,,\,y(1)\right)-$$

$$-\frac{bd}{\rho}\left\|y'(1)\right\|_H^2=\int_0^1\left\|y'(t)\right\|_H^2dt+\int_0^1(Ay(t)\,,\,y(t))_H\,dt+\frac{ac}{\rho}\left\|y(1)\right\|_H^2-$$

$$-\frac{ad}{\rho}\left(\left(y(1)\,,\,y'(1)\right)+\left(y'(1)\,,\,y(1)\right)\right)-\frac{bd}{\rho}\left\|y'(1)\right\|_H^2\geq\int_0^1\left\|y'(t)\right\|_H^2dt+\int_0^1\left\|y(t)\right\|_H^2dt.$$

The lemma is proved.

Denote the eigenvalues of the operator A by $\gamma_1\leq\gamma_2\leq\ldots$.

Under condition of compactness of A^{-1} in H, applying the Rellich theorem to the self-adjoint and positive definite operator L_0 we can show that the spectrum of L_0 is discrete. Since Q is bounded in L_2, L is also a discrete operator.

Suppose that the eigenvalues of A behave like

$$\gamma_k\sim gk^{\alpha},k\to\infty,g>0,\alpha>0. \qquad (6)$$

The following theorem is true.

Theorem 1. The eigenvalues of the operator L_0 form two sequences:

$$\lambda_k\sim-\frac{b}{a}+\frac{-c^2\pm c\sqrt{c^2+4d\,(b+d\gamma_k)}}{2d^2},$$

$$\lambda_{k,\,n} \sim \gamma_k + a_n^2,$$

where $\alpha_n = \pi n,\ \ n \in Z$.

Proof. From the spectral expansion of the self-adjoint operator A, we get the following problem for coefficients $y_k\left(t\right) = \left(y\left(t\right), \varphi_k\right)$:

$$-y_k''\left(t\right) = \left(\lambda - \gamma_k\right) y_k\left(t\right), \quad t \in \left(0,\, 1\right), \tag{7}$$

$$y_k'\left(0\right) = 0, \tag{8}$$

$$a y_k\left(1\right) + b y_k'\left(1\right) = \lambda \left(c y_k\left(1\right) - d y_k'\left(1\right)\right). \tag{9}$$

The solution of the problem (7), (8) is

$$y_k\left(t\right) = \cos\sqrt{\lambda - \gamma_k t}.$$

By virtue of condition (9), eigenvalues of the operator L_0 consist of those real $\lambda \neq \gamma_k$, which satisfy the equality

$$a\cos\sqrt{\lambda-\gamma_k} - b\sqrt{\lambda-\gamma_k}\sin\sqrt{\lambda-\gamma_k} = \lambda\left(c\cos\sqrt{\lambda-\gamma_k} + d\sqrt{\lambda-\gamma_k}\sin\sqrt{\lambda-\gamma_k}\right), \tag{10}$$

at least for one k.

Denote $z = \sqrt{\lambda - \gamma_k}$. Then the equation (10) becomes

$$a\cos z - bz\sin z = \left(z^2 + \gamma_k\right)\left(c\cos z + dz\sin z\right). \tag{11}$$

By taking $z = iy$ in (11), we get

$$\left(a - c\left(\gamma_k - y^2\right)\right) chy + y\left(b + d\left(\gamma_k - y^2\right)\right) shy = 0. \tag{12}$$

Denote

$$u_k\left(y\right) = a - c\left(\gamma_k - y^2\right) + y\left(b + \left(d\left(\gamma_k - y^2\right)\right)\right) thy, \quad y > 0.$$

Then

$$u_k'\left(y\right) = 2cy + \left(b + d\gamma_k - 3dy^2\right) thy + \frac{y\left(b + d\left(\gamma_k - y^2\right)\right)}{ch^2 y},$$

$$u_k\left(0\right) = a - c\gamma_k,$$

$$u_k\left(\sqrt{\gamma_k}\right) = a + b\sqrt{\gamma_k} th\sqrt{\gamma_k}.$$

If $c > 0$, then $u_k\left(0\right) < 0$, $u_k\left(\sqrt{\gamma_k}\right) > 0$, and if $c < 0$, then $u_k\left(0\right) > 0$, $u_k\left(\sqrt{\gamma_k}\right) < 0$.

Therefore, since the function $u_k(y)$ is decreasing on interval and assumes values of opposite signs at the end points, the equation (12) has exactly one root for each k.

Find the asymptotics of the roots of the equation (12). For large values of k we have

$$-dy^2 + cy + b + d\gamma_k + \mathrm{O}\left(\frac{1}{y}\right) = 0,$$

$$y_{1,2} \sim \frac{-c \pm \sqrt{c^2 + 4d\left(b + d\gamma_k\right)}}{-2d}.$$

Thus, by virtue of

$$\sqrt{\lambda - \gamma_k} \sim i\frac{-c \pm \sqrt{c^2 + 4d\left(b + d\gamma_k\right)}}{-2d},$$

we have

$$\lambda_k \sim -\frac{b}{d} + \frac{-c^2 \pm c\sqrt{c^2 + 4d\left(b + d\gamma_k\right)}}{2d^2}. \tag{13}$$

Now find the asymptotics of eigenvalues which are greater than γ_k, in other words, the real roots of the equation (11). Write this equation in the form

$$a - c\left(z^2 + \gamma_k\right) - z\left(b + d\left(z^2 + \gamma_k\right)\right) tgz = 0, \;\; z \in (0, \infty),$$

or

$$tgz = \frac{a - c\left(z^2 + \gamma_k\right)}{z\left(b + d\left(z^2 + \gamma_k\right)\right)}. \tag{14}$$

Consider the function

$$\frac{z\left(b + d\left(z^2 + \gamma_k\right)\right) tgz - a + c\left(z^2 + \gamma_k\right)}{z\left(b + d\left(z^2 + \gamma_k\right)\right)} = \frac{g_k(z)}{z\left(b + d\left(z^2 + \gamma_k\right)\right)}.$$

Since in each interval $\left(\frac{\pi}{2} + \pi n, \; \frac{3\pi}{2} + \pi n\right)$ $g_k(z)$ takes on all values from $-\infty$ to $+\infty$and

$$g_k'(z) = \frac{z\left(b + d\left(z^2 + \gamma_k\right)\right)}{\cos^2(z)} + \left(b + 3z^2 d + d\gamma_k\right) tg\, z + 2zc =$$

$$= \frac{z\left(b + d\left(z^2 + \gamma_k\right)\right) + \frac{1}{2}\left(b + 3dz^2 + d\gamma_k\right)\sin 2z + 2zc\cos^2 z}{\cos^2 z},$$

we obtain that if $d > 0$, then $g_k'(z) > 0$, and if $d < 0$, then $g_k'(z) < 0$.

Thus, $g_k(z)$ has only one zero in the above interval. Find the asymptotics of the zeros of this function. From (14) we get

$$z \sim \pi n. \tag{15}$$

The eigenvalues of the operator L_0 corresponding to these roots are

$$\lambda_{k,\,n} \sim \gamma_k + \alpha_n^2,$$

where $\alpha_n = \pi n,\ \ n \in Z.$

The theorem is proved.

The following relation is true for the resolvents of the operators L_0 and L:

$$R_\lambda\left(L\right) = R_\lambda\left(L_0\right) - R_\lambda\left(L\right) Q R_\lambda\left(L_0\right). \tag{16}$$

Theorem 2. Let $A = A^* > E$ in $H,\ \ A^{-1}$ be compact and the relation (6) hold. Then

$$\lambda_n\left(L_0\right) \sim \mu_n\left(L\right) \sim dn^\delta, \tag{17}$$

where

$$\delta = \begin{cases} \frac{2\alpha}{\alpha+2}, & \alpha > 2, \\ \frac{\alpha}{2}, & \alpha < 2, \\ 1, & \alpha = 2, \end{cases}$$

μ_n and λ_n are the eigenvalues of the operators L and L_0, respectively.

Proof. Denote the distribution function of L_0 by $N\left(\lambda,\, L_0\right)$. Then

$$N\left(\lambda,\, L_0\right) = \sum_{\lambda_m < \lambda} 1 = N_1\left(\lambda\right) + N_2\left(\lambda\right),$$

where

$$N_1\left(\lambda\right) = \sum_{\lambda_k < \lambda} 1\,, \quad N_2\left(\lambda\right) = \sum_{\lambda_{k,\,n} < \lambda} 1.$$

Since $\gamma_k \sim g k^\alpha$, we have $\lambda_k \sim c_1 k^{\frac{\alpha}{2}}$. That is

$$N_1 \sim C_1 \lambda^{\frac{2}{\alpha}}. \tag{18}$$

From the asymptotics of $x_{k,\,n}$ (Theorem 1) it follows that one can find a number c such that for large values of m the inequality

$$\left(\pi - c\right) m < x_{k,\,n} < \left(\pi + c\right) m,$$

holds. From (6) we obtain for a large k

$$\left(g - \varepsilon\right) k^\alpha < \gamma_k < \left(g + \varepsilon\right) k^\alpha.$$

From these inequalities it follows that $N_2\left(\lambda\right)$ is less than $N_2'\left(\lambda\right)$, where $N_2'\left(\lambda\right)$ is the number of positive integer valued pairs $\left(m,\, k\right)$ satisfying the inequality

$$\left(\pi - c\right)^2 m^2 + \left(g - \varepsilon\right) k^\alpha < \lambda.$$

Also, $N_2(\lambda)$ is greater than $N_2''(\lambda)$, where $N_2''(\lambda)$ is the number of positive integer valued pairs (m, k) for which

$$(\pi + c)^2 m^2 + (g + \varepsilon) k^\alpha < \lambda.$$

Thus,

$$N_2''(\lambda) < N_2(\lambda) < N_2'(\lambda).$$

We have

$$N_2'(\lambda) \le \frac{1}{\pi - c} \int_0^{\left(\frac{\lambda}{g-\varepsilon}\right)^{\frac{1}{\alpha}}} \sqrt{\lambda - (g-\varepsilon) x^\alpha} dx = \frac{\sqrt{\lambda}}{\pi - c} \int_0^{\left(\frac{\lambda}{g-\varepsilon}\right)^{\frac{1}{\alpha}}} \sqrt{1 - \frac{(g-\varepsilon) x^\alpha}{\lambda}} dx.$$

So, substituting

$$x = \left(\frac{\lambda}{g-\varepsilon} \sin^2 t\right)^{\frac{1}{\alpha}}, \quad dx = \frac{2 \sin t \cos t}{\alpha} \left(\frac{\lambda}{g-\varepsilon}\right)^{\frac{1}{\alpha}} \left(\sin^2 t\right)^{\frac{1-\alpha}{\alpha}} dt,$$

we get

$$N_2'(\lambda) \le \frac{2\lambda^{\frac{2+\alpha}{2\alpha}}}{\alpha(\pi - c)(g-\varepsilon)^{\frac{1}{\alpha}}} \int_0^{\frac{\pi}{2}} \cos^2 t \sin^{\frac{2}{\alpha}-1} t dt = \frac{2}{\alpha(\pi - c)(g-\varepsilon)^{\frac{1}{\alpha}}} \gamma \lambda^{\frac{2+\alpha}{2\alpha}},$$

where

$$\gamma = \int_0^{\frac{\pi}{2}} \cos^2 t \sin^{\frac{2}{\alpha}-1} t dt.$$

So we have

$$N_2''(\lambda) \ge \frac{2\gamma\lambda^{\frac{2+\alpha}{\alpha}}}{\alpha(\pi + c)(g+\varepsilon)^{\frac{1}{\alpha}}} - \left(\frac{\lambda}{g+\varepsilon}\right)^{\frac{1}{\alpha}} - \frac{\sqrt{\lambda}}{\pi + c}.$$

It follows

$$N_2(\lambda) \sim \frac{2\gamma\lambda^{\frac{2+\alpha}{\alpha}}}{\pi\alpha\sqrt{g}}. \tag{19}$$

From (18) and (19) we have

$$N(\lambda, L_0) \sim C_1 \lambda^{\frac{2}{\alpha}} + C_2 \lambda^{\frac{2+\alpha}{2\alpha}}.$$

For $\alpha > 2$

$$N(\lambda, L_0) \sim C_2 \lambda^{\frac{2+\alpha}{2\alpha}},$$

and, consequently,

$$\lambda_n(L_0) \sim dn^{\frac{2+\alpha}{2\alpha}}, d = C_2^{-\frac{2\alpha}{2+\alpha}}.$$

For $\alpha < 2$,

$$N(\lambda, L_0) \sim C_1 \lambda^{\frac{2}{\alpha}}, \lambda_n(L_0) \sim C_1^{-\frac{2}{\alpha}} n^{\frac{\alpha}{2}}.$$

For $\alpha = 2$, $N(\lambda, L_0) \sim (C_1 + C_2)\lambda$, hence $\lambda_n(L_0) \sim dn, d = (C_1 + C_2)^{-1}$.
By using (16) and the properties of $s-$numbers of compact operators (see [12], p. 44, 49), we get the following asymptotics for the eigenvalues of the operator L:

$$\mu_n(L) \sim dn^{\delta}.$$

The theorem is proved.

In [4], the above theorem was proved in the space $L_2(H, (0, b))$ $(0 < b < \infty)$ under the following conditions:

$$y'(0) + \lambda y(0) = 0,$$

$$y(b) = 0.$$

In our next research, we plan to calculate the regularized trace for the operator L.

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