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Global Bifurcation From Infinity in Nonlinear Elliptic Problems with Indefinite Weight

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Abstract. In this paper we consider global bifurcation of solutions in nonlinear eigenvalue problems for semi-linear elliptic partial differential equations with indefinite weight function. We prove the existence of two pairs of unbounded continua of solutions bifurcating from the points in $\mathbb{R}\times\{\infty\}$ corresponding to the positive and negative principal eigenvalues of the linear problem and such that the continua of each pair consists of positive and negative functions, respectively, in the neighborhood of these points.

Key Words and Phrases: nonlinear eigenvalue problem, bifurcation point, global continua, principal eigenvalue, indefinite weight function

2010 Mathematics Subject Classifications: 35J15, 35J65, 35P05, 35P30, 47J10, 47J15

1. Introduction

In this paper, we consider the following nonlinear eigenvalue problem

$$\begin{aligned} &Lu \equiv -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_i}\right) + c(x)\,u = \lambda a(x)u + g(x,u,\nabla u,\,\lambda) \text{ in } \Omega, \\ &u = 0 \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{1}$$

where Ω be a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$, $\nabla u = (\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \dots, \frac{\partial u}{\partial x_n})$ and λ is a real parameter. We assume that L is uniformly elliptic in $\overline{\Omega}$ and that the $a_{ij}(x) \in C^1(\bar{\Omega})$, $a_{ij}(x) = a_{ji}(x)$ for $x \in \overline{\Omega}$, $c(x) \in C(\bar{\Omega})$, $c(x) \geq 0$ for $x \in \overline{\Omega}$. Let $a(x) \in C(\overline{\Omega})$ such that $|\Omega_a^\sigma| > 0$ for $\sigma \in \{+,-\}$, where $\Omega_a^\sigma = \{x \in \Omega : \sigma a(x) > 0\}$ and $|\Omega_a^\sigma| = \text{meas}\{\Omega_a^\sigma\}$. Moreover, the nonlinear term $g \in C(\overline{\Omega}\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{R})$ and satisfies the following condition:

$$g(x,u,v,\lambda) = o(|u|+|s|) \quad \text{as} \quad |u|+|v| \to \infty, \tag{2}$$

uniformly in $x \in \overline{\Omega}$ and $\lambda \in \Lambda$, for every bounded interval $\Lambda \subset \mathbb{R}$.

Problem (1) with $a(x) > 0$, $x \in \overline{\Omega}$, and all the coefficients and the nonlinear terms are smooth was considered by Rabinowitz [9] in a more general case, where, in particular, it was shown that there exist two unbounded continua of solutions emanating from asymptotically bifurcation point corresponding to the first eigenvalue of the linear problem obtained from (1) by setting $g \equiv 0$ and contained in the classes of positive and negative functions in near of this point. In the future, Przybycin [8] and Rynne [10] extended the results of Rabinowitz [9] to the class of nonlinearizable eigenvalue problems for elliptic partial differential equations with a definite weight.

In the papers [3, 4], problem (1) was studied in the case when the nonlinear term g satisfies a $o(|u| + |\nabla u|)$ condition at $u = 0$. For such a problem, the authors show the existence of two pairs of unbounded continua of solutions bifurcating from points of the line of trivial solutions corresponding to the positive and negative principal eigenvalues of linear problem, and such that the continua of each pair are contained in the classes of positive and negative functions, respectively.

The purpose of the present paper is extend the result of Rabinowitz concerning the existence of branches of positive and negative solutions, [9], to the nonlinear problem (1) with indefinite weight function $a(x)$.

2. The classes P_σ^μ and principal eigenvalues of the corresponding linear problem

For $k \in \mathbb{N}$, and $\alpha \in (0,1)$ let $C^{k,\alpha}(\overline{\Omega})$ denote the Banach space of the functions in $C^k((\overline{\Omega})$ having all their derivatives of order k Hölder continuous with exponent α. We let $|\cdot|_k$ and $|\cdot|_{k,\alpha}$ denote the standard sup-norms on spaces $C^k((\overline{\Omega})$ and $C^{k,\alpha}(\overline{\Omega})$, respectively. For $p > 1$, let $W^{k,p}(\overline{\Omega})$ denote the standard Sobolev space of functions whose distributional derivatives, up to order k, belong to $L^p(\Omega)$. We let $||\cdot||_p$ and $||\cdot||_{k,p}$ denote the norm on $L_p(\Omega)$ and $W^{k,p}(\Omega)$, respectively.

It is known (see [1]) that, if $p > N$, then there exists a constant γ such that

$$||u||_{C^{1,1-n/p}} \leq \gamma\, ||u||_{W^{2,p}} \text{ for all } u \in W^{2,p}(\Omega).$$

Now let $\alpha \in (0,1)$ be the given number and p be a real number such that $p > n$ and $\alpha < 1 - n/p$. Then $W^{2,p}(\Omega)$ is compactly embedded in $C^{1,\alpha}(\overline{\Omega})$.

Let $E = \{u \in C^{1,\alpha}\left(\overline{\Omega}\right) : u = 0 \text{ on } \partial\Omega\}$ be the Banach space with the norm $||\cdot||_{C^{1,\alpha}}$. A pair (λ, u) is said to be a solution of problem (1) if $u \in W^{2,p}(\Omega)$ and (λ, u) satisfies (1). By virtue of compactly embedding $W^{2,p}(\Omega)$ in $C^{1,\alpha}(\overline{\Omega})$ we conclude that every solution of the nonlinear problem (1) belongs to $\mathbb{R} \times E$. Thus we may consider the structure of the set of solutions of problem (1) in $\mathbb{R} \times E$. Let $P_\sigma^+ = \{u \in E : u > 0 \text{ in } \Omega \text{ and } \frac{\partial u}{\partial n} < 0 \text{ on } \partial\Omega,\ \sigma \int\limits_\Omega a u^2 dx > 0\}$, where $\frac{\partial u}{\partial n}$ is the outward normal derivative of u on $\partial\Omega$.

Remark 1. It follows from the definition that for each $\sigma \in \{+\,,\,-\}$ the sets P_σ^+, $P_\sigma^- = -\,P_\sigma^+$ and $P_\sigma = P_\sigma^+ \cup P_\sigma^-$ are open subsets of E; for each $\sigma \in \{+\,,\,-\}$ the sets P_σ^+ and

P_σ^-, and for each $\nu \in \{+\,,\,-\}$ the sets P_+^ν and P_-^ν are disjoint. Moreover, if $u \in \partial P_\sigma^\nu$, $\sigma \in \{+\,,\,-\}$, $\nu \in \{+\,,\,-\}$, then the function u has either an interior zero in Ω or $\frac{\partial u}{\partial n} = 0$ at some point on $\partial\Omega$ or $\int\limits_\Omega au^2dx = 0$ [4].

Now we consider the linear eigenvalue problem obtained from (1) by setting $h \equiv 0$, i.e. the following spectral problem

$$\begin{aligned} &Lu = \lambda\, a(x)\, u \quad \text{in } \Omega, \\ &u = 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{3}$$

It should be noted that if the weight function $a(x)$ does not change sign in $\overline{\Omega}$, then (3) admits one principal eigenvalue [7], and if $a(x)$ changes sign in $\overline{\Omega}$, then problem (3) admits two principal eigenvalues; one positive and the other negative [3].

In [3] the authors obtained the following properties of the eigenfunctions corresponding to the principal eigenvalues of problem (3).

Theorem 1. (see [3, Lemmas 2.1-2.4, Theorems 2.1, 2.2 and Remark 2.1]) *The linear eigenvalue problem* (3) *have positive and negative principal eigenvalues* λ_1^+ *and* λ_1^-*, respectively, which are simple and given by the relations*

$$\lambda_1^\sigma = \inf\{R(u) : u \in H_0^1(\Omega),\ \sigma \int\limits_\Omega au^2dx > 0\} \text{ for } \sigma \in \{+\,,\,-\},$$

where $H_0^1(\Omega) = \{u \in W^{1,2}(\Omega) : u = 0 \text{ on } \partial\Omega\}$ *and* $R(u)$ *is the Rayleigh quotient* [2] *defined as follows:*

$$R(u) = \frac{\int\limits_\Omega a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}dx + \int\limits_\Omega cu^2dx}{\int\limits_\Omega au^2dx}.$$

Moreover, the corresponding eigenfunction $u_1^\sigma(x)$, $x \in \overline{\Omega}$, $\sigma \in \{+\,,\,-\}$, *can be chosen so that* $u_1^\sigma(x) > 0$ *for all* $x \in \Omega$ *and* $\frac{\partial u_1^\sigma(x)}{\partial n} < 0$ *for all* $x \in \partial\Omega$.

Remark 2. It follows from Theorem 1 that $u_1^\sigma \in P_\sigma^+$ for each $\sigma \in \{+\,,\,-\}$. It should be noted that u_1^σ is made unique by taking $||u_1^\sigma||_{C^{1,\alpha}} = 1$.

3. Global bifurcation of solutions of problem (1) from infinity

The closure of the set of nontrivial solutions of (1) will be denoted by $\mathcal{L}$. We say $(\lambda, \infty) \in \mathbb{R} \times \{\infty\}$ is a bifurcation point for problem (1) if any neighborhood of this point contains solutions of problem (1), i.e. there exists a sequence $\{(\lambda_n, u_n)\}_{n=1}^\infty \subset \mathcal{L}$ such that $\lambda_n \to \lambda$ and $|u_n|_{1,\alpha} \to \infty$ as $n \to \infty$ [6].

The main result of this paper is the following theorem.

Theorem 2. *For each $\sigma \in \{+, -\}$ and each $\nu \in \{+, -\}$ there exists a component $C^{\nu}_{1,\sigma}$ of $\mathcal{L}$ which contains $(\lambda_1^{\sigma}, \infty)$ and satisfies the conclusions of Theorem 1.6 and Corollary 1.8 from [9]. Moreover, the neighborhood Q of [9, Corollary 1.8] can be chosen such that*

$$(\mathcal{C}^{\nu}_{1,\sigma} \cap Q) \subset (\mathbb{R} \times P^{\nu}_{\sigma}) \cup \{(\lambda_1^{\sigma}, \infty)\}.$$

Proof. Step 1. We assume that $a_{ij} \in C^2(\overline{\Omega})$, c, $a \in C^1(\overline{\Omega})$ and $h \in C^1(\overline{\Omega}\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{R})$.

It follows from the L_p theory for uniformly elliptic partial differential equations [2] that there exists a unique $v = G(\lambda, u)$ satisfying

$$\begin{aligned} &Lv = \lambda a(x)\, u + g(x, u, \nabla u,\, \lambda)) \quad \text{in } \Omega, \\ &v = 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Since E is compactly embedding in $W_0^{2,p}(\Omega) = W^{2,p}(\Omega) \cap \{u : u = 0 \text{ on } \partial\Omega\}$ the Arzela-Ascoli Theorem imply that G is compact on $\mathbb{R} \times E$.

Denote by $w = \mathcal{L}u \in W_0^{2,p}(\Omega)$ the solution of the following problem

$$\begin{aligned} &Lw = a(x)\, u \text{ in } \Omega, \\ &w = 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Then from the above reasoning imply that $\mathcal{L}$ is a compact linear map on E. By the Theorem 1 it follows that λ_1^+ and λ_1^- are simple principal characteristic values of operator $\mathcal{L}$.

Suppose that $\mathcal{G}(\lambda, u) = G(\lambda, u) - \lambda\mathcal{L}u$. From the our above remarks it follows that (1) is equivalent to the following nonlinear eigenvalue problem

$$u = \lambda\mathcal{L}u + \mathcal{G}(\lambda, u). \tag{4}$$

Following the corresponding reasoning carried out in the proof of Theorem 2.28 from [9], we see that $\mathcal{G}(\lambda, u) = o(|u|_{1,\alpha})$ as $|u|_{1,\alpha} \to \infty$, uniformly on bounded λ intervals and $|u|^2_{1,\alpha}\mathcal{G}\left(\lambda, \frac{u}{|u|^2_{1,\alpha}}\right)$ is compact. Thus [9, Theorem 1.6 and Corollary 1.8] are applicable here. The verification of the last statement of this theorem follows as in [9, Theorem 2.4].

Step 2. To complete the proof of this theorem, we approximate equation (1) by "smoothed equations", as in [5, Section 4], and apply standard elliptic regularity results for elliptic operators [2].

References

[1] R. A. Adams, *Sobolev spaces*, Academic Press, New York, 1975.

[2] S. Agmon, A. Douglis, L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions*, I, Comm. Pure Appl. Math., **12**, 1959, 623-727.

[3] Z.S. Aliyev, Sh.M. Hasanova, *Global bifurcation of a second order nonlinear elliptic problem with an indefinite weight function*, Proc. IMM NAS Azerbaijan, **42(1)**, 2016, 116-125.

[4] Z.S. Aliyev, Sh.M. Hasanova, *Global bifurcation of positive solutions of semi-linear elliptic partial differential equations with indefinite weight*, Zeitschrift für Analysis und ihre Anwendungen, to appear.

[5] H. Berestycki, *On some nonlinear Sturm-Liouville problems*, J. Diff. Equat., **26**, 1977, 375-390.

[6] M.A. Krasnoselskii, *Topological methods in the theory of nonlinear integral equations*, Pergamon, London, 1974.

[7] M.G. Krein, M.A. Rutman, *Linear operators leaving invariant a cone in a Banach space*, Amer. Math. Soc. Transl., **10(1)**, 1962, 199-325.

[8] J. Przybycin, *Bifurcation from infinity for the special class of nonlinear differential equations*, J. Differential Equations, **65**, 1986, 235-239.

[9] P.H. Rabinowitz, *On bifurcation from infinity*, J. Differential Equations, **14**, 1973, 462-475.

[10] B.P. Rynne, M.A. Youngson, *Bifurcation of positive solutions from zero or infinity in elliptic problems which are not linearizable*, Nonlinear Analysis, **44(1)**, 2001, 21-31.

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