Caspian Journal of Applied Mathematics, Ecology and Economics
V. 7, No 1, 2019, July
ISSN 1560-4055

On the Solution of Generalized Fractional Kinetic Equations Involving Generalized M-Series

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Abstract. This paper refers to further generalizations of fractional kinetic equation. By using the generalized M-Series, solutions of unified fractional kinetic equations are obtained. Solutions are obtained in a compact form containing Wright hypergeometric function by using Laplace transform and Sumudu transform. Certain special cases of our main results are also pointed out.

Key Words and Phrases: generalized fractional kinetic equation, fractional calculus, Laplace transform, Sumudu transform, generalized M-series, special function.

2010 Mathematics Subject Classifications: 26A33, 44A10, 33E12, 34A08

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1. Introduction

The M-series was introduced by the mathematician M. Sharma [10], and defined as

$$ {}_{\mathrm{p}}\overset{\alpha}{\mathrm{M}}_{\mathrm{q}}\left(\mathrm{a}_1,\cdots,\mathrm{a}_{\mathrm{p}};\mathrm{b}_1,\cdots,\mathrm{b}_{\mathrm{q}};\ \mathrm{z}\right) = {}_{\mathrm{p}}\overset{\alpha}{\mathrm{M}}_{\mathrm{q}}\left(\mathrm{z}\right) = \sum_{\mathrm{k}=0}^{\infty} \frac{(\mathrm{a}_1)_{\mathrm{k}}\cdots(\mathrm{a}_{\mathrm{p}})_{\mathrm{k}}}{(\mathrm{b}_1)_{\mathrm{k}}\cdots(\mathrm{b}_{\mathrm{q}})_{\mathrm{k}}} \frac{\mathrm{z}^{\mathrm{k}}}{\Gamma(\alpha\,\mathrm{k}+1)}, \tag{1} $$

where $z,\alpha \in C, \Re(\alpha) > 0$ and $(a_i)_k, (b_j)_k\, (i = 1,\cdots,p; j = 1,\cdots,q)$ are the Pochhammer symbol given by $(\gamma)_n = \frac{\Gamma(\gamma+n)}{\Gamma(\gamma)}$.

The series in (1) is convergent for all z if $p \leq q$, also if $p = q+1$ its convergent absolutely or conditionally when $|z| = 1$, and divergent if $p > q+1$.

In 2009, the generalization of (1) was introduced and studied by Sharma and Jain [11], and given as

$$ \underset{\mathrm{p,q}}{\overset{\alpha,\beta}{\mathrm{M}}}\left(\mathrm{z}\right) = \sum_{\mathrm{k}=0}^{\infty} \frac{(\mathrm{a}_1)_{\mathrm{k}}\cdots(\mathrm{a}_{\mathrm{p}})_{\mathrm{k}}}{(\mathrm{b}_1)_{\mathrm{k}}\cdots(\mathrm{b}_{\mathrm{q}})_{\mathrm{k}}} \frac{\mathrm{z}^{\mathrm{k}}}{\Gamma(\alpha\mathrm{k}+\beta)}. \tag{2} $$

The series in (2) is convergent for all z if $p \leq q+\Re(\alpha)$, also it is convergent for $|z| < \delta = \alpha^{\alpha}$ if $p = q+\Re(\alpha)$ and divergent if $p > q+\Re(\alpha)$.

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Recently, a new generalization of M-series introduced by Faraj *et al.* [2] in the following manner:

$$\underset{\mathrm{p,\,q;\,m,\,n}}{\overset{\alpha,\,\beta}{\mathrm{M}}}(a_1,\cdots,a_p;b_1,\cdots,b_q;\ z) = \underset{\mathrm{p,\,q;\,m,\,n}}{\overset{\alpha,\,\beta}{\mathrm{M}}}(z) = \sum_{\mathrm{k=0}}^{\infty}\frac{(a_1)_{km}\cdots(a_p)_{km}}{(b_1)_{kn}\cdots(b_q)_{kn}}\frac{\mathrm{z^k}}{\Gamma(\alpha k+\beta)}, \quad (3)$$

where $z,\alpha,\beta\in C,\Re(\alpha)>0$ and m,n are non–negative real number.

The series in (3) is absolutely convergent for all values of z provided that $pm < qn+\Re(\alpha)$, moreover if $pm = qn+\Re(\alpha)$, the series converges for $|z|<\delta=\alpha^{\alpha}$.

For $m = n = 1$ *and* $m = n = \beta = 1$, equation (3) reduces to generalized M-series $\underset{\mathrm{p,q}}{\overset{\alpha,\beta}{\mathrm{M}}}(\mathrm{z})$ and M-series ${}_{\mathrm{p}}\overset{\alpha}{\mathrm{M}}_{\mathrm{q}}(\mathrm{z})$, respectively (see (1) and (2)).

Further, if we take $p = q = 1$, equation (3) reduces to generalized Mittag–Leffler function introduced by Salim and Faraj [4] and given as

$$\underset{1,1\,;\,\mathrm{m,\,n}}{\overset{\alpha,\beta}{\mathrm{M}}}(\mathrm{z}) = E\begin{array}{c} a_1,b_1,m\\ \alpha,\beta,n\end{array}(z) = \sum_{\mathrm{k=0}}^{\infty}\frac{(a_1)_{km}}{(b_1)_{kn}}\frac{\mathrm{z^k}}{\Gamma(\alpha k+\beta)}. \quad (4)$$

The generalized Wright hypergeometric function was introduced by Wright [14] and defined as

$$ {}_p\Psi_q(z) = {}_p\Psi_q\left[\begin{array}{c}(\mathrm{a}_i,\ \alpha_i)_{1,\,\mathrm{p}}\\ (\mathrm{b_j},\ \beta_{\mathrm{j}})_{1,\mathrm{q}}\end{array};\ \mathrm{z}\right] = \sum_{\mathrm{k=0}}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+\alpha_i k)}{\prod_{\mathrm{j=1}}^{q}\Gamma(b_j+\beta_j k)}\frac{\mathrm{z^k}}{\mathrm{k!}}, \quad (5)$$

where $z,a_i,b_j\in\mathbb{C}$ and $\alpha_i,\beta_j\in\mathbb{R}-\{0\}\,(i=1,\cdots,p;j=1,\cdots,q)$.

Haubold and Mathai [3] established a fractional differential equation between the rate of change of reaction, the destruction rate and the production rate as follows:

$$\frac{\mathrm{dN}}{\mathrm{dt}} = -\,\mathrm{d(N_t)}+\mathrm{p(N_t)}, \quad (6)$$

where $N=N(t)$ is the rate of reaction, $d(N_t)$ is the rate of destruction, $p(N_t)$ is the rate of production and N_t denotes the function defined by $N_t\,(t^*) = N\,(t-t^*)\,,\ t^*>0$.

A special case of (6), when spatial fluctuations or homogeneities in the quantity $N(t)$ are neglected, is given by the following differential equation:

$$\frac{\mathrm{dN}_i}{\mathrm{dt}} = -\,\mathrm{c}_i\ \mathrm{N}_i(\mathrm{t}) \quad (7)$$

with the initial condition that $N_i\,(t=0)=N_0$ is the number of density of species i at time $t=0$ and constant $c_i>0$. If we remove the index i and integrate the standard kinetic equation (7), we have

$$\mathrm{N(t)}-\mathrm{N_0} = -\,c\ {}_0\mathrm{D_t^{-1}}\mathrm{N(t)}, \quad (8)$$

where ${}_0\mathrm{D_t^{-1}}$ is the standard integral operator.

Houbold and Mathai [3], obtained the fractional generalization of the standard kinetic equation (7) as

$$\mathrm{N(t)} - \mathrm{N_0} = -\, c_0^{\nu}\mathrm{D}_{\mathrm{t}}^{-v}\mathrm{N(t)}, \tag{9}$$

where ${}_0\mathrm{D}_{\mathrm{t}}^{-v}$ is Riemann–Liouville fractional integral operator defined as follows [5]:

$${}_0\mathrm{D}_{\mathrm{t}}^{-v} f(t) = \frac{1}{\Gamma(\nu)} \int_0^{\mathrm{t}} (\mathrm{t}-\mathrm{s})^{\nu-1} f(\mathrm{s})\, \mathrm{ds}\ (\mathrm{t}>0,\, f(\nu)>0). \tag{10}$$

The solution of equation (8) is given by (See [3])

$$\mathrm{N(t)} = \mathrm{N_0} \sum_{k=0}^{\infty} \frac{(-1)^{\mathrm{k}}}{\Gamma(\nu\, \mathrm{k}+1)} (\mathrm{ct})^{\nu \mathrm{k}}. \tag{11}$$

Further, Saxena and Kalla [6] considered the following fractional kinetic equation

$$\mathrm{N(t)} - \mathrm{N_0}\, f(\mathrm{t}) = -\, \mathrm{c}_0^{\nu}\mathrm{D}_{\mathrm{t}}^{-v}\mathrm{N(t)},\ (\Re(\nu)>0), \tag{12}$$

where $N(t)$ denotes the number of density of a given species at time t, $N_0 = N(0)$ is the number of density of that species at time $t = 0$ and c is a constant.

2. Solution of generalized fractional kinetic equations by using the Laplace transform

In this section, we will establish and derive the solution of the generalized kinetic equations involving the generalized M-series (3) by applying the Laplace transform.

Laplace transform [12] of the function $f(\mathrm{t})$ is defined as

$$\mathrm{L}\left\{f(t):\mathrm{s}\right\} = \int_0^{\infty} \mathrm{e}^{-\mathrm{st}} f(\mathrm{t})\, \mathrm{dt},\ (\Re\,(\mathrm{s})>0). \tag{13}$$

and convolution theorem is given by

$$L\left\{f*g\right\}(s) = L\left\{\int_0^t f\,(t-\xi)\; g\,(\xi)\; \mathrm{d}\xi\right\} = L\left\{f\,(s)\right\}.L\left\{g\,(s)\right\}. \tag{14}$$

Laplace transform of the Riemann–Liouville fractional integral operator given by Erdélyi *et al.* [1] as

$$\mathrm{L}\left\{{}_0\mathrm{D}_{\mathrm{t}}^{-\mathrm{v}}\ \mathrm{N(t)} :\ \mathrm{s}\right\} =\ \mathrm{s}^{-\nu}\ \mathrm{N(s)}, \tag{15}$$

and also

$$\mathrm{L}\left\{\mathrm{N(t)}:\ \mathrm{s}\right\} = \mathrm{N(s)}. \tag{16}$$

The following Lemmas are required to prove our main results.

Lemma 1. *For $\Re(\gamma), \Re(\sigma), \Re(s) > 0$, the following Laplace transform of generalized M-series $\underset{p,q;m,n}{\overset{\alpha,\beta}{M}}(z)$ holds true:*

$$L\left\{t^{\gamma-1}\underset{p,q;m,n}{\overset{\alpha,\beta}{M}}(t^{\sigma}):s\right\}$$
$$= s^{-\gamma}\frac{\Gamma(b_1)\cdots\Gamma(b_q)}{\Gamma(a_1)\cdots\Gamma(a_p)}\ {}_{\mathrm{p+2}}\Psi_{\mathrm{q+1}}\left[\begin{array}{c}(\mathrm{a}_1,\mathrm{m}),\cdots,(\mathrm{a}_\mathrm{p},\ \mathrm{m}),(\gamma,\sigma),(1,1)\\(b_1,n),\cdots,(b_q,n),(\beta,\alpha)\end{array};s^{-\sigma}\right], \tag{17}$$

where ${}_{\mathrm{p+2}}\Psi_{q+1}(.)$ is given by (5)*.*

Proof. By taking (3) and (13) into account, we can easily obtain the required result (17) after a little simplification.

If we take $\gamma = \beta$ and $\sigma = \alpha$ in (17), then a special case of (17) is given by following lemma.

Lemma 2. *For* $\min\{\Re(s), \Re(\alpha), \Re(\beta)\} > 0$, *the Laplace transform of* (3) *is given by*

$$L\left\{t^{\beta-1}\underset{p,q;m,n}{\overset{\alpha,\beta}{M}}(t^{\alpha}):s\right\} = s^{-\beta}\frac{\Gamma(b_1)\cdots\Gamma(b_q)}{\Gamma(a_1)\cdots\Gamma(a_p)}\ {}_{\mathrm{p+1}}\Psi_{\mathrm{q}}\left[\begin{array}{c}(\mathrm{a}_1,\mathrm{m}),\cdots,(\mathrm{a}_\mathrm{p},\mathrm{m}),(1,1)\\(b_1,n),\cdots,(b_q,n),\end{array};s^{-\alpha}\right]. \tag{18}$$

Theorem 1. *Let $c,\ w,\nu,\gamma,\sigma\in\mathbb{R}^+;\alpha,\beta,t\in C;\ m,\ n>0;\Re(\alpha)>0$ and $pm\le qn+\Re(\alpha)$. Then, the solution of the following generalized fractional kinetic equation*

$$\mathrm{N(t)} - \mathrm{N}_0\,\mathrm{t}^{\gamma-1}\underset{p,q;m,n}{\overset{\alpha,\beta}{M}}(wt^{\sigma}) = -\mathrm{c}^{\nu}_0\mathrm{D}_\mathrm{t}^{-v}\mathrm{N(t)} \tag{19}$$

is given by

$$N(t) = \mathrm{N}_0\,t^{\gamma-1}\frac{\Gamma(b_1)\cdots\Gamma(b_q)}{\Gamma(a_1)\cdots\Gamma(a_p)}\sum_{r=0}^{\infty}(-c^{v}t^{v})^{r}$$
$$\times\ {}_{\mathrm{p+2}}\Psi_{\mathrm{q+2}}\left[\begin{array}{c}(\mathrm{a}_1,\mathrm{m}),\cdots,(\mathrm{a}_\mathrm{p},\mathrm{m}),\ (\gamma,\sigma),\ (1,1)\\(b_1,n),\cdots,(b_q,n),(\beta,\alpha),\ (\gamma+\nu r,\sigma)\end{array};wt^{\sigma}\right]. \tag{20}$$

Proof. Applying the Laplace transform on both sides of (19). Using (15) and (16) into account, we get

$$L\{N(t)\ :s\} - \mathrm{N}_0\,L\left\{t^{\gamma-1}\underset{p,q;\,m,n}{\overset{\alpha,\beta}{M}}(wt^{\sigma}):s\right\} = -c^{v}\,L\left\{{}_0D_t^{-\nu}N(t)\ :s\right\}$$

$$N(s) = \frac{\mathrm{N}_0}{1+\left(\frac{c}{s}\right)^{\nu}}L\left\{t^{\gamma-1}\underset{p,q;m,n}{\overset{\alpha,\beta}{M}}(wt^{\sigma}):s\right\}$$

Next, by using (17) and the following binomial series expansion

$$\left[1+\left(\frac{c}{s}\right)^{\nu}\right]^{-1} = \sum_{r=0}^{\infty}\ (-1)^{r}\ \left(\frac{c}{s}\right)^{vr}(c<\ |s|),$$

we obtain

$$N(s) = \mathrm{N}_0 \sum_{r=0}^{\infty} (-1)^r \left(\frac{c}{s}\right)^{\nu r} \sum_{k=0}^{\infty} \frac{(a_1)_{km} \cdots (a_p)_{km}}{(b_1)_{kn} \cdots (b_q)_{kn}} \frac{\Gamma(\gamma+\sigma \mathrm{k})\ \ \Gamma(\mathrm{k}+1)}{\Gamma(\alpha \mathrm{k}+\beta)} \frac{\mathrm{w^k}}{k!} \frac{1}{\mathrm{s}^{\gamma+\sigma \mathrm{k}}}$$

$$N(s) = \mathrm{N}_0 \sum_{r=0}^{\infty} (-c^v)^r \sum_{k=0}^{\infty} \frac{(a_1)_{km} \cdots (a_p)_{km}}{(b_1)_{kn} \cdots (b_q)_{kn}} \frac{\Gamma(\gamma+\sigma \mathrm{k})\Gamma(\mathrm{k}+1)}{\Gamma(\alpha \mathrm{k}+\beta)} \frac{\mathrm{w^k}}{\mathrm{k}!} \frac{1}{\mathrm{s}^{\gamma+\nu \mathrm{r}+\sigma \mathrm{k}}}. \tag{21}$$

Now, taking inverse Laplace transform of (21) and using $\mathrm{L}^{-1}\{\ \mathrm{s}^{-\nu}\text{: } \mathrm{t}\} = \frac{\mathrm{t}^{\nu-1}}{\Gamma(\nu)},\ (\Re\ (\mathrm{v}) > 0)$ and $\mathrm{L}^{-1}\{\mathrm{N}(s)\text{: } \mathrm{t}\} = N(t)$, we arrive at

$$L^{-1}\{N(s):t\} = \mathrm{N}_0 \sum_{r=0}^{\infty} (-c^v)^r \sum_{k=0}^{\infty} \frac{(a_1)_{km} \cdots (a_p)_{km}}{(b_1)_{kn} \cdots (b_q)_{kn}}$$

$$\times \frac{\Gamma(\gamma+\sigma \mathrm{k})\ \ \Gamma(\mathrm{k}+1)}{\Gamma(\alpha \mathrm{k}+\beta)} \frac{\mathrm{w^k}}{\mathrm{k}!}\ \ \mathrm{L}^{-1}\left\{\frac{1}{\mathrm{s}^{\gamma+\nu \mathrm{r}+\sigma \mathrm{k}}}\text{: } \mathrm{t}\right\}$$

or

$$N(t) = \mathrm{N}_0 \sum_{r=0}^{\infty} (-c^\nu)^r \sum_{k=0}^{\infty} \frac{(a_1)_{km} \cdots (a_p)_{km}}{(b_1)_{kn} \cdots (b_q)_{kn}} \frac{\Gamma(\gamma+\sigma \mathrm{k})\ \ \Gamma(\mathrm{k}+1)}{\Gamma(\alpha k+\beta)} \frac{\mathrm{w^k}}{\mathrm{k}!} \frac{\mathrm{t}^{\gamma+\nu \mathrm{r}+\sigma \mathrm{k}-1}}{\Gamma(\gamma+\nu \mathrm{r}+\sigma \mathrm{k})}$$

$$= \mathrm{N}_0\, t^{\gamma-1} \frac{\Gamma(b_1) \cdots \Gamma(b_q)}{\Gamma(a_1) \cdots \Gamma(a_p)} \sum_{r=0}^{\infty} (-c^v \mathrm{t}^\nu)^r$$

$$\times \sum_{\mathrm{k}=0}^{\infty} \frac{\Gamma(\mathrm{a_1}+\mathrm{mk}) \cdots \Gamma(\mathrm{a_p}+\mathrm{mk})}{\Gamma(\mathrm{b_1}+\mathrm{nk}) \cdots \Gamma(\mathrm{b_q}+\mathrm{nk})} \frac{\Gamma(\gamma+\sigma \mathrm{k})\, \Gamma(\mathrm{k}+1)}{\Gamma(\alpha \mathrm{k}+\beta)\, \Gamma(\gamma+\nu \mathrm{r}+\sigma \mathrm{k})} \frac{(\mathrm{wt}^\sigma)^{\mathrm{k}}}{\mathrm{k}!}.$$

Finally, by using (5), we get the desired result (20).
This complete the proof of Theorem 1.

If we set $m = n = 1$, then $\underset{\mathrm{p,q;m,n}}{\overset{\alpha,\beta}{\mathrm{M}}}(z)$ reduces to the generalized M-series $\underset{\mathrm{p,q}}{\overset{\alpha,\beta}{\mathrm{M}}}(z)$ [11], we get the generalized fractional kinetic equation with its solution given as follows:

Corollary 1. *Let* $c,\ w, \nu, \gamma, \sigma \in \mathbb{R}^+; \alpha, \beta, t \in C;\ m,\ n > 0\ ;\ \Re(\alpha) > 0$ *and* $p \le q + \Re(\alpha)$. *Then, the solution of the equation*

$$N(t) - \mathrm{N}_0\, t^{\gamma-1} \underset{p,q}{\overset{\alpha,\beta}{M}}(wt^\sigma) = -\mathrm{c}_0^\nu \mathrm{D}_\mathrm{t}^{-v} N(t) \tag{22}$$

is given by

$$N(t) = \mathrm{N}_0\ t^{\gamma-1} \frac{\Gamma(b_1) \cdots \Gamma(b_q)}{\Gamma(a_1) \cdots \Gamma(a_p)} \sum_{\mathrm{r}=0}^{\infty} (-c^v \mathrm{t}^\nu)^r$$

$$\times\ {}_{\mathrm{p}+2}\Psi_{\mathrm{q}+2}\left[\begin{matrix} (\mathrm{a_1},1), \cdots, (\mathrm{a_p},1), (\gamma,\sigma), (1,1) \\ (b_1,1), \cdots, (b_q,1), (\beta,\alpha),\ (\gamma+\nu r, \sigma) \end{matrix}\ ; wt^\sigma\right]. \tag{23}$$

If we take $\beta = 1$ in (22), the generalized M-series $\overset{\alpha,\beta}{\underset{\mathrm{p,q}}{\mathrm{M}}}(z)$ reduces to the M-series $_{\mathrm{p}}\overset{\alpha}{\mathrm{M}}_{\mathrm{q}}(z)$ [10], we arrive at

Corollary 2. *Let $c,\ w, \nu, \gamma, \sigma \in \mathbb{R}^+; \alpha, t \in C;\ m,\ n > 0\ ;\ \Re(\alpha) > 0$. Then, the solution of the equation*

$$N(t) = \mathrm{N}_0\, t^{\gamma-1}\, {}_{\mathrm{p}}\overset{\alpha}{\mathrm{M}}_{\mathrm{q}}(wt^{\sigma}) = -\mathrm{c}_0^{\nu}\mathrm{D}_{\mathrm{t}}^{-v}N(t) \tag{24}$$

is given by

$$N(t) = \mathrm{N}_0\, t^{\gamma-1}\frac{\Gamma(b_1)\cdots\Gamma(b_q)}{\Gamma(a_1)\cdots\Gamma(a_p)}\sum_{r=o}^{\infty}(-c^v t^v)^r$$

$$\times\ {}_{\mathrm{p+2}}\,\Psi_{\mathrm{q+2}}\left[\begin{matrix}(\mathrm{a}_1,\mathrm{l}),\cdots,(\mathrm{a}_{\mathrm{p}},1),(\gamma,\sigma),(1,1)\\(b_1,1),\cdots,(b_q,1),(1,\alpha),(\gamma+\nu r,\sigma)\end{matrix};wt^{\sigma}\right]. \tag{25}$$

Further, if we put $p = q = l$, then $\overset{\alpha,\beta}{\underset{\mathrm{p,q;m,n}}{\mathrm{M}}}(z)$ reduces to the generalized Mittag–Leffler function $E\begin{matrix}a_1, b_1, m\\ \alpha, \beta, n\end{matrix}$ (z) [4], we obtain

Corollary 3. *Let $c,\ w, \nu, \gamma, \sigma \in \mathbb{R}^+; \alpha, \beta, t \in C;\ m,\ n > 0\ ;\ \Re(\alpha) > 0$ and $m \leq n + \Re(\alpha)$. Then, the solution of the equation*

$$N(t) - \mathrm{N}_0\, t^{\gamma-1}E\begin{matrix}a_1, b_1, m\\ \alpha, \beta, n\end{matrix}\ (\mathrm{wt}^{\sigma})\ = -\mathrm{c}_0^{\nu}\mathrm{D}_{\mathrm{t}}^{-v}N(t) \tag{26}$$

is given by

$$N(t) = \mathrm{N}_0\, t^{\gamma-1}\frac{\Gamma(b_1)}{\Gamma(a_1)}\sum_{r=0}^{\infty}(-c^v t^v)^r\ {}_3\Psi_3\left[\begin{matrix}(\mathrm{a}_1,\mathrm{m}),\ (\gamma,\sigma),\ (1,1)\\(b_1,n),(\beta,\alpha),(\gamma+\nu r,\sigma)\end{matrix};wt^{\sigma}\right]. \tag{27}$$

Theorem 2. *Let $c,\ w, \nu \in \mathbb{R}^+; \alpha, \beta, t \in C;\ m,\ n > 0; \Re(\alpha) > 0$ and $pm \leq qn + \Re(\alpha)$. Then, the generalized fractional kinetic equation*

$$N(t) - \mathrm{N}_0\, t^{\beta-1}\overset{\alpha,\beta}{\underset{\mathrm{p,q;\ m,n}}{\mathrm{M}}}(\mathrm{wt}^{\alpha})\ = -\mathrm{c}_0^{\nu}\mathrm{D}_{\mathrm{t}}^{-v}N(t) \tag{28}$$

has the solution

$$N(t) = \mathrm{N}_0\, t^{\beta-1}\frac{\Gamma(b_1)\cdots\Gamma(b_q)}{\Gamma(a_1)\cdots\Gamma(a_p)}\sum_{r=0}^{\infty}(-c^v t^v)^r$$

$$\times\ {}_{p+1}\Psi_{q+1}\left[\begin{matrix}(\mathrm{a}_1,\mathrm{m}),\cdots,(a_{\mathrm{p}},m),(1,1)\\(b_1,n),\cdots,(b_q,n),(\beta+\nu r,\alpha)\end{matrix};wt^{\alpha}\right]. \tag{29}$$

Proof. The proof of result asserted by Theorem 2 runs parallel to that of Theorem 1. Here, we make use (18) instead of (17) into account. Therefore, we omit the details of the proof.

If we put $m = n = 1$, then $\overset{\alpha,\beta}{\underset{\mathrm{p,q;\ m,n}}{\mathrm{M}}}(z)$ reduces to $\overset{\alpha,\beta}{\underset{\mathrm{p,q}}{\mathrm{M}}}(z)$, we get the following corollary.

Corollary 4. *Let $c,\ w,\nu \in \mathbb{R}^{+}; \alpha,\beta,t \in C;\ m,\ n > 0; \Re(\alpha) > 0$ and $p \leq q + \Re(\alpha)$. Then, the equation*

$$N(t) - \mathrm{N_0}\, t^{\beta-1} \overset{\alpha,\beta}{\underset{\mathrm{p,q}}{\mathrm{M}}} (\mathrm{wt}^{\alpha}) = -\mathrm{c}_0^{\nu}\mathrm{D}_{\mathrm{t}}^{-v} N(t) \tag{30}$$

has the solution

$$N(t) = \mathrm{N_0}\, t^{\beta-1} \frac{\Gamma(b_1)\cdots\Gamma(b_q)}{\Gamma(a_1)\cdots\Gamma(a_p)} \sum_{r=0}^{\infty} (-c^{v}t^{v})^{r}$$

$$\times\ {}_{p+1}\Psi_{q+1}\left[\begin{array}{c} (\mathrm{a_1},1),\cdots,(a_{\mathrm{p}},1),\ (1,1) \\ (b_1,1),\cdots,(b_q,1),(\beta+\nu r,\alpha) \end{array} ; wt^{\alpha}\right]. \tag{31}$$

If we take $\beta = 1$ in (30), we have the solution of generalized fractional kinetic equation involving M-series ${}_{\mathrm{p}}\overset{\alpha}{\mathrm{M}}_{\mathrm{q}}(z)$ as follows:

Corollary 5. *Let $c,\ w,\nu \in \mathbb{R}^{+}; \alpha,t \in C;\ m,\ n > 0; \Re(\alpha) > 0$. Then, the equation*

$$N(t) - \mathrm{N_0}\, {}_{\mathrm{p}}\overset{\alpha}{\mathrm{M}}_{\mathrm{q}} (\mathrm{wt}^{\alpha}) = -\mathrm{c}_0^{\nu}\mathrm{D}_{\mathrm{t}}^{-v} N(t) \tag{32}$$

has the solution

$$N(t) = \mathrm{N_0} \frac{\Gamma(b_1)\cdots\Gamma(b_q)}{\Gamma(a_1)\cdots\Gamma(a_p)} \sum_{r=0}^{\infty} (-c^{v}t^{v})^{r}\ {}_{p+1}\Psi_{q+1}\left[\begin{array}{c} (\mathrm{a_1},1),\cdots,(a_{\mathrm{p}},1),(1,1) \\ (b_1,1),\cdots,(b_q,1),(1+\nu r,\alpha) \end{array} ; wt^{\alpha}\right]. \tag{33}$$

Further, if we set $p = q = 1$ in (28), then $\overset{\alpha,\beta}{\underset{\mathrm{p,q;\ m,n}}{\mathrm{M}}}(z)$ reduces to $E\begin{array}{c} a_1,b_1,m \\ \alpha,\beta,n \end{array}(\mathrm{z})$ we have the following corollary.

Corollary 6. *Let $c,\ w,\nu \in \mathbb{R}^{+}; \alpha,\beta,t \in C;\ m,\ n > 0; \Re(\alpha) > 0$ and $\mathrm{m} \leq n + \Re(\alpha)$. Then, the equation*

$$N(t) - \mathrm{N_0}\, t^{\beta-1}\, E\begin{array}{c} a_1,b_1,m \\ \alpha,\beta,n \end{array} (\mathrm{wt}^{\alpha}) = -\mathrm{c}_0^{\nu}\mathrm{D}_{\mathrm{t}}^{-v} N(t) \tag{34}$$

has the solution

$$N(t) = \mathrm{N_0}\, t^{\beta-1} \frac{\Gamma(b_1)}{\Gamma(a_1)} \sum_{r=0}^{\infty} (-c^{v}t^{v})^{r}\ {}_{2}\Psi_{2}\left[\begin{array}{c} (\mathrm{a_1},\mathrm{m}),(1,1) \\ (b_1,n),(\beta+\nu r,\alpha) \end{array} ; wt^{\alpha}\right]. \tag{35}$$

3. Solution of generalized fractional kinetic equations by using the Sumudu transform

In this section, we will discuss the solution of the generalized fractional kinetic equation (18) and (27) involving the generalized M-series [2] by applying another integral transform (i.e. Sumudu transform) technique.

Sumudu transform [13] of the function $f(t)$ is defined as

$$S\{f(t):u\} = \int_0^\infty e^{-t} f(ut)\,dt. \tag{36}$$

The convolution theorem for Sumudu transform is given by

$$S\{f*g:u\} \;=\; u\,S\{f:u\}\,S\{g:u\}. \tag{37}$$

If we apply (37) then, the Sumudu transform of the Riemann–Liouville fractional integral operator (10) is given by

$$S\left\{{}_0D_t^{-\nu} f\,(t):u\right\} = \,u\,S\left\{\frac{t^{\nu-1}}{\Gamma\,(\nu)}\right\}\,S\left\{f\,(t):u\right\} \tag{38}$$

and also

$$S\left\{N(t):u\right\} = N(u). \tag{39}$$

Now, we begin by stating and proving the following Lemmas.

Lemma 3. *For min*$\{\Re(\gamma),\Re(\sigma),\Re\,(u)\} > 0$, *the Sumudu transform of the generalized M-series* $\underset{p,q;\,m,n}{\overset{\alpha,\beta}{\mathrm{M}}}(z)$ *is given by*

$$\begin{aligned} &S\left\{t^{\gamma-1}\underset{p,q;m,n}{\overset{\alpha,\beta}{M}}(t^{\sigma}):u\right\} \\ &= u^{\gamma-1}\frac{\Gamma(b_1)\cdots\Gamma(b_q)}{\Gamma(a_1)\cdots\Gamma(a_p)}\;{}_{p+2}\Psi_{q+1}\left[\begin{array}{c}(a_1,m),\cdots,(a_p,m),\;\;(\gamma,\sigma),\;(1,1)\\ (b_1,n),\cdots,(b_q,n),(\beta,\alpha)\end{array};u^{\sigma}\right]. \end{aligned} \tag{40}$$

Proof. By taking (3) *and* (36) *into account, we can easily obtain* (40) *after a little simplification.*

If we take $\gamma=\beta$ and $\sigma=\alpha$ in (40), then a special case of the above Lemma 3 is given by

Lemma 4. *For min*$\{\Re(\alpha),\Re(\beta),\Re\,(u)\} > 0$, *the following Sumuda transform of generalized M-series* (3) *holds true:*

$$S\left\{t^{\beta-1}\underset{p,q;m,n}{\overset{\alpha,\beta}{M}}(t^{\alpha}):u\right\} = u^{\beta-1}\frac{\Gamma(b_1)\cdots\Gamma(b_q)}{\Gamma(a_1)\cdots\Gamma(a_p)}\;{}_{p+1}\Psi_{q}\left[\begin{array}{c}(a_1,m),\cdots,(a_p,m),(1,1)\\ (b_1,n),\cdots,(b_q,n),\end{array};u^{\alpha}\right]. \tag{41}$$

Discussion I. Let $c,\ w,v,\gamma,\sigma\in\mathbb{R}^+$ *and* $\Re\,(u)>0$ with $|u|<c^{-1}(c\;\neq w)$. Also $\alpha,\beta,t\in C$; $m,n>0$; $\Re(\alpha)>0$ *and* $pm\le qn+\Re(\alpha)$. Then, the solution of the generalized fractional kinetic equation (19) is given by (20).

By taking the Sumudu transform on both side of (19). Using (38) and (39), we have

$$\mathrm{S}\left\{\mathrm{N(t)}:\ \mathrm{u}\right\} - \mathrm{N}_0\,\mathrm{S}\left\{\mathrm{t}^{\gamma-1}\underset{\mathrm{p,q;m,n}}{\overset{\alpha,\beta}{\mathrm{M}}}(\mathrm{wt}^{\sigma}):\ \mathrm{u}\right\} = -\mathrm{c}^{\nu}\mathrm{S}\left\{{}_0\mathrm{D}_{\mathrm{t}}^{-v}N(t)\ :u\right\}$$

$$N(u) = \frac{N_0}{1+c^\nu u^\nu}\, S\left\{t^{\gamma-1} \underset{p,q;m,n}{\overset{\alpha,\beta}{M}}(wt^\sigma) \, : u\right\}$$

. Next, by using (39) and the binomial series expansion $(1+c^\nu u^\nu)^{-1} = \sum_{r=0}^{\infty}(-1)^r (cu)^{\nu r}$, we obtain

$$\mathrm{N(u)} = \mathrm{N}_0 \sum_{r=0}^{\infty}(-1)^r (cu)^{\nu r} \sum_{k=0}^{\infty} \frac{(a_1)_{km}\cdots(a_p)_{km}}{(b_1)_{kn}\cdots(b_q)_{kn}} \frac{\Gamma(\gamma+\sigma k)\ \Gamma(k+1)}{\Gamma(\alpha k+\beta)} \frac{w^k}{k!} u^{\gamma+\sigma k-1}$$

$$= \mathrm{N}_0 \sum_{r=0}^{\infty}(-c^\nu)^r \sum_{k=0}^{\infty} \frac{(a_1)_{km}\cdots(a_p)_{km}}{(b_1)_{kn}\cdots(b_q)_{kn}} \frac{\Gamma(\gamma+\sigma k)\ \Gamma(k+1)}{\Gamma(\alpha k+\beta)} \frac{w^k}{k!} u^{\gamma+vr+\sigma k-1}. \tag{42}$$

Now, taking inverse Sumudu transform of (42) and using

$$S^{-1}\left\{u^{v-1} \, : t\right\} = \frac{t^{\nu-1}}{\Gamma(\nu)}, (\min\{\Re(\nu), \Re(u)\} > 0)$$

and $S^{-1}\{N(u) \, : t\} = N(t)$, we get

$$S^{-1}\{N(u) : t\} = \mathrm{N}_0 \sum_{r=0}^{\infty}(-c^\nu)^r \sum_{k=0}^{\infty} \frac{(a_1)_{km}\cdots(a_p)_{km}}{(b_1)_{kn}\cdots(b_q)_{kn}} \frac{\Gamma(\gamma+\sigma k)\ \Gamma(k+1)}{\Gamma(\alpha k+\beta)} \frac{w^k}{k!}$$
$$\times \ \ S^{-1}\left\{u^{\gamma+\nu r+\sigma k-1} \colon \ t\right\}$$

or

$$N(t) = \mathrm{N}_0 \sum_{r=0}^{\infty}(-c^\nu)^r \sum_{k=0}^{\infty} \frac{(a_1)_{km}\cdots(a_p)_{km}}{(b_1)_{kn}\cdots(b_q)_{kn}} \frac{\Gamma(\gamma+\sigma k)\ \Gamma(k+1)}{\Gamma(\alpha k+\beta)} \frac{w^k}{k!} \frac{t^{\gamma+\nu r+\sigma k-1}}{\Gamma(\gamma+\nu r+\sigma k)}.$$

Finally, by using (5), we arrive at the desired result (20).

Discussion II. Let $c, w, v \in \mathbb{R}^+$ and $\Re(u) > 0$ with $|u| < c^{--1}(c \neq w)$. Also $\alpha, \beta, t \in C$; $m, n > 0$; $\Re(\alpha) > 0$ and $pm \leq qn + \Re(\alpha)$. Then, the solution of the generalized fractional kinetic equation (28) is given by (29).
As in the proof of the Theorem 2, we make use Sumudu transform instead of Laplace transform into account, then we can obtain desired result (29).

4. Conclusion

In this paper we have introduced a new fractional generalization of the standard kinetic equation and derived their solutions in view of generalized M-Series, M-series and generalized Mittag–Leffler function. We can also obtain the number of special functions as the special cases of our main results, being of general nature, are shown to be some unification and extension of many known results given, for example Saxena *et al.* [7, 8, 9], Saxena and Kalla [6] etc.

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Received: 22 March 2017
Accepted: 15 June 2019