Caspian Journal of Applied Mathematics, Ecology and Economics
V. 7, No 1, 2019, July
ISSN 1560-4055

On an Inverse Boundary Value Problem For a Third Order Partial Differential Equation With Non-classical Boundary Conditions

A.I. Ismayilov

Abstract. In this work the inverse boundary value problem with unknown time-dependent coefficient for a third-order partial differential equation with non-classical boundary conditions is studied. The definition of the classical solution of the stated problem is given. The essence of the problem is that it is required together with the solution to determine an unknown coefficient. The problem is considered in the rectangular domain. When solving the initial inverse boundary value problem, the transition from the initial inverse problem to some auxiliary inverse problem is performed. With the help of contraction mappings, the existence and uniqueness of the solution of an auxiliary problem are proved. Then the transition to the original inverse problem is made again, and as a result, a conclusion is made about the solvability of the initial inverse problem.
Key Words and Phrases: inverse problem, third order equations, existence and uniqueness of a classical solution.

1. Introduction

In the present work, by the inverse problem for partial differential equations we mean such a problem in which, together with the solution, it is required to determine the right-hand side or (and) one or another coefficient (coefficients) of the equation itself. Inverse problems arise in the most diverse areas of human activity, such as seismology, mineral exploration, biology, medicine, quality control of industrial products, etc., which puts them in a series of actual problems of modern mathematics. If in the inverse problem the solution and the right-hand side are unknown, then such as inverse problem will be linear; if the solution and at least one of the coefficients are unknown, then the inverse problem will be nonlinear

Various inverse problems for particular types of partial differential equations have been studied in many papers. We note here, first of all, the works of A.N. Tikhonov [1], M.M. Lavrent'ev [2,3], V.K. Ivanov [4] and their students. For more details, see the monograph by A.M. Denisov [5].

The goal of this paper is to prove the existence and uniqueness of the solution of an inverse boundary value problem for a third order differential equation with nonclassical boundary conditions.

 © *2013 CJAMEE* All rights reserved.

2. Statement of the inverse boundary value problem

Consider an inverse boundary value problem for the equation

$$u_{tt}(x,t) - a(t)u_{txx}(x,t) = p(t)u(x,t) + q(t)u_t(x,t) + f(x,t) \tag{1}$$

in the domain $D_T = \{(x,t) : 0 \le x \le 1,\ 0 \le t \le T\}$ with initial conditions

$$u(x,0) = \varphi(x)\,,\ u_t(x,0) = \psi(x)\ \ (0 \le x \le 1)\ , \tag{2}$$

with Dirichlet boundary condition

$$u(0,t) = 0\ \ (0 \le t \le T)\ , \tag{3}$$

with non-classical boundary condition

$$u_x(1,\ t) + du_{xx}(1,\ t) = 0\ \ (\,0 \le t \le T), \tag{4}$$

and with an additional condition

$$u(x_i,t) = h_i(t)\ \ \ (i = 1,2;\ 0 < x_1, x_2 < 1,\ \ x_1 \ne x_2,\ \ 0 \le t \le T), \tag{5}$$

where $d > 0$ is a given number, $a(t) > 0$, $f(x,t)$, $\varphi(x)$, $\psi(x)$, $h_i(t)$ $(i = 1,2)$-are given functions, $u(x,t)$, $p(t)$ and $q(t)$ are required functions.

Let us introduce the notation

$$\tilde{C}^{2,2}(D_T) = \ \{u(x,t) : u(x,t) \in C^2(D_T),\ u_{txx}(x,t) \in C^2(D_T)\}\,.$$

Definition 1. *Under the classical solution of the inverse problem* (1)-(5) *we mean the triple* $\{u(x,t), p(t)\,, q(t)\}$ *of the functions* $u(x,t)$, $p(t)$, $q(t)$, *if* $u(x,t) \in \tilde{C}^{2,2}(D_T)$, $p(t) \in C[0,T]$, $q(t) \in C[0,T]$ *and relations* (1) - (5) *are satisfied in the usual sense.*

First consider the following spectral problem [6,7] :

$$y''(x) + \lambda\, y\,(x) = 0\ \ (0 \le x \le 1),$$

$$y(0) = 0,\ \ y'(1) = d\lambda\, y\,(1),\ \ d > 0. \tag{6}$$

This problem has only eigenfunctions $y_k(x) = \sqrt{2}\sin\left(\sqrt{\lambda_k}x\right)$, $k = 0,\ 1,\ 2,\ ...$, with positive eigenvalues λ_k from the equation $ctg\sqrt{\lambda} = d\sqrt{\lambda}$. The zero index is assigned to any eigenfunction, and all the others are numbered in ascending order of eigenvalues.

The following theorem is true.

Theorem 1. *Let* $f(x,t) \in C(D_T)$, $\varphi(x), \psi(x) \in C[0,\ 1]$, $h_i(t) \in C^2[0,T]$ $\ (i = 1,2)$, $h(t) \equiv h_1(t)h_2'(t) - h_2(t)h_1'(t) \ne 0\ \ (0 \le t \le T)$, $\varphi'(1) + d\varphi''(1) = 0$,

$$\varphi(1) + \frac{1}{d\sin\sqrt{\lambda_0}}\int_0^1 \varphi(x)\sin(\sqrt{\lambda_0}x)dx = 0\,, \tag{7}$$

$$\psi(1)+\frac{1}{d\sin\sqrt{\lambda_0}}\int_0^1\psi(x)\sin(\sqrt{\lambda_0}x)dx=0\,, \tag{8}$$

$$f(1,\,t)+\frac{1}{d\sin\sqrt{\lambda_0}}\int_0^1 f(x,t)\sin(\sqrt{\lambda_0}x)dx=0\ (0\le t\le T), \tag{9}$$

and the conditions of matching are satisfied

$$\varphi(x_i)=h_i(0\,),\ \psi\,(x_i)=h_i'(0)\ \ (i=1,2)\,. \tag{10}$$

Then the problem of finding a classical solution of problem (1) - (5) *is equivalent to the problem of determining the functions* $u(x,t)\in\tilde{C}^{2,2}(D_T)$, $p(t)\in C[0,T]$, $q(t)\in C[0,T]$, *satisfying the equation* (1), *conditions* (2), (3) *and the conditions*

$$u(1,\ t)+\frac{1}{d\sin\sqrt{\lambda_0}}\int_0^1 u(x,t)\sin(\sqrt{\lambda_0}x)dx=0\ \ (\,0\le t\le T), \tag{11}$$

$$h_i''(t)-a(t)u_{txx}(x_i,t)=p(t)h_i(t)+q(t)h_i'(t)+f(x_i,t)\ (i=1,\,2;\ 0\le t\le T). \tag{12}$$

Proof. Let $\{u(x,t),p(t)\,,q(t)\}$ be any solution of problem (1) - (5). Then from equation (1), with considering (9), we have:

$$\left[u_{tt}(1,t)+\frac{1}{d\sin\sqrt{\lambda_0}}\int_0^1 u_{tt}(x,t)\sin(\sqrt{\lambda_0}x)dx\right]-$$

$$-a(t)\left[u_{txx}(1,t)+\frac{1}{d\sin\sqrt{\lambda_0}}\int_0^1 u_{txx}(x,t)\sin(\sqrt{\lambda_0}x)dx\right]=$$

$$=p(t)\left[u(1,t)+\frac{1}{d\sin\sqrt{\lambda_0}}\int_0^1 u(x,t)\sin(\sqrt{\lambda_0}x)dx\right]+$$

$$+q(t)\left[u_t(1,t)+\frac{1}{d\sin\sqrt{\lambda_0}}\int_0^1 u_t(x,t)\sin(\sqrt{\lambda_0}x)dx\right]\ (0\le t\le T). \tag{13}$$

Integrating in parts twice, in view of (3), with the help of easy transformations we find:

$$u_{xx}(1,t)+\frac{1}{d\sin\sqrt{\lambda_0}}\int_0^1 u_{xx}(x,t)\sin(\sqrt{\lambda_0}x)dx=\frac{1}{d}\left(u_x\,(1,t)+du_{xx}\,(1,t)\right)-$$

$$-\lambda_0\left[u(1,t)+\frac{1}{d\sin\sqrt{\lambda_0}}\int_0^1 u(x,t)\sin(\sqrt{\lambda_0}x)dx\right]. \tag{14}$$

Substituting (14) into (13), we get:

$$\left[u_{tt}(1,t)+\frac{1}{d\sin\sqrt{\lambda_0}}\int_0^1 u_{tt}(x,t)\sin(\sqrt{\lambda_0}x)dx\right]-a(t)\left[\frac{1}{d}\left(u_{tx}\,(1,t)+du_{txx}\,(1,t)\right)\right]=$$

$$=p(t)\left[u(1,t)+\frac{1}{d\sin\sqrt{\lambda_0}}\int_0^1 u(x,t)\sin(\sqrt{\lambda_0}x)dx\right]+$$

$$+(q(t)-\lambda_0 a(t))\left[u_t(1,t)+\frac{1}{d\sin\sqrt{\lambda_0}}\int_0^1 u_t(x,t)\sin(\sqrt{\lambda_0}x)dx\right]\ (0\le t\le T). \tag{15}$$

From (15), by virtue of (4), we find:

$$\omega''(t)-p(t)\omega(t)-q\,(t)-\lambda_0 a(t))\omega'(t)=0\ \ (0\le t\le T)\ , \tag{16}$$

where

$$\omega\,(t)\ \equiv u(1,t)+\frac{1}{d\sin\sqrt{\lambda_0}}\int_0^1 u(x,t)\sin(\sqrt{\lambda_0}x)dx\ \ (0\le t\le T)\ . \tag{17}$$

Further, by virtue of (2) and in view of (7), (8) we find :

$$\omega\,(0)\ =\varphi(1)+\frac{1}{d\sin\sqrt{\lambda_0}}\int_0^1 \varphi(x)\sin(\sqrt{\lambda_0}x)dx=0\,,$$

$$\omega'\,(0)\ =\psi(1)+\frac{1}{d\sin\sqrt{\lambda_0}}\int_0^1 \psi(x)\sin(\sqrt{\lambda_0}x)dx=0\,. \tag{18}$$

It is obvious that the problem (16), (18) has only a trivial solution, i.e. $\omega(t)=0\ \ (0\le t\le T)$. Therefore, it is clear from (17) that condition (11) is also satisfied.

Further, from (5) it is clear that

$$u_t(x_i,t)=h_i'(t),\quad u_{tt}(x_i,t)=h_i''(t)\ (i=1,\,2;\ 0\le t\le T)\,. \tag{19}$$

Supplying $x=x_i\,(i=1,2)$ in equation (1), we have

$$u_{tt}(x_i\,,t)-a(t)u_{txx}(x_i\,,t)=p(t)u(x_i\,,t)+q(t)u_t(x_i\,,t)+f(x_i\,,t)(i=1,2;\ 0\le t\le T). \tag{20}$$

From here, taking into account (5) and (19), we arrive at the fulfilment of (12).

Now, suppose that $\{u(x,t),p(t),q(t)\}$ is a solution to problem (1) - (3), (11), (12), and the condition of matching (10) is satisfied.

Then from (15), in view of (11) we have:

$$u_{tx}\,(1,t)+du_{txx}\,(1,t)\ =0. \tag{21}$$

By virtue of (2) and $\varphi'(1)+d\varphi''(1)=0$ it is obvious that

$$u_x\,(1,0)+du_{xx}\,(1,0)\ =\varphi'(1)+d\varphi''(1)=0. \tag{22}$$

From (21) and (22) we arrive at the fulfilment of (4).

Further, from (12) and (20) we obtain:

$$\frac{d^2}{dt^2}(u(x_i,t)-h_i(t))-q(t)\frac{d}{dt}(u(x_i,t)-h_i(t))$$
$$-p(t)(u(x_i,t)-h_i(t))=0\quad(i=1,2;\ 0\le t\le T). \tag{23}$$

By virtue of (2) and condition of matching (10), we have:

$$u(x_i,0)-h_i(0)=\varphi(x_i)-h_i(0\,)=0,$$
$$u_t(x_i,0)-h_i'(0)=\psi\,(x_i)-h_i'(0)=0\ (i=1,2). \tag{24}$$

From (23) and (24) we conclude that condition (5) is satisfied. The theorem is proved.

3. Auxiliary facts

Solving the homogeneous problem corresponding to problem (1) - (3), (11), (12), by the method of separation of variables we arrive at the spectral problem

$$y''(x) + \lambda\, y(x) = 0 \;\; (0 \le x \le 1),$$

$$y(0) = 0\,,\;\; y(1) + \frac{1}{d \sin\sqrt{\lambda_0}} \int_0^1 y(x) \sin(\sqrt{\lambda_0} x) dx = 0. \tag{25}$$

It is known [6] that the spectral problem (25) is equivalent to the spectral problem (6) without an eigenfunction corresponding to an eigenvalue λ_0. Consequently, the spectral problem (25) has only eigenfunctions $y_k(x) = \sqrt{2} \sin(\sqrt{\lambda_k} x)\,,\;\; k = 1,\; 2,\; ...$ with positive eigenvalues λ_k, defined from the equation $ctg\sqrt{\lambda} = d\sqrt{\lambda}$, numbered in increasing order.

Consequently, the spectral problem (25) has only eigenfunctions $y_k(x) = \sqrt{2} \sin(\sqrt{\lambda_k} x)\,,$ $k = 1,\; 2,\; ...$ with positive eigenvalues λ_k, determined from the equation $ctg\sqrt{\lambda} = d\sqrt{\lambda}$, numbered in increasing order.

The following statements were formulated and substantiated in [6,7].

Lemma 1. *Starting from some number N, the estimate*

$$0 < \sqrt{\lambda_k} - \pi k < (d\pi k)^{-1}\,. \tag{26}$$

Corollary 1. *Let $v_k(x) = \sqrt{2}\, \sin(\sqrt{\mu_k} x)$, where $\sqrt{\mu_k} = \pi k$, $k = 1, 2, 3, ...$ Then the following inequalities are true*

$$\sum_{k=N}^{\infty} \|y_k(x) - v_k(x)\|^2_{L_2(0,1)} \le 1/(9d^2). \tag{27}$$

Lemma 2. *Biorthogonally conjugated system $\{z_k(x)\}$ to the system $\{y_k(x)\}$, $k = 1, 2, 3, ...,$ is determined by the formula*

$$z_k(x) = \sqrt{2}(\sin(\sqrt{\lambda_k} x) - \sin\sqrt{\lambda_k}(\sin\sqrt{\lambda_0} x)/(\sin\sqrt{\lambda_0}))/(1 + d \sin^2 \sqrt{\lambda_k}). \tag{28}$$

Theorem 2. *Systems $\{y_k(x)\}$, $k = 1, 2, ...,$ form a Riesz basis for $L_2(0, 1)$.*

Now, let $\eta_k(x) = \sqrt{2} \cos(\sqrt{\lambda_k} x)$, $\xi_k(x) = \sqrt{2} \cos(\sqrt{\mu_k} x)$, $k = 1, 2, 3, ...$ Then, similarly to (27), the inequalities

$$\sum_{k=N}^{\infty} \|\eta_k(x) - \xi_k(x)\|^2_{L_2(0,1)} \le 1/(9d^2), \tag{29}$$

are true. Suppose that $g(x) \in L_2(0, 1)$. Then, in view of (27), we obtain

$$\left(\sum_{k=1}^{\infty} \left(\int_0^1 g(x) y_k(x) dx \right)^2 \right)^{1/2} \le M\, \|g(x)\|_{L_2(0,1)}\,, \tag{30}$$

where

$$M=\left[\sum_{k=1}^{N}\int_0^1 y_k^2(x)dx+2/\left(9d^2\right)+2\right]^{1/2}. \tag{31}$$

Similar to (30), taking into account (29), we find:

$$\left(\sum_{k=1}^{\infty}\left(\int_0^1 g(x)\eta_k(x)dx\right)^2\right)^{1/2}\le M\ \|g(x)\|_{L_2(0,1)}. \tag{32}$$

Since the functions $\{y_k(x)\}$, $k=1,2,3,...$, form a Riesz basis for space $L_2(0,1)$, then it is known that for any function $g(x)\in L_2(0,1)$ the equality

$$g(x)=\sum_{k=1}^{\infty} g_k y_k(x), \tag{33}$$

is true, where

$$g_k=\int_0^1 g(x)z_k(x)dx\,(k=1,2,....).$$

Further, it is not difficult to see that

$$g_k=\frac{\sqrt{2}}{\alpha_k}\left[\int_0^1 g(x)\sin\left(\sqrt{\lambda_k}x\right)dx-\frac{\cos\sqrt{\lambda_k}}{d\sqrt{\lambda_k}\,\sin\sqrt{\lambda_0}}\int_0^1 g(x)\sin\sqrt{\lambda_0}x dx\right], \tag{34}$$

where

$$\alpha_k=1+d\sin^2\sqrt{\lambda_k}>1.$$

Hence, in view of (30) we have:

$$\left(\sum_{k=1}^{\infty} g_k^2\right)^{1/2}\le M_0\ \|g(x)\|_{L_2(0,1)}, \tag{35}$$

where

$$M_0=\left[M+\frac{1}{d\left|\sin\sqrt{\lambda_0}\right|}\left(\sum_{k=1}^{\infty}\frac{1}{\lambda_k}\right)^{1/2}\right]\sqrt{2}. \tag{36}$$

Assume that $g(x)\in C[0,1]$, $g'(x)\in L_2(0,1)$, $g(0)=0$ and

$$J(g)\equiv g(1)+\frac{1}{d\sin\sqrt{\lambda_0}}\int_0^1 g(x)\sin(\sqrt{\lambda_0}x)dx=0.$$

Then from (34) we have:

$$g_k=\frac{\sqrt{2}}{\alpha_k}\frac{1}{\sqrt{\lambda_k}}\int_0^1 g'(x)\cos\left(\sqrt{\lambda_k}x\right)\,dx. \tag{37}$$

Hence, in view of (29) we obtain:

$$\left(\sum_{k=1}^{\infty}(\sqrt{\lambda_k}\,|g_k|\,)^2\right)^{1/2} \leq M\left\|g'(x)\right\|_{L_2(0,1)}. \tag{38}$$

Let $g(x) \in C^1[0,1]$, $g''(x) \in L_2(0,1)$, $g(0) = 0$ and $J(g) = 0$. Then from (37) we obtain:

$$g_k = -\frac{\sqrt{2}}{\alpha_k}\left[\frac{1}{\lambda_k}\int_0^1 g''(x)\sin(\sqrt{\lambda_k}x)dx - \frac{\cos\sqrt{\lambda_k}}{d\lambda_k\sqrt{\lambda_k}}g'(1)\right]. \tag{39}$$

Hence, we get:

$$\left(\sum_{k=1}^{\infty}(\lambda_k\,|g_k|\,)^2\right)^{1/2} \leq m\left|g'(0)\right| + \sqrt{2}M\left\|g''(x)\right\|_{L_2(0,1)}, \tag{40}$$

where $m = \frac{\sqrt{2}}{d}\left(\sum_{k=1}^{\infty}\frac{1}{\lambda_k}\right)^{1/2}$.

Now, suppose that $g(x) \in C^2[0,1]$, $g'''(x) \in L_2(0,1)$, $g(0) = 0$, $J(g) = 0$, $g''(0) = 0$ and $dg''(1) + g'(1) = 0$. Then from (39) we have:

$$g_k = -\frac{\sqrt{2}}{\alpha_k}\frac{1}{\lambda_k\sqrt{\lambda_k}}\int_0^1 g'''(x)\cos(\sqrt{\lambda_k}x)dx.$$

Hence, in view of (29) we have :

$$\left(\sum_{k=1}^{\infty}(\lambda_k\sqrt{\lambda_k}\,|g_k|\,)^2\right)^{1/2} \leq M\left\|g'''(x)\right\|_{L_2(0,1)}. \tag{41}$$

1. Denote by $B_{2,T}^{\frac{3}{2},\frac{3}{2}}$ [8], the set of all functions $u(x,t)$ of the form

$$u(x,t) = \sum_{k=1}^{\infty}u_k(t)y_k(x)\ ,$$

considering in D_T, where each of the functions $u_k(t)$ is continuously differentiable on $[0,T]$ and

$$I(u) \equiv \left\{\sum_{k=1}^{\infty}(\lambda_k\sqrt{\lambda_k}\,\|u_k(t)\|_{C[0,T]})^2\right\}^{\frac{1}{2}} + \left\{\sum_{k=1}^{\infty}(\lambda_k\sqrt{\lambda_k}\,\left\|u_k'(t)\right\|_{C[0,T]})^2\right\}^{\frac{1}{2}} < +\infty.$$

The norm on this set is defined as: $\|u(x,t)\|_{B_{2,T}^{\frac{3}{2},\frac{3}{2}}} = I(u)$.

2. By $E_T^{\frac{3}{2},\frac{3}{2}}$ denote the space consisting of the topological product $B_{2,T}^{\frac{3}{2},\frac{3}{2}} \times C[0,T] \times C[0,T]$. Norm of the element $z = \{u,p,q\}$ is defined by the formula

$$\|z\|_{E_T^{\frac{3}{2},\frac{3}{2}}} = \|u(x,t)\|_{B_{2,T}^{\frac{3}{2},\frac{3}{2}}} + \|p(t)\|_{C[0,T]} + \|q(t)\|_{C[0,T]}.$$

It is known that $B_{2,T}^{\frac{3}{2},\frac{3}{2}}$ and $E_T^{\frac{3}{2},\frac{3}{2}}$ are Banach spaces.

4. Solvability of an inverse boundary value problem

Taking into account Lemma 2 and Theorem 2, the first component $u(x,t)$ of the solution $\{u(x,t), p(t), q(t)\}$ of the problem (1) - (3), (11), (12) we will be sought in the form:

$$u(x,t) = \sum_{k=1}^{\infty} u_k(t) y_k(x) \ , \tag{42}$$

where

$$u_k(t) = \int_0^1 u(x,t) z_k(x) dx \ \ (k = 1, 2, ...).$$

We apply the method of separation of variables to determine the desired functions $u_k(t)$ $(k = 1, 2, ...;)$. Then from (1) and (2) we have:

$$u_k''(t) + \lambda_k a(t) u_k'(t) = F_k(t; u, p, q) \ \ (k = 1, 2, ...; \ 0 \leq t \leq T), \tag{43}$$

$$u_k(0) = \varphi_k, u_k'(0) = \psi_k \ (k = 1, 2, ...), \tag{44}$$

where

$$F_k(t; u, p, q) = f_k(t) + p(t) u_k(t) + q(t) u_k'(t), \ f_k(t) = \int_0^1 f(x,t) z_k(x) dx,$$

$$\varphi_k = \int_0^1 \varphi(x) z_k(x) dx, \quad \psi_k = \int_0^1 \psi(x) z_k(x) dx \ \ (k = 1, 2, ...).$$

Solving problem (43), (44), we find:

$$u_k(t) = \varphi_k + \psi_k \int_0^t e^{-\lambda_k \int_0^\tau a(s)ds} d\tau + \int_0^t F_k(\tau; u, p, q) \left(\int_\tau^t e^{-\lambda_k \int_\tau^\zeta a(s)ds} d\xi \right) d\tau. \tag{45}$$

Differentiating twice (45) we get:

$$u_k'(t) = \psi_k e^{-\lambda_k \int_0^t a(s)ds} + \int_0^t F_k(\tau; u, p, q) e^{-\lambda_k \int_\tau^t a(s)ds} d\tau \ (k = 1, 2, ...), \tag{46}$$

$$u_k''(t) = -\lambda_k a(t) \psi_k e^{-\lambda_k \int_0^t a(s)ds} -$$

$$-\lambda_k a(t) \int_0^t F_k(\tau; u, p, q) e^{-\lambda_k \int_\tau^t a(s)ds} d\tau \ + F_k(t; u, p, q) (k = 1, 2, ...). \tag{47}$$

After substituting the expression $u_k(t)$ $(k = 1, 2, \ldots)$ from (45) into (42), we have:

$$u(x,t) = \sum_{k=1}^{\infty} \left\{ \varphi_k + \psi_k \int_0^t e^{-\lambda_k \int_0^\tau a(s)ds} d\tau + \right.$$

$$\left. + \int_0^t F_k(\tau; u, p, q) \left(\int_\tau^t e^{-\lambda_k \int_\tau^\zeta a(s)ds} d\xi \right) d\tau \right\} y_k(x) \ . \tag{48}$$

Now from (12), in view of (42), we get:

$$p(t) = [h(t)]^{-1} \{h_2'(t)\,(h_1''(t) - f(x_1,t)) - h_1'(t)\,(h_2''(t) - f(x_2,t)) -$$
$$\left. -a(t) \sum_{k=1}^{\infty} \lambda_k u_k'(t)(h_2'(t) y_k(x_1) - h_1'(t) y_k(x_2)) \right\}, \tag{49}$$

$$q(t) = [h(t)]^{-1} \{h_1(t)\,(h_2''(t) - f(x_2,t)) - h_2(t)(h_1''(t) - f(x_1,t)) -$$
$$\left. -a(t) \sum_{k=1}^{\infty} \lambda_k u_k'(t)(h_1(t) y_k(x_2) - h_2(t) y_k(x_1)) \right\}, \tag{50}$$

where

$$h(t) \equiv h_1(t) h_2'(t) - h_2(t) h_1'(t) \neq 0 \, (0 \leq t \leq T).$$

In order to obtain the equation for the second and third components $p(t)$, $q(t)$ of the solution $\{u(x,t), p(t), q(t)\}$ of the problem (1)-(3), (11), (12) we substitute the expression $u_k'(t)$ from (46) into (49), (50) respectively, we have:

$$p(t) = [h(t)]^{-1} \{h_2'(t)\,(h_1''(t) - f(x_1,t)) - h_1'(t)\,(h_2''(t) - f(x_2,t)) -$$
$$-a(t) \sum_{k=1}^{\infty} \lambda_k (h_2'(t) y_k(x_1) - h_1'(t) y_k(x_2)) \times$$
$$\left. \times \left(\psi_k e^{-\lambda_k \int_0^t a(s)ds} + \int_0^t F_k(\tau; u, p, q) e^{-\lambda_k \int_\tau^t a(s)ds} d\tau \right) \right\}, \tag{51}$$

$$q(t) = [h(t)]^{-1} \{h_1(t)\,(h_2''(t) - f(x_2,t)) - h_2(t)(h_1''(t) - f(x_1,t)) -$$
$$-a(t) \sum_{k=1}^{\infty} \lambda_k \,(h_1(t) y_k(x_2) - h_2(t) y_k(x_1)) \times$$
$$\left. \times \left(\psi_k e^{-\lambda_k \int_0^t a(s)ds} + \int_0^t F_k(\tau; u, p, q) e^{-\lambda_k \int_\tau^t a(s)ds} d\tau \right) \right\}, \tag{52}$$

Thus, the solution of problem (1)-(3), (11), (12) was reduced to the solution of system (48), (51), (52) with respect to unknown functions, $u(x,t)$,$p(t)$ and $q(t)$.

To study the question of the uniqueness of the solution of problem (1) - (3), (11), (12), the following lemma plays an important role.

Lemma 3. *If* $\{u(x,t), p(t), q(t)\}$ *is any solution of the problem* (1)-(3), (11), (12), *then the functions*

$$u_k(t) = \int_0^1 u(x,t) z_k(x) dx (k = 1, 2, ...)$$

satisfy on $[0,T]$ *the system* (45).

Lemma 3 implies that the following holds.

Corollary 2. *Let system* (48), (51), (52) *have a unique solution. Then the problem* (1)-(3), (11), (12) *cannot have more than one solution, i.e. if problem* (1)-(3), (11), (12) *has a solution, then it is unique.*

Now consider the operator in space $E_T^{\frac{3}{2},\frac{3}{2}}$

$$\Phi(u,p,q) = \{\Phi_1(u,p,q), \Phi_2(u,p,q)\,,\ \Phi_3(u,p,q)\}\,,$$

where

$$\Phi_1(u,p,q) = \tilde{u}(x,t) \equiv \sum_{k=0}^{\infty} \tilde{u}_k(t) X_k(x), \Phi_2(u,p,q) = \tilde{p}(t),\ \Phi_3(u,p,q) = \tilde{q}(t),$$

and $\tilde{u}_k(t)$ $(k = 1,2,...)$, $\tilde{p}(t)$ and $\tilde{q}(t)$ are equal, respectively, right sides (45), (51) and (52).

Using easy transformations, we find that inequalities

$$\left(\sum_{k=1}^{\infty}\left(\lambda_k\sqrt{\lambda_k}\ \|\tilde{u}_k(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}} \le \sqrt{5}\left(\sum_{k=1}^{\infty}\left(\lambda_k\sqrt{\lambda_k}\ \ |\varphi_k|\right)^2\right)^{\frac{1}{2}} +$$

$$+\sqrt{5}\,T\left(\sum_{k=1}^{\infty}\left(\lambda_k\sqrt{\lambda_k}\ \ |\psi_k|\right)^2\right)^{\frac{1}{2}} + \sqrt{5}T\left[\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}\left(\lambda_k\sqrt{\lambda_k}\,|f_k(\tau)|\right)^2 d\tau\right)^{\frac{1}{2}} +\right.$$

$$+T\,\|p(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k\sqrt{\lambda_k}\ \ \|u_k(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}} +$$

$$\left. +\,T\,\|q(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k\sqrt{\lambda_k}\ \ \|u'_k(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}\right], \tag{53}$$

$$\left(\sum_{k=1}^{\infty}\left(\lambda_k\sqrt{\lambda_k}\ \|\tilde{u}'_k(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}} \le 2\left(\sum_{k=1}^{\infty}\left(\lambda_k\sqrt{\lambda_k}\ \ |\psi_k|\right)^2\right)^{\frac{1}{2}} +$$

$$+2\,T\left[\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}\left(\lambda_k\sqrt{\lambda_k}\,|f_k(\tau)|\right)^2 d\tau\right)^{\frac{1}{2}} +\right.$$

$$+T\,\|p(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k\sqrt{\lambda_k}\ \ \|u_k(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}} +$$

$$\left. +\,T\,\|q(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k\sqrt{\lambda_k}\ \ \|u'_k(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}\right], \tag{54}$$

$$\|\tilde{p}(t)\|_{C[0,T]} \leq \left\|[h(t)]^{-1}\right\|_{C[0,T]} \times$$
$$\times\{\left\|h_2'(t)\,(h_1''(t)-f(x_1,t))-h_1'(t)\,(h_2''(t)-f(x_2,t))\right\|_{C[0,T]}+$$
$$+\sqrt{2}\left(\sum_{k=1}^{\infty}\lambda_k^{-1}\right)^{\frac{1}{2}}\left\|a(t)(\left|h_1'(t)\right|+\left|h_2'(t)\right)\right|\|_{C[0,T]}\left[\left(\sum_{k=1}^{\infty}\left(\lambda_k\sqrt{\lambda_k}\ \ |\psi_k|\right)^2\right)^{\frac{1}{2}}+\right.$$
$$+\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}\left(\lambda_k\sqrt{\lambda_k}\ |f_k(\tau)|\right)^2 d\tau\right)^{\frac{1}{2}}+$$
$$+T\,\|p(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k\sqrt{\lambda_k}\ \ \|u_k(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}+$$
$$\left.+\ T\,\|q(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k\sqrt{\lambda_k}\ \ \left\|u_k'(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}\right]\Bigg\}, \tag{55}$$

$$\|\tilde{q}(t)\|_{C[0,T]} \leq \left\|[h(t)]^{-1}\right\|_{C[0,T]} \times$$
$$\times\{\left\|h_1(t)\,(h_2''(t)-f(x_2,t))-h_2(t)(h_1''(t)-f(x_1,t))\right\|_{C[0,T]}+$$
$$+\sqrt{2}\left(\sum_{k=1}^{\infty}\lambda_k^{-1}\right)^{\frac{1}{2}}\left\|a(t)(|h_1(t)|+|h_2(t))\right|\|_{C[0,T]}\left[\left(\sum_{k=1}^{\infty}\left(\lambda_k\sqrt{\lambda_k}\ \ |\psi_k|\right)^2\right)^{\frac{1}{2}}+\right.$$
$$+\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}\left(\lambda_k\sqrt{\lambda_k}\ |f_k(\tau)|\right)^2 d\tau\right)^{\frac{1}{2}}+$$
$$+T\,\|p(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k\sqrt{\lambda_k}\ \ \|u_k(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}+$$
$$\left.+\ T\,\|q(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k\sqrt{\lambda_k}\ \ \left\|u_k'(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}\right]\Bigg\}, \tag{56}$$

are true. Suppose that the data of the problem (1) - (3), (11), (12) satisfy the following conditions:

1) $\varphi(x)\in C^2\,[0,1]\,,\varphi'''(x)\in L_2(0,1),\,\varphi(0)=0\ ,\ J(\varphi)=0\ ,\varphi''(0)=0\,,$

$$d\varphi''(1)+\varphi'(1)=0\,.$$

2) $\psi(x)\in C^2\,[0,1]\,,\psi''(x)\in L_2(0,1),\,\psi(0)=0\ ,\ J(\psi)=0\ ,\psi''(0)=0\,,$

$$d\psi''(1)+\psi'(1)=0\,.$$

3) $f(x,t), f_x(x,t),\ f_{xx}(x,t) \in C(D_T),\ f_{xxx}(x,t) \in L_2(D_T),\ f\ (0,t) = 0, J\,(f) = 0,$ $f_{xx}(0,t) = 0,$

$$df_{xx}(1,t) + f_x(1,t) = 0\ \ (0 \le t \le T)\,.$$

4) $0 < a(t) \in C[0,T]$, $h_i(t) \in C^1[0,T]$ $(i = 1,2)$,

$$h(t) \equiv h_1(t)h_2'(t) - h_2(t)h_1'(t) \neq 0\,(0 \le t \le T).$$

Then from (53) - (56), in view of (41), respectively, we obtain:

$$\|\tilde{u}(x,t)\|_{B^{\frac{3}{2},\frac{3}{2}}_{2,T}} \le A_1(T) + B_1(T)(\|p(t)\|_{C[0,T]} + \|q(t)\|_{C[0,T]})\,\|u(x,t)\|_{B^{\frac{3}{2},\frac{3}{2}}_{2,T}}\ , \qquad (57)$$

$$\|\tilde{p}(t)\|_{C[0,T]} \le A_2(T) + B_2(T)(\|p(t)\|_{C[0,T]} + \|q(t)\|_{C[0,T]})\,\|u(x,t)\|_{B^{\frac{3}{2},\frac{3}{2}}_{2,T}}\ , \qquad (58)$$

$$\|\tilde{q}(t)\|_{C[0,T]} \le A_3(T) + B_3(T)(\|p(t)\|_{C[0,T]} + \|q(t)\|_{C[0,T]})\,\|u(x,t)\|_{B^{\frac{3}{2},\frac{3}{2}}_{2,T}}\ , \qquad (59)$$

where

$$A_1(T) = \sqrt{5}M\left\|\varphi'''(x)\right\|_{L_2(0,1)} + (\sqrt{5}T+2)M\left\|\varphi'''(x)\right\|_{L_2(0,1)} +$$
$$+(\sqrt{5}T+2)\sqrt{T}M\left\|f_{xxx}(x,t)\right\|_{L_2(D_T)}, B_1\,(T) = (\sqrt{5}T+2)T,$$

$$A_2(T) = \left\|[h(t)]^{-1}\right\|_{C[0,T]} \times$$
$$\times\{\left\|h_2'(t)\,(h_1''(t) - f(x_1,t)) - h_1'(t)\,(h_2''(t) - f(x_2,t))\right\|_{C[0,T]} +$$
$$+\sqrt{2}\left(\sum_{k=1}^{\infty}\lambda_k^{-1}\right)^{\frac{1}{2}}\left\|a(t)(\left|h_1'(t)\right| + \left|h_2'(t)\right)\right|\Big\|_{C[0,T]}\left[M\left\|\psi'''(x)\right\|_{L_2(0,1)} +\right.$$
$$\left. +\ \sqrt{T}M\left\|f_{xxx}(x,t)\right\|_{L_2(D_T)}\right]\Big\},$$

$$B_2(T) = \sqrt{2}\left\|h^{-1}(t)\right\|_{C[0,T]} T\left(\sum_{k=1}^{\infty}\lambda_k^{-1}\right)^{\frac{1}{2}}\left\|a(t)(\left|h_1'(t)\right| + \left|h_2'(t)\right)\right|\Big\|_{C[0,T]},$$

$$A_3(T) = \left\|[h(t)]^{-1}\right\|_{C[0,T]} \times$$
$$\times\left\{\left\|h_1(t)\left(f(x_2,t) - a_1(t)h_2'(t)\right) - h_2(t)\left(f(x_1,t) - a_1(t)h_1'(t)\right)\right\|_{C[0,T]} +\right.$$
$$+\sqrt{2}\left(\sum_{k=1}^{\infty}\lambda_k^{-1}\right)^{\frac{1}{2}}\left\|a(t)(\left|h_1(t)\right| + \left|h_2'(t)\right)\right|\Big\|_{C[0,T]}\left[M\left\|\psi'''(x)\right\|_{L_2(0,1)} +\right.$$
$$\left.\left. +\ \sqrt{T}M\left\|f_{xxx}(x,t)\right\|_{L_2(D_T)}\right]\right\},$$

$$B_3(T) = \sqrt{2}\left\|h^{-1}(t)\right\|_{C[0,T]} T\left(\sum_{k=1}^{\infty}\lambda_k^{-1}\right)^{\frac{1}{2}}\left\|a(t)(\left|h_1'(t)\right| + \left|h_2'(t)\right)\right|\Big\|_{C[0,T]}.$$

From inequalities (57) - (59) we conclude:

$$\|\tilde{u}(x,t)\|_{B^{\frac{3}{2},\frac{3}{2}}_{2,T}} + \|\tilde{p}(t)\|_{C[0,T]} + \|\tilde{q}(t)\|_{C[0,T]} \le$$

$$\le A(T) + B(T)(\|p(t)\|_{C[0,T]} + \|q(t)\|_{C[0,T]})\,\|u(x,t)\|_{B^{\frac{3}{2},\frac{3}{2}}_{2,T}} \,, \tag{60}$$

where

$$A(T) = A_1(T) + A_2(T) + A_3(T), \quad B(T) = B_1(T) + B_2(T) + B_3(T).$$

So, we can prove the following theorem:

Theorem 3. *Let the conditions 1)- 4) be fulfilled and*

$$B(T)(A(T)+2)^2 < 1. \tag{61}$$

Then the problem (1)-(3), (11), (12) *has the only solution in a ball* $K = K_R(\|z\|_{E^{\frac{3}{2},\frac{3}{2}}_{T}} \le R = A(T)+2)$ *from space* $E^{\frac{3}{2},\frac{3}{2}}_{T}$.

Proof. In space $E^{\frac{3}{2},\frac{3}{2}}_{T}$ we consider the equation

$$z = \Phi z, \tag{62}$$

where $z = \{u, p, q\}$, components $\Phi_i(u,p,p\,)(i = 1,2,3)$ of operator $\Phi(u,p,q)$ are defined by the right-hand sides of equations (48), (51), (52), respectively.

Consider the operator $\Phi(u,p,q)$ in the ball $K = K_R(\|z\|_{E^{\frac{3}{2},\frac{3}{2}}_{T}} \le R = A(T)+2)$ from $E^{\frac{3}{2},\frac{3}{2}}_{T}$. Similarly to (60), we obtain that for any $z, z_1, z_2 \in K_R$ valid the following estimates:

$$\|\Phi z\|_{E^{\frac{3}{2},\frac{3}{2}}_{T}} \le A(T) + B(T)(\|p(t)\|_{C[0,T]} + \|q(t)\|_{C[0,T]})\,\|u(x,t)\|_{B^{\frac{3}{2},\frac{3}{2}}_{2,T}} \le$$

$$\le A(T) + B(T)(A(T)+2)^2 \,, \tag{63}$$

$$\|\Phi z_1 - \Phi z_2\|_{E^{\frac{3}{2},\frac{3}{2}}_{T}} \le B(T)R\Big(\|p_1(t) - p_2(t)\|_{C[0,T]} +$$

$$+ \|q_1(t) - q_2(t)\|_{C[0,T]} + \|u_1(x,t) - u_2(x,t)\|_{B^{\frac{3}{2},\frac{3}{2}}_{2,T}}\Big). \tag{64}$$

Then, from estimates (63) and (64), taking into account (61), it follows that the operator Φ acts in a ball $K = K_R$ and is contractive. Therefore, in the ball $K = K_R$, the operator Φ has a unique fixed point $\{u, p, q\}$, which is the unique solution of equation (62), i.e. is the unique solution of the system (48), (51), (52) in the ball $K = K_R$.

The function $u(x,t)$, as an element of space $B^{\frac{3}{2},\frac{3}{2}}_{2,T}$, is continuous and has continuous derivatives $u_x(x,t)$, $u_{xx}(x,t)$, $u_{tx}(x,t)$, $u_{txx}(x,t)$ in D_T.

From (43), by virtue of (38), it is not difficult to see that

$$\left(\sum_{k=1}^{\infty}(\sqrt{\lambda_k}\left\|u_k''(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}} \leq \sqrt{2}\left\|a(t)\right\|_{C[0,T]}\left\{\left(\sum_{k=1}^{\infty}(\lambda_k\sqrt{\lambda_k}\left\|u_k'(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}+\right.$$

$$\left.+M\left\|\left\|f_x(x,t)+p(t)u_x(x,t)+p(t)u_{tx}(x,t)\right\|_{C[0,T]}\right\|_{L_2(0,1)}\right\}.$$

It follows that $u_{tt}(x,t)$ is continuous in D_T.

It is easy to verify that equation (1) and conditions (2), (3), (11) and (12) are satisfied in the usual sense. Consequently, $\{u(x,t), p(t), q(t)\}$ is the solution of the problem (1) - (3), (11), (12). By virtue of Corollary 2 of Lemma 3, it is unique in the ball $K = K_R$. The theorem is proved.

Using Theorem 1, we prove the following

Theorem 4. *Let all the conditions of Theorem 3 be satisfied and the conditions of matching*

$$\varphi(x_i) = h_i(0), \ \psi(x_i) = h_i'(0) \ \ (i = 1, 2).$$

Then problem (1) - (5) *has a unique classical solution in ball* $K = K_R(\|z\|_{E_T^{\frac{3}{2},\frac{3}{2}}} \leq R = A(T)+2)$ *of space* $E_T^{\frac{3}{2},\frac{3}{2}}$.

References

[1] A.I. Tikhonov, *On the stability of inverse problems*, Dokl. AN SSSR, **39(5)**, 1943, 195-198.

[2] M. M. Lavrentiev, *An inverse problem for a wave equation*, Dokl. AN SSSR, **157(3)**, 1964, 520-521.

[3] M.M. Lavrentiev, V.G. Romanov, S.P. Shishatskii, *Ill-Posed Problems in Mathematical Physics and Analysis,* Nauka, Moscow, 1980.

[4] V.K. Ivanov, V.V. Vasin, V.P. Tanina, *The theory of linear ill-posed problems and its applications*, Nauka, Moscow, 1978, 206 p.

[5] A.M. Denisov, *Introduction to the theory of inverse problems*, M., MSU, 1994. 206 p.

[6] N.Yu. Kapustin, E.I. Moiseev, *Convergence of spectral expansions for functions of the Hölder class for two problems with a spectral parameter in the boundary condition*, Differ. Uravn., **36(8)**, 2000, 1069–1074; Differ. Equ., **36(8)**, 2000, 1182–1188.

[7] N.Yu. Kapustin, E.I. Moiseev, *On the singularities of the root space of a spectral problem with a spectral parameter in the boundary condition*, Dokl. AN, **385(1)**, 2002, 20–24.

[8] K.I. Khudaverdiev, A.A. Veliyev, *Investigation of a one-dimensional mixed problem for a class of third-order pseudohyperbolic equations with nonlinear operator right*, Baku, Chashyogly, 2010, 168 p.

Arif I. Ismailov
Baku State University, Baku, Azerbaijan
Email: ismayilovarif@icloud.com

Received: 28 January 2019
Accepted: 17 June 2019