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Properties of Eigenvalues and Eigenfunctions of a Spectral Problem With Discontinuity Point

G.V. Maharramova

Abstract. In this paper we obtain the asymptotics of eigenvalues and eigenfunctions of one spectral problem for a discontinuous second-order differential operator with a spectral parameter in discontinuity conditions which arises by solving the problem on vibrations of a loaded string with fixed ends.

Key Words and Phrases: eigenvalues, eigenfunctions, asymptotic formulae, spectral problem.

2010 Mathematics Subject Classifications: 34B05, 34B24, 34L10, 34L20

1. Introduction

Consider the following boundary value problem

$$-y'' + q(x)\,y = \lambda y \quad , x \in \left(0, \frac{1}{3}\right) \bigcup \left(\frac{1}{3}, 1\right), \tag{1}$$

$$\left.\begin{array}{l} y(0) = y(1) = 0, \\ y(-\frac{1}{3}) = y(+\frac{1}{3}), \\ y'(-\frac{1}{3}) - y'(+\frac{1}{3}) = \lambda m y(\frac{1}{3}), \end{array}\right\} \tag{2}$$

which arises by solving the problem on vibrations of a loaded string with the fixed ends[1-3]. In the case when a load is placed in the middle of the string, this problem was investigated in[4,5]. Similar questions for the problem on vibrations of a loaded string when the load is fixed in one or two ends of a string, are investigated by [8-11].

For the case $q(x) \equiv 0$ the asymptotics of eigenvalues and eigenfunctions, also the basis properties of eigenfunctions were investigated completely in [7].

2. The asymptotic of eigenvalues and eigenfunctions

We denote $\lambda = \rho^2$, $Im\rho = \tau$. Suppose that $q(x)$ is a complex valued summable function on $(-1,1)$. Denote by $y_1(x,\lambda)$ the solution of (1) satisfying the initial conditions

$$\left.\begin{array}{l} y_1(0) = 0, \\ y_1{}'(0) = \rho, \end{array}\right\} \tag{3}$$

and by $y_2(x,\lambda)$ the solution of (1) satisfying the initial conditions

$$\left.\begin{array}{l} y_2(1) = 0, \\ y_2{}'(1) = -\rho. \end{array}\right\} \tag{4}$$

Lemma 1. *The following integral representations hold:*

$$y_1(x,\lambda) = \sin\rho x + \frac{1}{\rho}\int_0^x \sin\rho(x-t)\,q(t)\,y_1(t,\lambda)\,dt, \quad 0 < x < \frac{1}{3}, \tag{5}$$

$$y_1'(x,\lambda) = \rho\cos\rho x + \int_0^x \cos\rho(x-t)\,q(t)\,y_1(t,\lambda)\,dt, \quad 0 < x < \frac{1}{3}, \tag{6}$$

$$y_2(x,\lambda) = \sin\rho(1-x) + \frac{1}{\rho}\int_x^1 \sin\rho(t-x)\,q(t)\,y_2(t,\lambda)\,dt, \quad \frac{1}{3} < x < 1, \tag{7}$$

$$y_2'(x,\lambda) = -\rho\cos\rho(1-x) - \int_x^1 \cos\rho(t-x)\,q(t)\,y_2(t,\lambda)\,dt, \quad \frac{1}{3} < x < 1. \tag{8}$$

Proof. Since $y_1(x,\lambda)$ satisfies (1), then

$$\int_0^x \sin\rho(x-t)\,q(t)\,y_1(t)\,dt = \int_0^x \sin\rho(x-t)\,y_1{}''(t,\lambda)\,dt + \rho^2\int_0^x \sin\rho(x-t)\,y_1(t,\lambda)\,dt.$$

Integrating by part the first integral in the right-hand side of the last equation twice and taking into account (3), we find

$$\int_0^x \sin\rho(x-t)\,q(t)\,y_1(t)\,dt = -\rho\sin\rho x + \rho y_1(x,\lambda),$$

i.e. the equality (5).

The equality (6) is obtained by differentiating the equality (5).

The equalities (7) and (8) are obtained similarly.

Lemma 2. *The following asymptotic formulas hold when $\rho \to \infty$*

$$y_1(x,\lambda) = O\left(e^{|\tau|x}\right), \tag{9}$$

$$y_2(x,\lambda) = O\left(e^{|\tau|(1-x)}\right), \tag{10}$$

more precisely

$$y_1(x,\lambda) = \sin\rho x + O\left(\frac{e^{|\tau|x}}{|\rho|}\right), \tag{11}$$

$$y_2(x,\lambda) = \sin\rho\,(1-x) + O\left(\frac{e^{|\tau|(1-x)}}{|\rho|}\right). \tag{12}$$

All estimates are satisfied uniformly on x for $y_1(x,\lambda)$ when $0 \le x \le \frac{1}{3}$ and for $y_2(x,\lambda)$ when $\frac{1}{3} \le x \le 1$.

The proof repeats that lemma in [6] word for word.

Denote

$$q_1(x) = \frac{1}{2}\int_0^x q(t)\,dt,$$

$$q_2(x) = \frac{1}{2}\int_x^1 q(t)\,dt.$$

Theorem 1. *The spectrum of problem* (1)-(2) *consists of three sequences $\lambda_{i,n} = \rho_{i,n}^2$, $i = 1,2,3$; $n = 1,2,...,$ of asymptotically simple eigenvalues:*

$$\begin{array}{l} \rho_{1,n} = 3\pi n + \frac{\alpha_1}{n} + o\left(\frac{1}{n}\right) \\ \rho_{2,n} = 3\pi n + \frac{\alpha_2}{n} + o\left(\frac{1}{n}\right) \\ \rho_{3,n} = 3\pi n + \frac{3\pi}{2} + \frac{\alpha_3}{n} + o\left(\frac{1}{n}\right), \end{array}$$

where α_i, $i = 1,2,3$ are different numbers expressed by the values of the functions $q_1(x)$ and $q_2(x)$ at the point $\frac{1}{3}$.

Proof. Substitute asymptotics for $y_1(x)$ from (11) in the right-hand side of (5):

$$y_1(x) = \sin\rho x + \frac{1}{\rho}\int_0^x \sin\rho(x-t)\,q(t)\left[\sin\rho t + O\left(\frac{e^{|\tau|}}{\rho}\right)\right]dt =$$

$$= \sin\rho x + \frac{1}{\rho}\int_0^x \sin\rho(x-t)\sin\rho t\cdot q(t)\,dt +$$

$$+\frac{1}{\rho^2}\int_0^x \sin\rho(x-t)\,q(t)\,O\left(e^{|\tau|t}\right)dt =$$

$$= \sin\rho x + \frac{1}{2\rho}\int_0^x [\cos\rho(x-2t) - \cos\rho x]\,q(t)\,dt +$$

$$+\frac{1}{\rho^2}\int_0^x \sin\rho(x-t)\,q(t)\,O\left(e^{|\tau|t}\right) =$$

$$= \sin\rho x + \frac{1}{2\rho}\int_0^x \cos\rho(x-2t)\,q(t)\,dt - \frac{\cos\rho x}{2\rho}\int_0^x q(t)\,dt +$$

$$+\frac{1}{\rho^2}\int_0^x \sin\rho\,(x-t)\,q\,(t)\,O\left(e^{|\tau|t}\right)dt=$$
$$=\sin\rho x+\frac{1}{2\rho}\int_0^x \cos\rho\,(x-2t)\,q\,(t)\,dt-$$
$$-\frac{1}{\rho}\cos\rho x\left(\frac{1}{2}\int_0^x q\,(t)\,dt\right)+$$
$$++\frac{e^{|\tau|x}}{\rho^2}\int_0^x \frac{\sin\rho\,(x-t)}{e^{|\tau|(x-t)}}dt.$$

Hence,

$$y_1\,(x)=\sin\rho x-\frac{1}{\rho}q_1\,(x)\cos\rho x+\frac{1}{2\rho}\int_0^x \cos\rho\,(x-2t)\,q\,(t)\,dt+O\left(\frac{e^{|\tau|x}}{|\rho|^2}\right) \tag{13}$$

Substitute asymptotics for $y_1\,(x)$ from (11) in the right-hand side of (6):

$$y_1'\,(x)=\rho\cos\rho x+\int_0^x \cos\rho\,(x-t)\,q\,(t)\,y_1\,(t,\lambda)\,dt=$$
$$=\rho\cos\rho x+\int_0^x \cos\rho\,(x-t)\left[\sin\rho t+O\left(\frac{e^{|\tau|t}}{|\rho|}\right)\right]q\,(t)\,dt=$$
$$=\rho\cos\rho x+\frac{1}{2}\int_0^x\left[\sin\rho x+\sin\rho\,(2t-x)\right]q\,(t)\,dt+$$
$$+\int_0^x \cos\rho\,(x-t)\,O\left(\frac{e^{|\tau|t}}{|\rho|}\right)q\,(t)\,dt=\rho\cos\rho x+$$
$$++\frac{1}{2}\int_0^x \sin\rho x q\,(t)\,dt+\frac{1}{2}\int_0^x \sin\rho\,(2t-x)\,q\,(t)\,dt+$$
$$+\int_0^x \cos\rho\,(x-t)\,O\left(\frac{e^{|\tau|t}}{|\rho|}\right)dt=\rho\cos\rho x+q_1\,(x)\sin\rho x+$$
$$+\frac{1}{2}\int_0^x \sin\rho\,(2t-x)\,q\,(t)\,dt+\frac{e^{|\tau|x}}{|\rho|}\int_0^x \frac{\cos\rho\,(x-t)}{e^{|\tau|(x-t)}}O\,(1)\,q\,(t)\,dt$$
$$=\rho\cos\rho x+q_1\,(x)\sin\rho x+\frac{1}{2}\int_0^x \sin\rho\,(2t-x)\,q\,(t)\,dt+$$
$$+O\left(\frac{e^{|\tau|x}}{|\rho|}\right).$$

Hence,

$$y_1'\,(x)=\rho\cos\rho x+q_1\,(x)\sin\rho x+\frac{1}{2}\int_0^x \sin\rho\,(2t-x)\,q\,(t)\,dt+O\left(\frac{e^{|\tau|x}}{|\rho|}\right) \tag{14}$$

The following asymptotic equalities are obtained analogously:

$$\begin{array}{l} y_2(x)=\sin\rho(1-x)-\frac{1}{\rho}q_2(x)\cos\rho(1-x)+ \\ +\frac{1}{2\rho}\int_x^1\cos\rho(2t-x-1)\,q(t)\,dt+O\left(\frac{e^{|\tau|(1-x)}}{|\rho|^2}\right) \end{array} \tag{15}$$

and

$$\begin{array}{l} y_2'(x)=-\rho\cos\rho(1-x)-q_2(x)\sin\rho(1-x)- \\ -\frac{1}{2}\int_x^1 q(t)\sin\rho(1+x-2t)\,dt+O\left(\frac{e^{|\tau|(1-x)}}{|\rho|}\right). \end{array} \tag{16}$$

Obviously, for any $\lambda\neq 0$ the solution $y(x,\lambda)$ of the problem (1)-(2) have to be in the form

$$y(x)=\begin{cases} C_1y_1(x), & for \quad 0<x<\frac{1}{3}, \\ C_2y_2(x), & for \quad \frac{1}{3}<x<1, \end{cases}$$

here C_1 and C_2 are complex numbers. $\lambda\neq 0$ is an eigenvalue of the problem (1)-(2) if and only if

C_1 and C_2 are nontrivial solutions of following homogeneous system of linear equations:

$$\begin{cases} C_1\left(\sin\frac{1}{3}\rho-\frac{1}{\rho}q_1\left(\frac{1}{3}\right)\cos\frac{1}{3}\rho+\frac{1}{2\rho}\int_0^x\cos\rho\left(\frac{1}{3}-2t\right)q(t)\,dt+O\left(\frac{e^{\frac{1}{3}|\tau|}}{|\rho|^2}\right)\right)- \\ -C_2\left(\sin\frac{2}{3}\rho-\frac{1}{\rho}q_2\left(\frac{1}{3}\right)\cos\frac{2}{3}\rho+\frac{1}{2\rho}\int_{\frac{1}{3}}^1\cos\left(2t-\frac{4}{3}\right)q(t)\,dt+O\left(\frac{e^{\frac{2}{3}|\tau|}}{|\rho|^2}\right)\right)=0 \\ C_1\left(\rho\cos\frac{1}{3}\rho+q_1\left(\frac{1}{3}\right)\sin\frac{1}{3}\rho+\frac{1}{2}\int_0^{\frac{1}{3}}\sin\rho\left(2t-\frac{1}{3}\right)q(t)\,dt+O\left(\frac{e^{\frac{1}{3}|\tau|}}{|\rho|}\right)\right)- \\ -C_2\left(-\rho\cos\frac{2}{3}\rho-q_2\left(\frac{1}{3}\right)\sin\frac{2}{3}\rho-\frac{1}{2}\int_{\frac{1}{3}}^1 q(t)\sin\left(\frac{4}{3}-2t\right)dt+O\left(\frac{e^{\frac{2}{3}|\tau|}}{|\rho|}\right)\right)= \\ =C_1\rho^2m\left(\sin\frac{1}{3}\rho-\frac{q_1\left(\frac{1}{3}\right)}{\rho}\cos\frac{1}{3}\rho+\frac{1}{2\rho}\int_0^{\frac{1}{3}}\cos\rho\left(\frac{1}{3}-2t\right)q(t)\,dt+O\left(\frac{e^{\frac{1}{3}|\tau|}}{|\rho|^2}\right)\right) \end{cases}$$

To define eigenvalues we obtain following equation

$$\Delta(\lambda)=\begin{vmatrix} a_{11}(\rho) & a_{12}(\rho) \\ a_{21}(\rho) & a_{22}(\rho) \end{vmatrix}=0,$$

here

$$\begin{array}{l} a_{11}(\rho)=\sin\frac{1}{3}\rho-\frac{1}{\rho}q_1\left(\frac{1}{3}\right)\cos\frac{1}{3}\rho+\frac{1}{2\rho}\int_0^{\frac{1}{3}}\cos\rho\left(\frac{1}{3}-2t\right)q(t)\,dt+O\left(\frac{e^{\frac{1}{3}|\tau|}}{|\rho|^2}\right) \\ a_{12}(\rho)=-\sin\frac{2}{3}\rho+\frac{1}{\rho}q_2\left(\frac{1}{3}\right)\cos\frac{2}{3}\rho-\frac{1}{2\rho}\int_{\frac{1}{3}}^1\cos\rho\left(2t-\frac{4}{3}\right)q(t)\,dt-O\left(\frac{e^{\frac{2}{3}|\tau|}}{|\rho|^2}\right) \\ a_{21}(\rho)=\left(\rho\cos\frac{1}{3}\rho-\rho^2m\sin\frac{1}{3}\rho\right)+\left(q_1\left(\frac{1}{3}\right)\sin\frac{1}{3}\rho+\rho mq_1\left(\frac{1}{3}\right)\cos\frac{1}{3}\rho\right)+ \\ +\left(\frac{1}{2}\int_0^{\frac{1}{3}}\sin\rho\left(2t-\frac{1}{3}\right)q(t)\,dt-\frac{\rho m}{2}\int_0^{\frac{1}{3}}\cos\rho\left(\frac{1}{3}-2t\right)q(t)\,dt\right)+O\left(e^{\frac{1}{3}|\tau|}\right) \\ a_{22}(\rho)=\rho\cos\frac{2}{3}\rho+q_2\left(\frac{1}{3}\right)\sin\frac{2}{3}\rho+\frac{1}{2}\int_{\frac{1}{3}}^1 q(t)\sin\left(\frac{4}{3}-2t\right)dt-O\left(\frac{e^{\frac{2}{3}|\tau|}}{|\rho|}\right) \end{array}$$

Using that, for any complex number z .

$$|\sin z|\leq e^{|Imz|}$$

and

$$|\cos z| \leq e^{|Imz|},$$

we can write

$$\begin{array}{ll} \left|\cos\rho\left(\frac{1}{3}-2t\right)\right| \leq e^{\frac{1}{3}|\tau|}, & for \quad 0 \leq t \leq \frac{1}{3}, \\ \left|\cos\rho\left(2t-\frac{4}{3}\right)\right| \leq e^{\frac{4}{3}|\tau|}, & for \quad \frac{1}{3} \leq t \leq 1, \\ \left|\sin\rho\left(2t-\frac{1}{3}\right)\right| \leq e^{\frac{1}{3}|\tau|}, & for \quad 0 \leq t \leq \frac{1}{3}, \\ \left|\sin\rho\left(\frac{4}{3}-2t\right)\right| \leq e^{\frac{4}{3}|\tau|}, & for \quad \frac{1}{3} \leq t \leq 1. \end{array}$$

Taking into account the last inequalities, for $|\rho| \to \infty$ we obtain:

$$\begin{array}{l} \int_0^{\frac{1}{3}} q(t)\cos\rho\left(\frac{1}{3}-2t\right)dt = O\left(e^{\frac{1}{3}|\tau|}\right), \\ \int_{\frac{1}{3}}^1 q(t)\cos\rho\left(2t-\frac{4}{3}\right)dt = O\left(e^{\frac{4}{3}|\tau|}\right), \\ \int_0^{\frac{1}{3}} q(t)\sin\rho\left(2t-\frac{1}{3}\right)dt = O\left(e^{\frac{1}{3}|\tau|}\right), \\ \int_{\frac{1}{3}}^1 q(t)\sin\left(\frac{4}{3}-2t\right)dt = O\left(e^{\frac{4}{3}|\tau|}\right). \end{array}$$

From the last asymptotic formulas we obtain that $\Delta(\lambda)$ can be written as the form:

$$\Delta(\lambda) = \begin{vmatrix} \sin\frac{1}{3}\rho & -\sin\frac{2}{3}\rho \\ \rho\cos\frac{1}{3}\rho - \rho^2 m\sin\frac{1}{3}\rho & \rho\cos\frac{2}{3}\rho \end{vmatrix} +$$

$$+\begin{vmatrix} \sin\frac{1}{3}\rho & \frac{1}{\rho}q_2\left(\frac{1}{3}\right)\cos\frac{2}{3}\rho \\ \rho\cos\frac{1}{3}\rho - \rho^2 m\sin\frac{1}{3}\rho & q_2\left(\frac{1}{3}\right)\sin\frac{2}{3}\rho \end{vmatrix} +$$

$$+\begin{vmatrix} \sin\frac{1}{3}\rho & -\frac{1}{2\rho}\int_{\frac{1}{3}}^1 q(t)\cos\rho\left(2t-\frac{4}{3}\right)dt \\ \rho\cos\frac{1}{3}\rho - \rho^2 m\sin\frac{1}{3}\rho & \frac{1}{2}\int_{\frac{1}{3}}^1 q(t)\sin\left(\frac{4}{3}-2t\right)dt \end{vmatrix} +$$

$$+\begin{vmatrix} \sin\frac{1}{3}\rho & -O\left(\frac{e^{\frac{2}{3}|\tau|}}{|\rho|^2}\right) \\ \rho\cos\frac{1}{3}\rho - \rho^2 m\sin\frac{1}{3}\rho & O\left(\frac{e^{\frac{2}{3}|\tau|}}{|\rho|}\right) \end{vmatrix} +$$

$$+\begin{vmatrix} -\frac{q_1\left(\frac{1}{3}\right)}{\rho}\cos\frac{1}{3}\rho & -\sin\frac{2}{3}\rho \\ q_1\left(\frac{1}{3}\right)\sin\frac{1}{3} + \rho m q_1\left(\frac{1}{3}\right)\cos\frac{1}{3}\rho & \rho\cos\frac{2}{3}\rho \end{vmatrix} +$$

$$+\begin{vmatrix} -\frac{q_1\left(\frac{1}{3}\right)}{\rho}\cos\frac{1}{3}\rho & \frac{1}{\rho}q_2\left(\frac{1}{3}\right)\cos\frac{2}{3}\rho \\ q_1\left(\frac{1}{3}\right)\sin\frac{1}{3} + \rho m q_1\left(\frac{1}{3}\right)\cos\frac{1}{3}\rho & q_2\left(\frac{1}{3}\right)\sin\frac{2}{3}\rho \end{vmatrix} +$$

$$+\begin{vmatrix} -\frac{q_1\left(\frac{1}{3}\right)}{\rho}\cos\frac{1}{3}\rho & -\frac{1}{2\rho}\int_{\frac{1}{3}}^1 q(t)\cos\rho\left(2t-\frac{4}{3}\right)dt \\ q_1\left(\frac{1}{3}\right)\sin\frac{1}{3} + \rho m q_1\left(\frac{1}{3}\right)\cos\frac{1}{3}\rho & \frac{1}{2}\int_{\frac{1}{3}}^1 q(t)\sin\left(\frac{4}{3}-2t\right)dt \end{vmatrix} +$$

$$+\begin{vmatrix} -\frac{q_1\left(\frac{1}{3}\right)}{\rho}\cos\frac{1}{3}\rho & -O\left(\frac{e^{\frac{2}{3}|\tau|}}{|\rho|^2}\right) \\ q_1\left(\frac{1}{3}\right)\sin\frac{1}{3} + \rho m q_1\left(\frac{1}{3}\right)\cos\frac{1}{3}\rho & O\left(\frac{e^{\frac{2}{3}|\tau|}}{|\rho|}\right) \end{vmatrix} +$$

$$+\begin{vmatrix} \frac{1}{2\rho}\int_0^{\frac{1}{3}} q(t)\cos\rho\left(\frac{1}{3}-2t\right)dt & -\sin\frac{2}{3}\rho \\ \frac{1}{2}\int_0^{\frac{1}{3}1} q(t)\sin\rho\left(2t-\frac{1}{3}\right)dt-\frac{\rho m}{2}\int_0^{\frac{1}{3}} q(t)\cos\rho\left(\frac{1}{3}-2t\right)dt & \rho\cos\frac{2}{3}\rho \end{vmatrix}+$$

$$+\begin{vmatrix} \frac{1}{2\rho}\int_0^{\frac{1}{3}} q(t)\cos\rho\left(\frac{1}{3}-2t\right)dt & \frac{1}{\rho}q_2\left(\frac{1}{3}\right)\cos\frac{2}{3}\rho \\ \frac{1}{2}\int_0^{\frac{1}{3}1} q(t)\sin\rho\left(2t-\frac{1}{3}\right)dt-\frac{\rho m}{2}\int_0^{\frac{1}{3}} q(t)\cos\rho\left(\frac{1}{3}-2t\right)dt & q_2\left(\frac{1}{3}\right)\sin\frac{2}{3}\rho \end{vmatrix}+$$

$$+\begin{vmatrix} O\left(\frac{e^{\frac{1}{3}|\tau|}}{|\rho|^2}\right) & -\sin\frac{2}{3}\rho \\ O\left(e^{\frac{1}{3}|\tau|}\right) & \rho\cos\frac{2}{3}\rho \end{vmatrix}+O\left(\frac{e^{|\tau|}}{|\rho|}\right)$$

Opening all determinants in the last equality, we obtain the following for the function $\Delta(\lambda)$:

$$\Delta(\lambda)=\cos^3\frac{1}{3}\rho\left(2\rho^2 m-4q_2-4q_1-2mq_1q_2\right)+$$
$$+\sin^3\frac{1}{3}\left(-4\rho-2\rho mq_2-2\rho mq_1+\frac{4}{\rho}q_1q_2\right)+$$
$$+\sin\frac{1}{3}\rho\left(3\rho+\rho mq_2+2\rho mq_1-\frac{3}{\rho}q_1q_2\right)+$$
$$+\cos\tfrac{1}{3}\rho\left(-2\rho^2 m+3q_2+3q_1+mq_1q_2\right)+\sin\tfrac{1}{3}\rho\times$$
$$\times\left(\begin{array}{l} \frac{1}{2}\int_{\frac{1}{3}}^{1} q(t)\sin\left(\frac{4}{3}-2t\right)dt-\frac{\rho m}{2}\int_{\frac{1}{3}}^{1} q(t)\cos\rho\left(2t-\frac{4}{3}\right)dt+O\left(\frac{e^{\frac{2}{3}|\tau|}}{|\rho|}\right)- \\ -mO\left(e^{\frac{2}{3}|\tau|}\right)+\frac{1}{2\rho}q_1\int_{\frac{1}{3}}^{1} q(t)\cos\rho\left(2t-\frac{4}{3}\right)dt+q_1O\left(\frac{e^{\frac{2}{3}|\tau|}}{|\rho|^2}\right) \end{array}\right)+$$
$$+\cos\frac{1}{3}\rho\left(\begin{array}{l} \frac{1}{2}\int_{\frac{1}{3}}^{1} q(t)\cos\rho\left(2t-\frac{4}{3}\right)dt+O\left(\frac{e^{\frac{2}{3}|\tau|}}{|\rho|}\right)-\frac{1}{2\rho}q_1\int_{\frac{1}{3}}^{1} q(t)\sin\rho\left(\frac{4}{3}-2t\right)dt+ \\ +\frac{m}{2}q_1\int_{\frac{1}{3}}^{1} q(t)\cos\rho\left(2t-\frac{4}{3}\right)dt-q_1O\left(\frac{e^{\frac{2}{3}|\tau|}}{|\rho|^2}\right)+mq_1O\left(\frac{e^{\frac{2}{3}|\tau|}}{|\rho|}\right) \end{array}\right)+$$
$$+\sin\frac{1}{3}\rho\cos\frac{1}{3}\rho\left(\begin{array}{l} \int_0^{\frac{1}{3}} q(t)\sin\rho\left(2t-\frac{1}{3}\right)dt-\rho m\int_0^{\frac{1}{3}} q(t)\cos\rho\left(\frac{1}{3}-2t\right)dt+ \\ +\frac{1}{\rho}q_2\left(\frac{1}{3}\right)\int_0^{\frac{1}{3}} q(t)\cos\rho\left(\frac{1}{3}-2t\right)dt \end{array}\right)+$$
$$+\cos\frac{2}{3}\rho\left(\begin{array}{l} \frac{1}{2}\int_0^{\frac{1}{3}} q(t)\cos\rho\left(\frac{1}{3}-2t\right)dt-\frac{1}{2\rho}q_2\left(\frac{1}{3}\right)\int_0^{\frac{1}{3}} q(t)\sin\rho\left(2t-\frac{1}{3}\right)dt+ \\ +\frac{m}{2}q_2\left(\frac{1}{3}\right)\int_0^{\frac{1}{3}} q(t)\cos\rho\left(\frac{1}{3}-2t\right)dt+O\left(\frac{e^{\frac{2}{3}|\tau|}}{|\rho|}\right) \end{array}\right)+$$
$$+O\left(\frac{e^{|\tau|}}{|\rho|}\right). \tag{17}$$

Circle the points $\tilde{\rho}_k=3\pi k,\ k=1,2,.$ by the circles with radius $\frac{\pi}{4}$. Out of these circles the inequality

$$|\delta(\rho)|\geq C|\rho|^2 e^{2|\tau|}$$

holds for the function

$$\delta(\rho) = \rho \sin \frac{\rho}{3} \left(-\rho m \sin \frac{2\rho}{3} + 2 \cos \frac{2\rho}{3} + 1 \right)$$

here $C > 0$is a constant. Since modules of remained summands of the right-hand side of equality(17) don't exceed $A|\rho| e^{2|\tau|}$ (here $A > 0$ is a constant), then by Rouchet theorem for sufficiently large k function $\Delta(\lambda)$ possesses exactly three zeroes multiplicity taking into account in $|Im\rho| \le h$, here h is a positive constant.
Since, all zeroes of $\Delta(\rho^2)$ belong to strip $|Im\rho| \le h$ in a sequel assume that ρ runs only in this strip. Under this assumptation the following asymptotic equalities are true for $|\rho| \to +\infty$:

$$\left.\begin{array}{l} O\left(\frac{e^{|\tau|}}{\rho}\right) = O\left(\frac{e^{\frac{2}{3}|\tau|}}{\rho}\right) = O\left(\frac{e^{\frac{2}{3}|\tau|}}{\rho}\right) = O\left(\frac{1,}{\rho}\right) \\ O\left(\frac{e^{\frac{2}{3}|\tau|}}{\rho^2}\right) = O\left(\frac{1}{\rho^2}\right), \\ O\left(e^{|\tau|}\right) = O(1) \end{array}\right\} \tag{18}$$

In the other hand, in the strip $|Im\rho| \le h$

$$\begin{array}{l} \int_0^{\frac{1}{3}} q(t) \cos\rho\left(\frac{1}{3} - 2t\right) dt = \\ = \int_{\frac{1}{3}}^1 q(t) \cos\rho\left(2t - \frac{4}{3}\right) dt = \\ \int_0^{\frac{1}{3}} q(t) \sin\rho\left(2t - \frac{1}{3}\right) dt = \\ \int_{\frac{1}{3}}^1 q(t) \sin\left(\frac{4}{3} - 2t\right) dt = o(1) \end{array} \tag{19}$$

for $|\rho| \to +\infty$.

Theorem is proved.

Now lets pass to study the asymptotic behavior of eigenfunctions of the problem (1)-(2).

Theorem 2. *Let the function $q(x)$ satisfies the conditions of the Theorem 1. Then the eigenfunctions $y_{1,n}(x)$ corresponding to eigenvalues $\lambda_{1,n} = (\rho_{1,n})^2$, the eigenfunctions $y_{2,n}(x)$corresponding to eigenvalues $\lambda_{2,n} = (\rho_{2,n})^2$ and the eigenfunctions $y_{3,n}(x)$ corresponding to eigenvalues $\lambda_{3,n} = (\rho_{3,n})^2$ satisfies the following asymptotic equalities:*

$$y_{1,n}(x) = \left\{ \begin{array}{l} \sin 3\pi nx + O\left(\frac{1}{n}\right), \ x \in \left[0, \frac{1}{3}\right], \\ \gamma_{1,n} \sin 3\pi nx + O\left(\frac{1}{n}\right), \ x \in \left[\frac{1}{3}, 1\right], \end{array} \right.$$

$$y_{2,n}(x) = \left\{ \begin{array}{l} \sin 3\pi nx + O\left(\frac{1}{n}\right), \ x \in \left[0, \frac{1}{3}\right], \\ \gamma_{2,n} \sin 3\pi nx + O\left(\frac{1}{n}\right), \ x \in \left[\frac{1}{3}, 1\right], \end{array} \right.$$

$$y_{3,n}(x) = \left\{ \begin{array}{l} O\left(\frac{1}{n}\right), \ x \in \left[0, \frac{1}{3}\right], \\ m \cos 3\pi(n + \frac{1}{2})x + O\left(\frac{1}{n}\right), \ x \in \left[\frac{1}{3}, 1\right]. \end{array} \right.$$

Proof. From the asymptotic equalities obtained for $\rho_{1,n}$,$\rho_{2,n}$ and $\rho_{3,n}$ and asymptotic expression for $A_{22}(\rho)$ for the sufficiently large n we have

$$a_{22}(\rho_{1,n}) \neq 0, \quad a_{22}(\rho_{2,n}) \neq 0 \quad and \quad a_{22}(\rho_{3,n}) \neq 0.$$

Hence, for the sufficiently large n the eigenfunction of the problem (1)-(2) corresponding to eigenvalue $\lambda_{1,n} = (\rho_{1,n})^2$ will be

$$y_{1,n}(x) = \begin{cases} \frac{1}{\rho_{1,n}} a_{22}(\rho_{1,n})\, y_1(x, \lambda_{1,n}), & for \quad x \in \left[0, \frac{1}{3}\right], \\ -\frac{1}{\rho_{1,n}} a_{21}(\rho_{1,n})\, y_2(x, \lambda_{1,n}), & for \quad x \in \left[\frac{1}{3}, 1\right], \end{cases}$$

and the eigenfunction corresponding to eigenvalue $\lambda_{2,n} = (\rho_{2,n})^2$ and $\lambda_{3,n} = (\rho_{3,n})^2$ will be

$$y_{2,n}(x) = \begin{cases} \frac{1}{\rho_{2,n}} a_{22}(\rho_{2,n})\, y_1(x, \lambda_{2,n}), & for \quad x \in \left[0, \frac{1}{3}\right], \\ -\frac{1}{\rho_{2,n}} a_{21}(\rho_{2,n})\, y_2(x, \lambda_{2n}), & for \quad x \in \left[\frac{1}{3}, 1\right], \end{cases}$$

$$y_{3,n}(x) = \begin{cases} \frac{1}{\rho_{3n}} a_{22}(\rho_{3,n})\, y_1(x, \lambda_{3,n}), & for \quad x \in \left[0, \frac{1}{3}\right], \\ -\frac{1}{\rho_{3,n}} a_{21}(\rho_{3,n})\, y_2(x, \lambda_{3n}), & for \quad x \in \left[\frac{1}{3}, 1\right]. \end{cases}$$

Let $x \in \left[0, \frac{1}{3}\right]$. Since,

$$\begin{array}{l} \cos z = 1 + O\left(z^2\right), \quad z \to 0, \\ \sin z = z + O\left(z^3 = O(z)\right), \quad z \to 0. \end{array}.$$

Then we have:

$$\begin{array}{l} \frac{1}{\rho_{1,n}} a_{22}(\rho_{1,n}) = \cos\frac{2}{3}\left(3\pi n + \frac{\alpha_1}{n} + o\left(\frac{1}{n}\right)\right) + \\ +O\left(\frac{1}{n}\right) = \cos\left(2\pi n + \frac{2\alpha_1}{3n} + o\left(\frac{1}{n}\right)\right) + O\left(\frac{1}{n}\right) = \\ = 1 + O\left(\frac{1}{n}\right) \end{array}$$

$$y_1(x, \lambda) = \sin\rho_{1,n}x + O\left(\frac{1}{n}\right),$$

$$\begin{array}{l} \sin\rho_{1,n}x = \sin\left(3\pi n + \frac{\alpha_1}{n} + o\left(\frac{1}{n}\right)\right)x = \\ = \sin\left(3\pi n x + O\left(\frac{1}{n}\right)\right) = \sin 3\pi n x + O\left(\frac{1}{n}\right), \end{array}$$

$$-\frac{1}{\rho_{1,n}} a_{21}(\rho_{1,n}) = -\cos\frac{1}{3}\rho_{1,n} + \rho_{1,n} m \sin\frac{1}{3}\rho_{1n} -$$

$$-mq_1\cos\frac{1}{3}\rho_{1,n} + o(1) = -\left(1 + mq_1\left(\frac{1}{3}\right)\right) \times$$

$$\times \cos\frac{1}{3}\left(3\pi n + \frac{\alpha_1}{n} + o\left(\frac{1}{n}\right)\right) +$$

$$+m\left(3\pi n + \frac{\alpha_1}{n} + o\left(\frac{1}{n}\right)\right)\sin\frac{1}{3}\left(3\pi n + \frac{\alpha_1}{n} + o\left(\frac{1}{n}\right)\right) + o(1) =$$

$$= -\left(1 + mq_1\left(\frac{1}{3}\right)\right)\cos\left(\pi n + \frac{\alpha_1}{3n} + o\left(\frac{1}{n}\right)\right) +$$

$$+m\left(3\pi n+\frac{\alpha_1}{n}+o\left(\frac{1}{n}\right)\right)\sin\left(\pi n+\frac{\alpha_1}{3n}+o\left(\frac{1}{n}\right)\right)=$$

$$=(-1)^{n+1}\,m\left(3\pi n+\frac{\alpha_1}{n}+o\left(\frac{1}{n}\right)\right)\left(\frac{\alpha_1}{n}+o\left(\frac{1}{n}+O\left(\frac{1}{n^3}\right)\right)\right)+o\left(1\right)=$$

$$=(-1)^{n+1}\left(1+mq_1\left(\frac{1}{3}\right)-m\alpha_1\pi\right)+o\left(1\right).$$

By the same way as for $y_1\left(x,\lambda_{1,n}\right)$ we can prove, that
$y_2\left(x,\lambda_{1,n}\right)=(-1)^{n+1}\sin 3\pi nx+O\left(\frac{1}{n}\right)$ $y_2\left(x,\lambda_{1,n}\right)=(-1)^{n+1}\sin 3\pi nx+O\left(\frac{1}{n}\right).$
Finally for $x\in\left[\frac{1}{3},1\right]$ we have

$$y_{1,n}\left(x\right)=\gamma_{1,n}\sin 3\pi nx+O\left(\frac{1}{n}\right),$$

here

$$\gamma_{1,n}=\left(1+mq_1\left(\frac{1}{3}\right)-m\alpha_1\pi\right)+o\left(1\right).$$

The following asymptotic equality for the eigenfunction $y_{2,n}\left(x\right)$ proves analogously:

$$y_{2,n}\left(x\right)=\begin{cases}\sin 3\pi nx+O\left(\frac{1}{n}\right), \ x\in\left[0,\frac{1}{3}\right],\\ \gamma_{2,n}\sin 3\pi nx+O\left(\frac{1}{n}\right), \ x\in\left[\frac{1}{3},1\right],\end{cases}$$

here

$$\gamma_{2,n}=1+mq_1\left(\frac{1}{3}\right)-m\alpha_2\pi+o\left(1\right).$$

Now we derive formulae for $y_{3,n}\left(x\right)$. At first let $x\in\left[0,\frac{1}{3}\right]$. In this case we obtain

$$\frac{1}{\rho_{3,n}^2}a_{22}\left(\rho_{3,n}\right)=\frac{1}{3\pi\left(n+\frac{1}{2}\right)+O\left(\frac{1}{n}\right)}\times$$

$$\times\cos\frac{2}{3}\left(3\pi\left(n+\frac{1}{2}\right)+\frac{\alpha_3}{n}+o\left(\frac{1}{n}\right)\right)+O\left(\frac{1}{n}\right)=O\left(\frac{1}{n}\right)$$

Consequently, for $x\in\left[0,\frac{1}{3}\right]$ we obtain

$$y_{3n}\left(x\right)=O\left(\frac{1}{n}\right).$$

Now let $x\in\left[\frac{1}{3},1\right]$. In this case we obtain

$$-\frac{1}{\rho_{3,n}^2}a_{21}\left(\rho_{1,n}\right)=-\frac{1}{\rho_{3,n}}\cos\frac{1}{3}\rho_{3,n}+$$

$$+m\sin\frac{1}{3}\rho_{3,n}-\frac{q_1}{\rho_{3,n}}\sin\frac{1}{3}\rho_{3,n}-$$

$$-\frac{mq_1}{\rho_{3,n}^2}\cos\frac{1}{3}\rho_{3,n}+o\left(1\right)=$$

$$=m\sin\left(\pi n+\frac{\pi}{2}+O\left(\frac{1}{n}\right)\right)+O\left(\frac{1}{n}\right)=$$

$$=-m\cos\left(\pi n+O\left(\frac{1}{n}\right)\right)+O\left(\frac{1}{n}\right)=m\left(-1\right)^{n+1}+O\left(\frac{1}{n}\right).$$

$$\begin{array}{l}\sin\rho_{3n}(1-x)=\sin\left(3\pi\left(n+\frac{1}{2}\right)+O\left(\frac{1}{n}\right)\right)(1-x)=\\ =(-1)^{n+1}\cos\left(3\pi\left(n+\frac{1}{2}\right)\right)x+O\left(\frac{1}{n}\right).\end{array}$$

Consequently, for $x\in\left[\frac{1}{3},1\right]$we obtain

$$y_{3,n}\left(x\right)=\left(m\left(-1\right)^{n+1}+O\left(\frac{1}{n}\right)\right)\cdot\left(\left(-1\right)^{n+1}\cos\left(3\pi\left(n+\frac{1}{2}\right)x+O\left(\frac{1}{n}\right)\right)\right)=$$

$$=m\cos 3\pi\left(n+\frac{1}{2}\right)x+O\left(\frac{1}{n}\right).$$

Thus,

$$y_{3,n}\left(x\right)=\left\{\begin{array}{l}O\left(\frac{1}{n}\right),\quad x\in\left[0,\frac{1}{3}\right],\\ m\cos 3\pi\left(n+\frac{1}{2}\right)x+O\left(\frac{1}{n}\right),\ x\in\left[\frac{1}{3},1\right].\end{array}\right.$$

Theorem is proved.

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Guler V. Maharramova
Institute of Mathematics and Mechanics of NAS of Azerbaijan, Baku, Azerbaijan
Email: g.meherremova.89@mail.ru

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