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On the Properties of Operator Generated by the Direct Value of the Derivative of Simple Layer Logarithmic Potential

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Abstract. The existence of the derivative of simple layer logarithmic potential is shown and some properties of the operator generated by the derivative of simple layer logarithmic potential are studied in generalized Hölder spaces.

Key Words and Phrases: Lyapunov curve, derivative of simple layer logarithmic potential, curvilinear singular integral, generalized Hölder spaces.

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1. Introduction

As is known (see [1]), the boundary value problems for vector Laplace equations are reduced to a singular integral equation which depends on the derivative of simple layer logarithmic potential

$$V(x) = \int_L \overrightarrow{grad_x} \Phi(x,\, y)\, \rho(y) dL_y, \quad x = (x_1,\, x_2) \in L, \tag{1}$$

where $L \subset R^2$ is a simple closed Lyapunov curve with the index $0 < \alpha \leq 1$, $\rho(y)$ is a continuous function on the curve L, $\Phi(x, y)$ is a fundamental solution of the Laplace equation $\Delta u = 0$, i.e.

$$\Phi(x,\, y) = \frac{1}{2\pi} \ln \frac{1}{|x-y|}, x, y \in R^2, x \neq y,$$

and Δ is a Laplace operator.

Counterexamples provided by Lyapunov show (see [2]) that the derivatives for the simple and double layer potentials with continuous density do not exist in general. It should be noted that in [3], the boundedness of the operator generated by the direct value of the derivative of simple layer acoustic potential was proved in generalized Hölder spaces, and in [4], the acceptable formula for the calculation of derivative of the double layer acoustic potential was obtained and the basic properties of the operator generated by the derivative of double layer acoustic potential were studied in generalized Hölder spaces. Besides, based on these results, the approximate solutions of integral equations of boundary value problems for the Helmholtz equation were studied in [5, 6, 7, 8]. However, some basic properties of the operator $(A\rho)(x) = V(x),\ x \in L$ in generalized Hölder spaces have not been studied yet. This work is just dedicated to this matter.

2. Main Results

We denote by $C(L)$ a space of all continuous functions on L with the norm $\|\rho\|_\infty = \max_{x\in L} |\rho(x)|$, and we introduce a modulus of continuity of the form

$$\omega(\varphi,\delta) = \delta \sup_{\tau\geq\delta} \frac{\bar{\omega}(\varphi,\tau)}{\tau}, \quad \delta > 0,$$

for the function $\varphi(x) \in C(L)$, where $\bar{\omega}(\varphi,\tau) = \max_{\substack{|x-y|\leq\tau \\ x,y\in L}} |\varphi(x) - \varphi(y)|$.

Theorem 1. *Let L be a simple closed Lyapunov curve with the index $0 < \alpha \leq 1$ and*

$$\int_0^{diam\,L} \frac{\omega(\rho,t)}{t} dt < +\infty.$$

Then the integral (1) exists in the sense of the Cauchy principal value, with

$$\sup_{x\in L} |V(x)| \leq M^* \left(\|\rho\|_\infty + \int_0^{diam\,L} \frac{\omega(\rho,t)}{t} dt \right).$$

Proof. Let $V(x) = (V_1(x), V_2(x))$, where

$$V_m(x) = \int_L \frac{\partial \Phi(x,y)}{\partial x_m} \rho(y)\, dL_y, \quad x = (x_1, x_2) \in L \quad (m = 1, 2).$$

Simple calculation yields

$$V_m(x) = \frac{1}{2\pi} \int_L \frac{y_m - x_m}{|x-y|^2} \rho(y)\, dL_y.$$

Let $d > 0$ be a radius of a standard circle for L (see [9]), and $\vec{n}(x)$ be an outer unit normal at the point $x \in L$. Then, for every point $x \in L$, the neighborhood $L_d(x) = \{y \in L : |y - x| < d\}$ either intersects the line parallel to the normal $\vec{n}(x)$ at one point

*Hereinafter M denotes a positive constant which can be different in different inequalities.

only or does not intersect it at all, i.e. the set $L_d(x)$ is uniquely projected onto the interval $\Omega_d(x)$ lying on the line $\Gamma(x)$ tangent to L at the point x. On some part of $L_d(x)$, we choose a local rectangular coordinate system (u,v) centered at the point x, where the axis v is directed along the normal $\vec{n}(x)$, and the axis u is directed in the positive direction of the tangent line $\Gamma(x)$. It is known that the coordinates of the point x are (0, 0). Besides, in this coordinate system the neighborhood $L_d(x)$ can be given by the equation $v = f(u)$, $u \in \Omega_d(x)$, where $f \in H_{1,\alpha}(\Omega_d(x))$ and $f(0) = 0$, $f'(0) = 0$. Here $H_{1,\alpha}(\Omega_d(x))$ denotes the linear space of all continuously differentiable functions f on $\Omega_d(x)$, which satisfy the condition

$$\left|f'(u_1) - f'(u_2)\right| \le M_f \left|u_1 - u_2\right|^{\alpha}, \forall u_1, u_2 \in \Omega_d(x),$$

where M_f is a positive constant depending on f, but not on u_1 and u_2. Let $\Gamma_d(x)$ be a part of the tangent line $\Gamma(x)$ at the point $x \in L$ lying inside a circle of radius d centered at x. Besides, let $\tilde{y} \in \Gamma(x)$ be a projection of the point $y \in L_d(x)$. Then (see [10])

$$|x - \tilde{y}| \le |x - y| \le C_1 \, |x - \tilde{y}| \,, \;\; mesL_d(x) \le C_2 \, mes\Gamma_d(x),$$

where C_1 and C_2 are positive constants depending only on L, and $mesL_d(x)$ denotes the length of the curve $L_d(x)$.

Obviously,

$$\begin{aligned}\int\limits_{L} \frac{y_m - x_m}{|x-y|^2}\,\rho\,(y)\;dL_y &= \int\limits_{L\backslash L_d(x)} \frac{y_m - x_m}{|x-y|^2}\rho\,(y)\;dL_y + \\ &+ \int\limits_{L_d(x)} \frac{y_m - x_m}{|x-y|^2}\;(\rho(y) - \rho(x))\;dL_y + \\ &+\rho\,(x) \int\limits_{L_d(x)} \frac{y_m - x_m}{|x-y|^2}\,dL_y\,, \;\; x \in L \quad (m = 1,\,2)\,.\end{aligned} \tag{2}$$

As we can see, the first integral on the right-hand side of the last equality exists as a proper integral, while the second one converges as an improper integral, with

$$\left|\int\limits_{L/\,L_d(x)} \frac{y_m - x_m}{|x-y|^2}\,\rho\,(y)\;dL_y\right| \le M\,\|\rho\|_{\infty}\,, \;\; \forall x \in L \qquad (m = 1,\,2) \tag{3}$$

and

$$\begin{aligned}&\left|\int\limits_{L_d(x)} \frac{y_m - x_m}{|x-y|^2}\;(\rho(y) - \rho(x))\;dL_y\right| \le \\ &\le M \int\limits_{0}^{diamL} \frac{\omega\,(\rho,\,t)}{t}\,dt < +\infty, \;\; \forall x \in L \;\; (m = 1,\,2)\,.\end{aligned} \tag{4}$$

It remains to prove that the integral

$$\int\limits_{L_d(x)} \frac{y_m - x_m}{|x-y|^2}\, dL_y \quad (m = 1,\, 2)$$

exists in the sense of the Cauchy principal value. Let $d_0 = d/C_1$. It is clear that $(-d_0, d_0) \subset \Omega_d(x)$. Using the calculation formula for curvilinear integral, we obtain

$$\int\limits_{L_d(x)} \frac{y_1 - x_1}{|x-y|^2}\, dL_y = \int\limits_{\Omega_d(x)\backslash(-d_0,d_0)} \frac{u\sqrt{1+(f'(u))^2}}{u^2+(f(u))^2}\, du + \int\limits_{-d_0}^{d_0} \frac{du}{u} +$$

$$+ \int\limits_{-d_0}^{d_0} \frac{u\left(\sqrt{1+(f'(u))^2}-1\right)}{u^2+(f(u))^2} du + \int\limits_{-d_0}^{d_0} u\left(\frac{1}{u^2+(f(u))^2} - \frac{1}{u^2}\right) du.$$

Denote the integrals on the right-hand side of the last equality by A_1,A_2, A_3 and A_4, respectively.

As we can see, the integral A_1 exists as a proper integral, while the integral A_2 exists in the sense of the Cauchy principal value and is equal to zero. Besides, taking into account that

$$\left|f'(u)\right| \le M\ |u|^{\,\alpha} \tag{5}$$

(see [9]), we find

$$|A_3| = \left|\int\limits_{-d_0}^{d_0} \frac{u\ (f'(u))^2}{\left(u^2+(f(u))^2\right)\left(1+\sqrt{1+(f'(u))^2}\right)} du\right| \le M \int\limits_{-d_0}^{d_0} |u|^{2\alpha-1}\, du \le M.$$

As

$$|\,f(u)| = |\,f(u)\ - f(0)\,| \le M\ |u|^{\,1+\alpha}\,, \tag{6}$$

we have

$$|\,A_4\,| = \left|\int\limits_{-d_0}^{d_0} \frac{u\ (f(u))^2}{u^2\left(u^2+(f(u))^2\right)}\, du\right| \le M \int\limits_{-d_0}^{d_0} |u|^{2\alpha-1}\, du \le M$$

and

$$\left|\int\limits_{L_d(x)} \frac{y_2 - x_2}{|x-y|^2}\, dL_y\right| = \left|\int\limits_{\Omega_d(x)} \frac{f\,(u)\,\sqrt{1+(f'\,(u))^2}}{u^2+(f(u))^2}\, du\right| \le M \int\limits_{\Omega_d(x)} |u|^{\alpha-1}\, du \le M.$$

So we obtain

$$\left|\rho(x)\int\limits_{L_d(x)}\frac{y_m-x_m}{|x-y|^2}\,dL_y\right|\le M\,\|\rho\|_\infty\,,\quad\forall x\in L\quad(m=1,\,2)\,. \tag{7}$$

Considering the inequalities (3), (4) and (7) in (2), we finish the proof of the theorem.

Now let's show the validity of the Zygmund estimate for the direct value of the derivative of simple layer logarithmic potential.

Theorem 2. *Let L be a simple closed Lyapunov curve with the index $0<\alpha\le 1$ and*

$$\int\limits_0^{diam\,L}\frac{\omega(\rho,\,t)}{t}dt<+\infty.$$

Then for every $m=\overline{1,2}$ and for any two points $x'\,,\;x''\in L$ the following estimates hold:

$$\left|V_m(x')-V_m(x'')\right|\le$$

$$\le M_\rho\left(h^\alpha+\omega(\rho,h)+\int\limits_o^h\frac{\omega(\rho,t)}{t}dt+h\int\limits_h^{diamL}\frac{\omega(\rho,t)}{t^2}dt\right)\quad if\ \ 0<\alpha<1,$$

$$\left|V_m(x')-V_m(x'')\right|\le$$

$$M_\rho\left(h\,|\ln h|+\omega(\rho,h)+\int\limits_o^h\frac{\omega(\rho,t)}{t}dt+h\int\limits_h^{diamL}\frac{\omega(\rho,t)}{t^2}dt\right)\quad if\ \alpha=1,$$

where $h=|\,x'-x''\,|$, and M_ρ is a positive constant depending only on L and ρ.

Proof. Let $0<\alpha<1$ and $m=1$. Consider any two points $x',\;x''\in L$ such that h is sufficiently small. It is not difficult to see that

$$V_1(x')-V_1(x'')=\frac{1}{2\pi}\int\limits_L\left(\frac{(y_1-x_1')\,\left(\rho(y)-\rho(x')\right)}{|x'-y|^2}-\right.$$

$$\left.-\frac{(y_1-x_1'')\,\left(\rho(y)-\rho(x'')\right)}{|x''-y|^2}\right)dL_y+$$

$$+\left(\frac{\rho(x')}{2\pi}\int\limits_L\frac{y_1-x_1'}{|x'-y|^2}\,dL_y-\frac{\rho(x'')}{2\pi}\int\limits_L\frac{y_1-x_1''}{|x''-y|^2}\,dL_y\right).$$

Denote two terms on the right-hand side of the last equality by $F(x',x'')$ and $G(x',x'')$, respectively.

Estimate the integral $F(x',\, x'')$.

$$F(x',\, x'') = \int\limits_{L\backslash L_d(x')} \left(\frac{(y_1 - x_1')\ (\rho\,(y) - \rho\,(x'))}{2\pi\ |x' - y|^2} - \frac{(y_1 - x_1'')\ (\rho\,(y) - \rho\,(x''))}{2\pi\ |x'' - y|^2} \right) dL_y +$$

$$+ \int\limits_{L_{h/2}(x')} \frac{(y_1 - x_1')\ (\rho\,(y) - \rho\,(x'))}{2\pi\ |x' - y|^2} dL_y - \int\limits_{L_{h/2}(x'')} \frac{(y_1 - x_1'')\ (\rho\,(y) - \rho\,(x''))}{2\pi\ |x'' - y|^2} dL_y -$$

$$- \int\limits_{L_{h/2}(x')} \frac{(y_1 - x_1'')\ (\rho\,(y) - \rho\,(x''))}{2\pi\ |x'' - y|^2} dL_y + \int\limits_{L_{h/2}(x'')} \frac{(y_1 - x_1')\ (\rho\,(y) - \rho\,(x'))}{2\pi\ |x' - y|^2} dL_y +$$

$$+ \int\limits_{L_d(x')\backslash\left(L_{h/2}(x') \bigcup L_{h/2}(x'')\right)} \left(y_1 - x_1'\right)\ \left(\rho\,(y) - \rho\,(x')\right)\ \times$$

$$\times \left(\frac{1}{2\pi\ |x' - y|^2} - \frac{1}{2\pi\ |x'' - y|^2} \right) dL_y +$$

$$+ \int\limits_{L_d(x')\backslash\left(L_{h/2}(x') \bigcup L_{h/2}(x'')\right)} \frac{(x_1'' - x_1')\ (\rho\,(y) - \rho\,(x'))}{2\pi\ |x'' - y|^2} dL_y +$$

$$+ \left(\rho\left(x''\right) - \rho\left(x'\right)\right) \int\limits_{L_d(x')\backslash\left(L_{h/2}(x') \bigcup L_{h/2}(x'')\right)} \frac{y_1 - x_1''}{2\pi\ |x'' - y|^2} dL_y.$$

Denote the terms on the right-hand side of the last equality by $F_1(x', x'')$, $F_2(x', x'')$, $F_3(x', x'')$, $F_4(x', x'')$, $F_5(x', x'')$, $F_6(x', x'')$, $F_7(x', x'')$ and $F_8(x', x'')$, respectively.

Obviously, $|F_1(x',\, x'')| \le M\ \|\rho\|_\infty\ h$.

Using the calculation formula for curvilinear integral, we have

$$\left|F_2(x', x'')\right| \le M \int\limits_0^h \frac{\omega\ (\rho, t)}{t} dt,\ \ \left|F_3(x', x'')\right| \le M \int\limits_0^h \frac{\omega\ (\rho, t)}{t} dt.$$

Besides, considering the inequalities

$$h/2 \le \left|y - x''\right| \le 3\,h/2, \quad y \in L_{h/2}(x'),$$

we obtain

$$\left|F_4(x',\, x'')\right| \le M\, \frac{\omega\ (\rho,\, 3h/2)}{h} mes L_{h/2}(x') \le M\,\omega\ (\rho,\, h)\,.$$

Similarly, taking into account the inequality

$$h/2 \le \left|y - x'\right| \le 3\,h/2, \ \ y \in L_{h/2}(x''),$$

we obtain $|F_5(x',\, x'')| \leq M\,\omega\;(\rho,\, h)$.

For every $y \in L_d(x') \backslash (L_{h/2}(x') \bigcup L_{h/2}(x''))$ we have

$$\left|x' - y\right| \leq \left|x' - x''\right| + \left|x'' - y\right| \leq 3\,\left|x'' - y\right|$$

and

$$\left|x'' - y\right| \leq 3\,\left|x' - y\right|,$$

then

$$\left|F_6(x', x'')\right| \leq Mh \int\limits_h^d \frac{\omega\;(\rho, t)}{t^2} dt, \;\; \left|F_7(x', x'')\right| \leq Mh \int\limits_h^d \frac{\omega\;(\rho, t)}{t^2} dt.$$

Let's estimate the term $F_8(x',\, x'')$. To do so, we choose on some part of $L_d(x')$ a local rectangular coordinate system $(u,\, v)$ centered at the point x', where the axis v is directed along the normal $\vec{n}(x')$, and the axis u is directed in the positive direction of the tangent line $\Gamma(x')$. The coordinates of the point x' are $(0,\, 0)$, and the coordinates of the point x'' are denoted by $(u'',\, f(u''))$. Let $h_0 = |u''|$ and $\Omega_{h/2}\,(x', x'')$ denote the projection of the set $L_{h/2}\,(x') \bigcup L_{h/2}\,(x'')$ onto the tangent line $\Gamma(x')$.

By the calculation formula for curvilinear integral, we obtain

$$F_8(x',\, x'') = \frac{\rho\,(x'') - \rho\,(x')}{2\pi} \int\limits_{\Omega_d(x') \backslash \Omega_{h/2}(x',x'')} \frac{u\left(\sqrt{1 + \left(f'\,(u)\right)^2} - 1\right)}{u^2 + \left(f(u)\right)^2}\, du +$$

$$+ \frac{\rho\,(x'') - \rho\,(x')}{2\pi} \int\limits_{\Omega_d(x') \backslash \Omega_{h/2}(x',x'')} \left(\frac{1}{u^2 + \left(f(u)\right)^2} - \frac{1}{u^2}\right) u\, du +$$

$$+ \frac{\rho\,(x'') - \rho\,(x')}{2\pi} \int\limits_{\Omega_d(x') \backslash \Omega_{h/2}(x',x'')} \frac{du}{u}.$$

Taking into account (5), we find

$$\sqrt{1 + \left(f'\,(u)\right)^2} - 1 \leq M\; |u|^{2\alpha}, \;\; \forall u \in \Omega_d(x').$$

Besides, by virtue of (6) we obtain

$$\left| \frac{1}{u^2 + \left(f(u)\right)^2} - \frac{1}{u^2} \right| \leq M\; |u|^{2\alpha - 2}, \;\; \forall u \in \Omega_d(x') \backslash 0.$$

Then

$$\left| \frac{\rho\,(x'') - \rho\,(x')}{2\pi} \int\limits_{\Omega_d(x') \backslash \Omega_{h/2}(x',x'')} \frac{u\left(\sqrt{1 + \left(f'\,(u)\right)^2} - 1\right)}{u^2 + \left(f(u)\right)^2}\, du \right| \leq M\omega\;(\rho, h)$$

and

$$\left|\frac{\rho\left(x''\right)-\rho\left(x'\right)}{2\pi}\int\limits_{\Omega_d(x')\backslash\Omega_{h/2}(x',x'')}\left(\frac{1}{u^2+\left(f(u)\right)^2}-\frac{1}{u^2}\right)u\,du\right|\leq M\omega\ (\rho,h)\,.$$

As

$$\int\limits_{(-d_0,d_0)\,\backslash\,(-2h,2h)}\frac{du}{u}=\int\limits_{-d_0}^{-2h}\frac{du}{u}+\int\limits_{2h}^{d_0}\frac{du}{u}=0,$$

we have

$$\left|\frac{\rho\left(x''\right)-\rho\left(x'\right)}{2\pi}\int\limits_{\Omega_d(x')\backslash\Omega_{h/2}(x',x'')}\frac{du}{u}\right|=$$

$$=\left|\frac{\rho\left(x''\right)-\rho\left(x'\right)}{2\pi}\left(\int\limits_{\Omega_d(x')\backslash(-d_0,d_0)}\frac{du}{u}+\int\limits_{(-2h,2h)\backslash\Omega_{h/2}(x',x'')}\frac{du}{u}\right)\right|\leq$$

$$\leq\frac{\omega\left(\rho,h\right)}{2\pi}\left(M+M\int\limits_{h/C_1}^{2h}\frac{du}{u}\right)\leq M\omega\left(\rho,h\right),$$

and, consequently, $|F_8(x',\, x'')|\leq M\,\omega\,(\rho,h)$.

As a result, summing up the estimates obtained above for $F_j(x',\, x''),\ j=\overline{1,\,8}$, we find:

$$\left|F\left(x',\,x''\right)\right|\leq M\left(\|\rho\|_\infty\,h+\omega\left(\rho,h\right)+\int\limits_o^h\frac{\omega\ (\rho,t)}{t}dt+h\int\limits_h^{diamL}\frac{\omega\left(\rho,t\right)}{t^2}dt\right).$$

Now let's estimate the expression $G(x',\, x'')$. It is clear that

$$G(x',\,x'')=\frac{\rho\left(x'\right)-\rho\left(x''\right)}{2\,\pi}\int\limits_L\frac{y_1-x_1'}{|x'-y|^2}\,dL_y+$$

$$+\frac{\rho\left(x''\right)}{2\pi}\left(\int\limits_{L\backslash L_d(x')}\frac{y_1-x_1'}{|x'-y|^2}\,dL_y-\int\limits_{L\backslash L_d(x')}\frac{y_1-x_1''}{|x''-y|^2}\,dL_y\right)+$$

$$+\frac{\rho\left(x''\right)}{2\pi}\left(\int\limits_{L_d(x')}\frac{y_1-x_1'}{|x'-y|^2}\,dL_y-\int\limits_{L_d(x')}\frac{y_1-x_1''}{|x''-y|^2}\,dL_y\right).$$

Denote the terms on the right-hand side of the last equality by $G_1(x', x'')$, $G_2(x', x'')$ and $G_3(x', x'')$, respectively.

As the integral

$$\int_L \frac{y_1 - x_1'}{|x' - y|^2}\, dL_y$$

converges in the sense of the Cauchy principal value, we have

$$\left|G_1(x',\, x'')\right| \le M\, \omega\,(\rho, h)\,.$$

Besides, it is clear that

$$\left|G_2(x',\, x'')\right| \le M\, \left\|\rho\right\|_\infty h.$$

As is known, the following relations are true in the sense of the Cauchy principal value:

$$\int_{-d_0}^{d_0} \frac{du}{u} = 0 \text{ and } \int_{u''-d_0+h_0}^{u''+d_0-h_0} \frac{du}{u - u''} = 0.$$

Then the term $G_3(x',\, x'')$ can be represented as follows:

$$G_3(x',\, x'') = \frac{\rho(x'')}{2\pi} \left[- \int_{(-d_0,d_0)\setminus(u''-d_0+h_0,u''+d_0-h_0)} \frac{du}{u - u''} + \right.$$

$$+ \int_{\Omega_d(x')\setminus(-d_0,d_0)} \left(\frac{u}{u^2 + (f(u))^2} - \frac{u - u''}{(u - u'')^2 + (f(u) - f(u''))^2} \right) \sqrt{1 + (f'(u))^2} du +$$

$$+ \int_{(-d_0,d_0)\setminus((-h_0/2,h_0/2)\bigcup(u''-h_0/2,u''+h_0/2))} \frac{u'' \left(\sqrt{1 + (f'(u))^2} - 1 \right)}{u^2 + (f(u))^2} du +$$

$$+ \int_{(-d_0,d_0)\setminus((-h_0/2,h_0/2)\bigcup(u''-h_0/2,u''+h_0/2))} \left(\sqrt{1 + (f'(u))^2} - 1 \right) \times$$

$$\times \frac{(u - u'') \left((u - u'')^2 - u^2 + (f(u) - f(u''))^2 - (f(u))^2 \right)}{\left(u^2 + (f(u))^2 \right) \left((u - u'')^2 + (f(u) - f(u''))^2 \right)} du +$$

$$+ \int_{-h_0/2}^{h_0/2} \frac{u \left(\sqrt{1 + (f'(u))^2} - 1 \right)}{u^2 + (f(u))^2} du +$$

$$+\int\limits_{u''-h_0/2}^{u''+h_0/2}\frac{u\left(\sqrt{1+\left(f'(u)\right)^2}-1\right)}{u^2+\left(f(u)\right)^2}du-\int\limits_{-h_0/2}^{h_0/2}\frac{(u-u'')\left(\sqrt{1+\left(f'(u)\right)^2}-1\right)}{(u-u'')^2+\left(f(u)-f(u'')\right)^2}du-$$

$$-\frac{\sqrt{1+\left(f'(u'')\right)^2}-1}{1+\left(f'(u'')\right)^2}\int\limits_{u''-h_0/2}^{u''+h_0/2}\frac{du}{u-u''}-$$

$$-\int\limits_{u''-h_0/2}^{u''+h_0/2}\frac{(u-u'')\left(\sqrt{1+\left(f'(u)\right)^2}-\sqrt{1+\left(f'(u'')\right)^2}\right)}{(u-u'')^2+\left(f(u)-f(u'')\right)^2}du-$$

$$-\left(\sqrt{1+\left(f'(u'')\right)^2}-1\right)\int\limits_{u''-h_0/2}^{u''+h_0/2}\frac{1}{u-u''}\left(\frac{(u-u'')^2}{(u-u'')^2+\left(f(u)-f(u'')\right)^2}-\right.$$

$$\left.-\frac{1}{1+\left(f'(u'')\right)^2}\right)du+\int\limits_{(-d_0,d_0)\setminus(u''-d_0+h_0,u''+d_0-h_0)}\left(u\left(\frac{1}{u^2+\left(f(u)\right)^2}-\frac{1}{u^2}\right)-\right.$$

$$\left.-\left(u-u''\right)\left(\frac{1}{(u-u'')^2+\left(f(u)-f(u'')\right)^2}-\frac{1}{(u-u'')^2}\right)\right)du+$$

$$+\int\limits_{(u''-d_0+h_0,u''+d_0-h_0)\setminus((-h_0/2,h_0/2)\bigcup(u''-h_0/2,u''+h_0/2))}u''\left(\frac{1}{u^2+\left(f(u)\right)^2}-\frac{1}{u^2}\right)du+$$

$$+\int\limits_{(u''-d_0+h_0,u''+d_0-h_0)\setminus((-h_0/2,h_0/2)\bigcup(u''-h_0/2,u''+h_0/2))}\left(\left(\frac{1}{u^2+\left(f(u)\right)^2}-\right.\right.$$

$$\left.\left.-\frac{1}{(u-u'')^2+\left(f(u)-f(u'')\right)^2}\right)+\left(\frac{1}{(u-u'')^2\left(1+\left(f'(u'')\right)^2\right)}-\frac{1}{u^2}\right)\right)\times$$

$$\times\left(u-u''\right)du+\int\limits_{-h_0/2}^{h_0/2}\left(\frac{1}{u^2+\left(f(u)\right)^2}-\frac{1}{u^2}\right)u\,du+$$

$$+\int\limits_{u''-h_0/2}^{u''+h_0/2}\left(\frac{1}{u^2+\left(f(u)\right)^2}-\frac{1}{u^2}\right)u\,du+\int\limits_{-h_0/2}^{h_0/2}\left(\frac{1}{(u-u'')^2\left(1+\left(f'(u'')\right)^2\right)}-\right.$$

$$\left.-\frac{1}{(u-u'')^2+\left(f(u)-f(u'')\right)^2}\right)\left(u-u''\right)du+$$

$$+\int_{u''-h_0/2}^{u''+h_0/2}\left(\frac{1}{(u-u'')^2\left(1+(f'(u''))^2\right)}-\frac{1}{(u-u'')^2+(f(u)-f(u''))^2}\right)(u-u'')\,du\Bigg].$$

As there exists a point $u_* = u'' + \theta\,(u-u'')$ such that

$$f(u)-f(u'') = f'(u_*)\,(u-u''),$$

where $\theta \in (0,\,1)$, it is not difficult to show that

$$\left|G_3(x',\,x'')\right| \le M\ \|\rho\|_\infty\, h^\alpha.$$

Consequently,

$$\left|G(x',\,x'')\right| \le M\ \left(\omega(\rho,\,h) + \|\rho\|_\infty\, h^\alpha\right).$$

Now, taking into account the estimates derived above for $F(x',\,x'')$ and $G(x',\,x'')$, we arrive at the conclusion that if $0<\alpha<1$, then

$$\left|V_1(x')-V_1(x'')\right| \le M_\rho\left(h^\alpha+\omega\,(\rho,h)+\int_o^h\frac{\omega\,(\rho,t)}{t}dt+h\int_h^{diamL}\frac{\omega\,(\rho,t)}{t^2}dt\right).$$

Similarly, it is not difficult to prove that

$$\left|V_2(x')-V_2(x'')\right| \le M_\rho\left(h^\alpha+\omega\,(\rho,h)+\int_o^h\frac{\omega\,(\rho,t)}{t}dt+h\int_h^{diamL}\frac{\omega\,(\rho,t)}{t^2}dt\right).$$

It follows from the proof of the theorem that if $\alpha=1$, then

$$\left|V_m(x')-V_m(x'')\right| \le M_\rho\left(h\,|\ln h|+\omega\,(\rho,h)+\int_o^h\frac{\omega\,(\rho,t)}{t}dt+\right.$$

$$\left.+\,h\int_h^{diamL}\frac{\omega\,(\rho,t)}{t^2}dt\right),\quad m=\overline{1,2}.$$

Theorem is proved.

Theorem 3. *Let L be a simple closed Lyapunov curve with the index $0<\alpha\le 1$ and*

$$\int_0^{diam\,L}\frac{\omega(\rho,\,t)}{t}dt < +\infty.$$

Then the following estimates hold:
$\omega\,(V,h) \le M_\rho\left(h^\alpha+\omega\,(\rho,h)+\int_o^h\frac{\omega\,(\rho,t)}{t}dt+h\int_h^{diamL}\frac{\omega(\rho,t)}{t^2}dt\right)$ *if* $0<\alpha<1$,
$\omega\,(V,h) \le M_\rho\left(h\,|\ln h|+\omega\,(\rho,h)+\int_o^h\frac{\omega(\rho,t)}{t}dt+h\int_h^{diamL}\frac{\omega\,(\rho,t)}{t^2}dt\right)$ *if* $\alpha=1$,
where M_ρ is a positive constant depending only on L and ρ.

Proof. Consider the function

$$\psi(h)=\begin{cases} h^{\alpha}+\omega(\rho,h)+\int_o^h \frac{\omega(\rho,t)}{t}dt+h\int_h^{diamL}\frac{\omega(\rho,t)}{t^2}dt, & if\ 0<\alpha<1,\\ h\,|\ln h|+\omega(\rho,h)+\int_o^h \frac{\omega(\rho,t)}{t}dt+h\int_h^{diamL}\frac{\omega(\rho,t)}{t^2}dt, & if\ \alpha=1. \end{cases}$$

It is not difficult to show that $\lim\limits_{h\to 0}\psi(h)=0$, the function $\psi(h)$ is non-decreasing, and the function $\psi(h)/h$ is non-increasing. Then, using Theorem 2, we finish the proof of the theorem.

Introduce the following classes of functions on $(0, diamL]$:

$$\chi=\left\{\varphi:\quad \varphi\uparrow, \lim_{\delta\to 0}\varphi(\delta)=0, \varphi(\delta)/\delta\downarrow\right\},$$

$$J_0(S)=\left\{\varphi\in\chi:\quad \int\limits_0^{diamL}\frac{\varphi(t)}{t}dt<+\infty\right\}.$$

Also consider the function

$$Z(h,\varphi)=\begin{cases} h^{\alpha}+\varphi(h)+\int\limits_o^h\frac{\varphi(t)}{t}dt+h\int_h^{diamL}\frac{\varphi(t)}{t^2}dt, & if\ \ 0<\alpha<1,\\ h\,|\ln h|+\varphi(h)+\int\limits_o^h\frac{\varphi(t)}{t}dt+h\int\limits_h^{diamL}\frac{\varphi(t)}{t^2}dt, & if\ \ \alpha=1. \end{cases}$$

Where there is no misunderstanding, we will sometimes write $Z(h)$, $Z(\varphi)$ instead of $Z(h,\varphi)$. It is clear that $\lim\limits_{h\to 0}Z(h)=0$, the function $Z(h)$ is non-decreasing, and the function $Z(h)/h$ is non-increasing.

Let $\varphi\in\chi$. Denote by $H(\varphi)$ the linear space of all continuous functions ρ on L which satisfy the condition

$$|\rho(x)-\rho(y)|\le C_\rho\varphi(|x-y|), x,y\in L,$$

where C_ρ is a positive constant depending on L and ρ, but not on x and y. It is known (see [11]) that the space $H(\varphi)$ is a Banach space equipped with the norm

$$\|\rho\|_{H(\varphi)}=\sup_{x\in L}|\rho(x)|+\sup_{\substack{x,y\in L\\ x\neq y}}\frac{|\rho(x)-\rho(y)|}{\varphi(|x-y|)}.$$

Theorem 3 implies

Theorem 4. *Let $\varphi\in J_0(L)$. Then the operator $(A\rho)(x)=V(x)$, $x\in L$, acts boundedly from $H(\varphi)$ to $H(Z(\varphi))$, and*

$$\|V\|_{H(Z(\varphi))}\le M\ \|\rho\|_{H(\varphi)}.$$

Denote by $H_\beta(L)$ the space of all continuous functions f on L which satisfy the Hölder condition

$$|f(x) - f(y)| \le M_f |x-y|^\beta, \forall x, y \in L,$$

where $0 < \beta \le 1$ and M_f is a positive constant depending on f, but not on x and y. It is known (see [11]) that the space $H_\beta(L)$ is a Banach space equipped with the norm

$$\|f\|_\beta = \sup_{x\in L} |f(x)| + \sup_{\substack{x,y\in L \\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|^\beta}.$$

Corollary 1. *Let L be a simple closed Lyapunov curve with the index $0 < \alpha \le 1$ and $\rho \in H_\beta(L)$, $0 < \beta \le 1$. The following assertions are true:*
(a) *if $\alpha < \beta$, then $V \in H_\alpha(L)$ and $\|V\|_\alpha \le M \|\rho\|_\beta$;*
(b) *if $\beta \le \alpha < 1$, then $V \in H_\beta(L)$ and $\|V\|_\beta \le M \|\rho\|_\beta$;*
(c) *if $\alpha = 1$, $\beta < 1$, then $V \in H_\beta(L)$ and $\|V\|_\beta \le M \|\rho\|_\beta$;*
(d) *if $\alpha = 1$, $\beta = 1$, then $V \in H_\gamma(L)$ and $\|V\|_\gamma \le M \|\rho\|_1$, where $\gamma \in (0,1)$.*

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