Caspian Journal of Applied Mathematics, Ecology and Economics
V. 7, No 1, 2019, July
ISSN 1560-4055

The Properties of the Eigenvalues and Eigenfunctions of a Vibration Boundary Value Problem

S.B. Guliyeva

Abstract. In the present paper we consider the eigenvalue problem for ordinary differential equation of fourth order with a spectral parameter contained linearly in the two of boundary conditions. The basic properties of the eigenvalues and eigenfunctions of this spectral problem are investigated.

Key Words and Phrases: spectral problem, eigenvalue, eigenfunction, oscillatory properties of eigenfunctions.

2010 Mathematics Subject Classifications: 34B09, 34B24, 34C23, 34L10, 47B50

1. Introduction

We consider the following boundary-value problem

$$\ell(y)(x) \equiv y^{(4)}(x) - (q(x)y'(x))' = \lambda y(x), \;\; 0 < x < 1, \tag{1}$$

$$y(0) = y'(0) = 0, \tag{2}$$

$$y''(1) - (a_1\lambda + b_1)\, y'(1) = 0 \tag{3}$$

$$Ty(1) - (a_2\lambda + b_2)y(1) = 0, \tag{4}$$

where $\lambda \in \mathbb{C}$ is spectral parameter, $Ty \equiv y''' - qy'$, $q(x)$ is positive and absolutely continuous function on $[0, l]$, a_1, a_2, b_1 and b_2 are real constants such that $a_1 > 0$ and $a_2 < 0$.

The problem (1)-(4) describes small bending vibrations of a homogeneous rod, in cross-sections of which the longitudinal force acts, the left end is fixed rigidly, the right end is fixed elastically and in this end is concentrated the inertial mass (see [6, Ch. 8, § 5]).

The spectral properties of the eigenvalue problem (1)-(4) in the case $b_2 = 0$ was investigated in [4] (see also [3]). In these papers, the oscillation properties of eigenfunctions and their derivatives, the basis properties of the system of eigenfunctions in $L_p(0, 1)$, $1 < p < \infty$, are studied. Similar questions in the case when the spectral parameter is contained in one of the boundary conditions are studied in detail in the papers [1, 2, 5, 7].

Recall that boundary-value problem (1)-(4) reduces to a spectral problem for a self-adjoint operator in the Hilbert space $H = L_2(0,1) \oplus \mathbb{C}^2$ (see [3, 4]). Hence all eigenvalues of this problem are real and simple.

In this paper, we study the general characteristic of the location of eigenvalues on the real axis, the oscillatory properties of eigenfunctions of problem (1)-(4) and their derivatives.

2. Operator interpretation of the boundary problem (1)-(4)

Let $H = L_2(0,1) \oplus \mathbb{C}^2$ be the Hilbert space with inner product

$$(\hat{u}, \hat{v}) = (\{u, m, k\}, \{v, s, t\}) = \int_0^1 u(x)\overline{v(x)}\, dx + \ |a_1|^{-1} m\bar{s} \ + \ |a_2|^{-1} k\bar{t}.$$

It is well known (see [3, 4]) that the boundary-value problem (1)-(4) reduces to the spectral problem for the linear operator L in the Hilbert space H, where

$$L\hat{y} = L\{u, m, k\} = \{(Ty(x))', \, y''(1) - b_1 y'(1), \, Ty(1) - b_2 y(1)\},$$

is an operator with the domain

$$D(L) = \{\{y\ (x),\, m,\, k\}\ :\ y \in W_2^4(0,1),\ (Ty(x))' \in L_2(0,1),$$
$$y(0) = y'(0) = 0,\ m = a_1 y'(1),\ n = a_2 y(1)\}\,.$$

It is obvious that the operator L is well defined in H and problem (1)-(4) take the form

$$L\hat{y} = \lambda \hat{y},\ \ \hat{y} \in D(L).$$

This means that the eigenvalues λ_{nk}, $nk \in \mathbb{N}$, of problem (1)-(4) and of the operator L coincide, and between the eigenvectors, there is a one-to-one correspondence

$$y_n(x) \leftrightarrow \ \{y_n(x),\, m_n,\, k_n\},\ \ m_n = a_1 y_n'(1),\ \ k_n = a_2 y_n\,(1).$$

Theorem 1. *L is a self-adjoint discrete lower-semibounded operator in H. The system of eigenvectors $\{\hat{y}_k\}_{k=1}^{\infty}$, $\hat{y}_k = \{y_k(x),\, m_k,\, n_k\}$, $m_k = a_1 y_k'(1)$, $n_k = a_2 y_k(1)$, of the operator L forms forms an orthogonal basis in the space H (forms a Riesz basis (after normalization) in H).*

The proof of this theorem is similar to that of [3, Theorem 5.1].

Corollary 1. *The eigenvalues of the operator L are real, simple and form an unboundedly increasing sequence $\{\lambda_k\}_{k=1}^{\infty}$.*

3. Oscillatory properties of eigenfunctions of the boundary-value problem (1)-(4) and their derivatives

Theorem 2. *For each fixed* $\lambda \in \mathbb{C}$ *there exists a unique nontrivial solution* $y(x,\lambda)$ *of the boundary-value problem* (1), (2), (4) *up to a constant factor. The solution* $y(x,\lambda)$ *for each fixed* $x \in [0,1]$ *is an entire function of* λ *.*

The proof of this theorem is similar to that of [3, Theorem 3.1].

Remark 1. It follows from [7, Theorem 2.2] that the eigenvalues of the boundary value problem

$$\begin{array}{l} \ell(y)(x) = \lambda y(x),\ x \in (0,1), \\ y(0) = y'(0) = 0, \\ y'(1)\cos\gamma + y''(1)\sin\gamma = 0,\ \gamma \in [0, \frac{\pi}{2}], \\ Ty(1) - (a_2\lambda + b_2)y(1) = 0, \end{array} \tag{5}$$

are real, simple and form an infinitely increasing sequence $\{\lambda_k^{(\gamma)}\}_{k=1}^{\infty}$; $\lambda_k(\gamma) > 0$ for $k \geq 2$ and there for each fixed γ exists $b_2(\gamma) < 0$ such that $\lambda_1(\gamma) > 0$ for $b_2 > b_2(\gamma)$, $\lambda_1(\gamma) = 0$ for $b_2 = b_2(\gamma)$ and $\lambda_1(\gamma) < 0$ for $b_2 < b_2(\gamma)$. Moreover, the eigenfunction $u_k^{(\gamma)}(x)$ corresponding to the eigenvalue $\lambda_k(\gamma)$ for $k \geq 2$ has $k-1$ simple zeros in the interval $(0,1)$; the eigenfunction $u_1^{(\gamma)}(x)$ has no zeros for $b_2 \geq b_2(\gamma)$ and has arbitrary number of zeros in $(0,1)$ for $b_2 < b_2(\gamma)$.

Denote: $B_k = (\lambda_{k-1}(0), \lambda_k(0))$, $k \in \mathbb{N}$, where $\lambda_0(0) = -\infty$.

Clearly, the eigenvalues $\lambda_k(0)$ and $\lambda_k(\pi/2)$, $k \in \mathbb{N}$, of problem (5) for $\gamma = 0$ and $\gamma = \pi/2$ are zeros of the entire functions $y'(1,\lambda)$ and $y''(1,\lambda)$, respectively. It is obvious that the function

$$F(\lambda) = y''(1,\lambda)/y'(1,\lambda)$$

is will defined for

$$\lambda \in B \equiv \left(\bigcup_{k=1}^{\infty} B_k\right) \bigcup (\mathbb{C}\backslash\mathbb{R}),$$

and is meromorphic function of finite order. $\lambda_k(0)$ and $\lambda_k(\pi/2)$, $k \in \mathbb{N}$, are the poles and zeros of the function $F(\lambda)$, respectively.

Lemma 1. *The following relations*

$$\frac{dF(\lambda)}{d\lambda} = -\frac{1}{y'^2(1,\lambda)}\left\{\int_0^1 y^2(x,\lambda)dx - a_2 y^2(1,\lambda)\right\},\ \lambda \in B. \tag{6}$$

$$\lim_{\lambda \to -\infty} F(\lambda) = +\infty, \tag{7}$$

are true.

The proof of this lemma is similar to that of [3, Lemmas 3.3 and 3.4].

It follows from the maximal minimal property of eigenvalues and (6) that

$$\lambda_1\left(\frac{\pi}{2}\right) < \lambda_1(0) < \lambda_2\left(\frac{\pi}{2}\right) < \lambda_2(0) < \ \dots \ . \tag{8}$$

Note that the eigenvalues of problem (1)-(4) are the roots of the following equation

$$y''(1,\lambda) - (a_1\lambda + b_1)\, y'(1,\lambda) = 0. \tag{9}$$

Lemma 2. *Let λ be an eigenvalue of the boundary-value problem* (1)-(4)*. Then* $y'(1,\lambda) \neq 0$*.*

Proof. Let λ be an eigenvalue of problem (1)-(4) and $y'(1,\lambda) = 0$. Then it follows from boundary condition (3) that $y''(1,\lambda) = 0$. Hence λ is an eigenvalue of problem (5) for $\gamma = 0$ and $\gamma = \frac{\pi}{2}$, which contradicts to relation (8). Thus $y'(1,\lambda) \neq 0$ if λ is an eigenvalue of problem (1)-(4). The proof of this lemma is complete.

Remark 2. It follows from Lemma 2 that each root of (9) is also root of the equation

$$F(\lambda) = a_1\lambda + b_1,$$

as well.

Theorem 3. *There exists an unboundedly increasing sequence of eigenvalues $\lambda_1 < \lambda_2 < \ldots < \lambda_k < \ldots$ of the boundary value problem* (1)-(4)*. Moreover, we have the following location of these eigenvalues on the real axis:*

(i) if $\lambda_1(0) > 0$, then there exists a real number $b_{1,0}$ such that $\operatorname{sign} b_{1,0} = \operatorname{sign} \lambda_1(\pi/2)$ *and $\lambda_1 < 0$ for $b_1 > b_{1,0}$, $\lambda_1 = 0$ for $b_1 = b_{1,0}$, $\lambda_1 > 0$ for $b_1 < b_{1,0}$ and $\lambda_k > 0$ for $k \geq 2$;*

(ii) if $\lambda_1(0) = 0$, then $\lambda_1 < 0$ and $\lambda_k > 0$ for $k \geq 2$;

(iii) if $\lambda_1(0) < 0$, then $\lambda_1 < 0$ and there exists a real number $b_{1,1} > 0$ such that $\lambda_1 < 0$ for $b_1 > b_{1,1}$, $\lambda_1 = 0$ for $b_1 = b_{1,1}$, $\lambda_1 > 0$ for $b_1 < b_{1,1}$ and $\lambda_k > 0$ for $k \geq 3$.

The proof of this theorem is similar to that of first part of [3, Theorem 4.1] with use of Corollary 1, Theorem 2, Lemmas 1, 2 (relations (6) and (7)) and Remarks 1, 2.

Now let us take up the question of the number of zeros contained in the interval $(0,1)$ of the functions $y(x,\lambda)$ and $y'(x,\lambda)$.

Remark 3. Following the corresponding reasoning carried out in [3, Lemmas 3.1, 3.2 and 3.6] we can show that the zeros contained in $(0,1]$ of the functions $y(x,\lambda)$ and $y'(x,\lambda)$ are simple and C^1 functions of λ. Moreover, for $\lambda > 0$ between consecutive zeros of the function $y'(x,\lambda)$ in $(0,1]$, there is exactly one zero of function $y(x,\lambda)$; as $\lambda > 0$ increases the number of zeros contained in $(0,1)$ does not decrease.

We denote by $\epsilon(\lambda)$ and $\varkappa(\lambda)$ the number of zeros contained in $(0,1)$ of the functions $y(x,\lambda)$ and $y'(x,\lambda)$, respectively.

Theorem 4. *The functions $y(x,\lambda)$ and $y'(x,\lambda)$ have the following oscillation properties depending on the parameter $\lambda > 0$:*

(i) if $\lambda_1(0) \geq 0$, then
$\epsilon(\lambda) = \varkappa(\lambda) = 0$ *for* $\lambda \in (0, \lambda_1(0)]$,
$\epsilon(\lambda) = k-2$ *or* $\epsilon(\lambda) = k-1$ *for* $\lambda \in (\lambda_{k-1}(0), \lambda_k(\pi/2))$ *at* $k \geq 2$,
$\epsilon(\lambda) = k-1$ *for* $\lambda \in [\lambda_k(\pi/2), \lambda_k(0)]$ *at* $k \geq 2$,
$\varkappa(\lambda) = k-1$ *for* $\lambda \in (\lambda_{k-1}(0), \lambda_k(0)]$ *at* $k \geq 2$;
(ii) if $\lambda_1(0) < 0$, then
$\epsilon(\lambda) = \varkappa(\lambda) = 0$ *for* $\lambda \in (0, \lambda_2(0)]$,
$\epsilon(\lambda) = k-3$ *or* $\epsilon(\lambda) = k-2$ *for* $\lambda \in (\lambda_{k-1}(0), \lambda_k(\pi/2))$ *at* $k \geq 3$,
$\epsilon(\lambda) = k-2$ *for* $\lambda \in [\lambda_k(\pi/2), \lambda_k(0)]$ *at* $k \geq 3$,
$\varkappa(\lambda) = k-2$ *for* $\lambda \in (\lambda_{k-1}(0), \lambda_k(0)]$ *at* $k \geq 3$.

The proof of this theorem is similar to that of [3, Theorem 3.2] with use of Theorem 3 and Remarks 1, 3.

By virtue of [3, Corollary 3.1] as $\lambda < 0$ varies, the functions $y(x,\lambda)$ and $y'(x,\lambda)$ can lose or gain zeros only by these zeros leaving or entering the interval $[0,1]$ only through the endpoint $x = 0$. If these zeros pass through the point $x = 0$, then $x = 0$ would be a triple zero of function $y(x,\lambda)$, i.e. $y(0,\lambda) = y'(0,\lambda) = y''(0,\lambda) = 0$.

Assume that $\lambda < 0$ and μ is a real eigenvalue of the following spectral problem

$$\begin{aligned} &\ell(y)(x) = \lambda y(x), \ x \in (0,1), \\ &y(0) = y'(0) = y''(0) = 0, \\ &Ty(1) - (a_2\lambda + b_2)y(1) = 0. \end{aligned} \tag{10}$$

The oscillation index of μ which we denote by $i(\lambda)$ is the difference between the number of zeros of the function $y(x,\lambda)$ for $\lambda = \mu - 0$ contained in the interval $(0,1)$ and the number of the same zeros for $\lambda = \mu + 0$ (see [4, 5]). It follows from this definition that the number of zeros of the solution $y(x,\lambda)$ of problem (1), (2), (4) contained in $(0,1)$ is equal to the sum of the oscillation indices of all eigenvalues of the spectral problem (10) contained in the interval $(\lambda, 0)$.

Assume that $i(\mu_k)$ is an oscillation index of the eigenvalue μ_k, $k \in \mathbb{N}$, of problem (10), which is negative and simple [4, Lemma 4.2]. If $\lambda < 0$, then by condition (2) we have

$$\epsilon(\lambda) = \sum_{\mu_k \in (\lambda,0)} i\,(\mu_k), \tag{11}$$

$$\begin{aligned} \varkappa(\lambda) &= \sum_{\mu_k \in (\lambda,0)} i\,(\mu_k) \text{ for } \lambda_1(0) \geq 0, \\ \varkappa(\lambda) &= \sum_{\mu_k \in (\lambda,0)} i\,(\mu_k) + H(\lambda - \lambda_1(0)) \text{ for } \lambda_1(0) < 0. \end{aligned} \tag{12}$$

Theorem 5. *The eigenfunctions $y_k(x)$, $k = 1, 2, \ldots$ of the boundary value problem* (1)-(4) *and their derivatives have the following oscillation properties:*

i) if $\lambda_1(0) > 0$, then:

the functions $y_1(x)$ and $y_1'(x)$ have no zeros in the case $\lambda_1 \geq 0$, have $\sum\limits_{\mu_k \in (\lambda_1, 0)} i(\mu_k)$ simple zeros in the interval $(0,1)$ in the case $\lambda_1 < 0$,

the function $y_k(x)$ for $k \geq 2$ has either $k-2$ or $k-1$ simple zeros in the interval $(0,1)$,

the function $y_k'(x)$ has $k-1$ simple zeros in the interval $(0,1)$;

ii) if $\lambda_1(0) = 0$, then:

the functions $y_1(x)$ and $y_1'(x)$ have $\sum\limits_{\mu_k \in (\lambda_1, 0)} i(\mu_k)$ simple zeros in the interval $(0,1)$,

the function $y_k(x)$ for $k \geq 2$ has either $k-3$ or $k-2$ simple zeros in the interval $(0,1)$,

the function $y_k'(x)$ has $k-2$ simple zeros in the interval $(0,1)$;

iii) if $\lambda_1(0) < 0$, then:

the function $y_1(x)$ has $\sum\limits_{\mu_k \in (\lambda_1, 0)} i(\mu_k)$ and the function $y_1'(x)$ has $\sum\limits_{\mu_k \in (\lambda_1, 0)} i(\mu_k) + H(\lambda_1 - \lambda_1(0))$ simple zeros in the interval $(0,1)$,

$y_2(x)$ and $y_2'(x)$ have no zeros in the case $\lambda_2 \geq 0$, have $\sum\limits_{\mu_k \in (\lambda_1, 0)} i(\mu_k)$ simple zeros in the interval $(0,1)$ in the case $\lambda_2 < 0$,

the function $y_k(x)$ for $k \geq 3$ has either $k-3$ or $k-2$ simple zeros in the interval $(0,1)$,

the function $y_k'(x)$ has $k-2$ simple zeros in the interval $(0,1)$.

The proof of this theorem follows directly from Theorems 3, 4 and formulas (11), (12).

Remark 4. Using oscillation Theorems 3, 5 and applying the technique carried out in [3], it is possible to establish sufficient conditions for the subsystems of eigenfunctions of problem (1)-(4) to form a basis in the space $L_p(0,1)$, $1 < p < \infty$.

References

[1] Z.S. Aliyev, *Basis properties of a fourth order differential operator with spectral parameter in the boundary condition*, Cent. Eur. J. Math., **8(2)**, 2010, 378-388.

[2] Z.S. Aliev, *Basis properties in L_p of systems of root functions of a spectral problem with spectral parameter in a boundary condition*, Differential Equations, **47(6)**, 2011, 766-777.

[3] Z.S. Aliev, S.B. Guliyeva, *Properties of natural frequencies and harmonic bending vibrations of a rod at one end of which is concentrated inertial load*, J. Differential Equations, **263(9)**, 2017, 5830-5845.

[4] Z.S. Aliev, S.B. Guliyeva, *Spectral properties of a fourth order eigenvalue problem with spectral parameter in the boundary conditions*, Filomat, **32(7)**, 2018, 2421-2431.

[5] J. Ben Amara, A.A. Vladimirov, *On oscillation of eigenfunctions of a fourth-order problem with spectral parameters in the boundary conditions*, J. Math. Sci., **150(5)**, 2008, 2317-2325.

[6] B.B. Bolotin, *Vibrations in technique: Handbook in 6 volumes, The vibrations of linear systems*, I, Engineering Industry, Moscow, 1978.

[7] N.B. Kerimov, Z.S. Aliev, *On the basis property of the system of eigenfunctions of a spectral problem with spectral parameter in a boundary condition*, Differential Equations, **43(7)**, 2007, 905-915.

Sevinc B. Guliyeva
Ganja State University, Ganja, Azerbaijan
E-mail: bakirovna89@mail.ru

Received 18 February 2019
Accepted 20 May 2019