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On Some Class of Extremal Manifolds

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Abstract. In this paper it is studied some class of extremal manifolds given by a system of smooth functions. V. G. Sprindzuk in [11] put question on obtaining the conditions in which a manifold is extremal. In this paper it is given such a condition in the terms of convergence exponent for some improper integrals like the special integral of Terry's problem.

Key Words and Phrases: measure theory, extremal manifold, transcendental number, Lebesgue measure, measurable function.

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1. Introduction.

In 1932 Mahler K. had advanced a conjecture about S-numbers. To formulate this conjecture let's introduce some notations. We shall denote by Π a following set of polynomials with integral coefficients of degree not exceeding n:

$$\Pi = \{f(x) = \sum_{i=0}^{n} a_i x^i \neq 0 | a_i \in \mathbb{Z}\}.$$

The number

$$H(f) = max(|a_0|, |a_1|, ..., |a_n|)$$

is called to be the height of the polynomial

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n$$

with real coefficients. Let α be a transcendental number. Then $f(\alpha) \neq 0$. Consider some real number $H > 0$, and take all polynomials from the class Π with the heights not exceeding $H(f) \leq H$. Mahler had proven that the inequality

$$\|f(\alpha)\| > H^{-n\kappa}; h(f) \leq H$$

is satisfied for all polynomials in the class Π with the height not exceeding H for almost all real transcendental numbers, in the Lebesgue sense. The value firstly established for

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the constant κ by Mahler was $4+\varepsilon$, with arbitrarily small positive constant ε. Mahler had conjectured that it is possible to take $\kappa = 1+\varepsilon$. This conjecture was proven by Sprindzuk V. G. in 1965 by the method of essential and non-essential domains (see [11]).

For a given real number $H > 0$ the number of polynomials with heights doesn't exceeding H is finite. Denote by $\omega_n(\alpha)$ the supremum of that positive numbers $\gamma > 0$, for which the inequality

$$|f(\alpha)| < H^{-\gamma};\ H = H(f) \qquad (1)$$

is satisfied for infinite number of polynomials from Π, when $H \to \infty$. It means that for arbitrary $\varepsilon > 0$ there is a non-bounded from above sequence $H_1, H_2, ...$ such that (1) is satisfied for all such H_m with

$$\gamma = \omega_n(\alpha) + \varepsilon.$$

This number is defined for every given n, and, by this reason, one can define the number (finite or infinite)

$$g = \overline{\lim_{n\to\infty}} \frac{\omega_n(\alpha)}{n}.$$

Note that for transcendental numbers due to Dirichlet's principle we always have $\omega_n(\alpha) \geq n$ and therefore, $g \geq 1$. The Mahler hypothesis is consisted in the statement that $\omega_n(\alpha) = n$ for almost all transcendental numbers α.

Consider now the system of inequalities

$$\max\left(\|\alpha_1 q\|, \|\alpha_2 q\|, ..., \|\alpha_n q\|\right) < q^{-u}, u > 0. \qquad (2)$$

Let $u(\alpha_1, ..., \alpha_n)$ be defined as a *sup* of such $u > 0$ for which (2) is satisfied for infinite set of natural numbers q. It is not difficult to show that $u(\alpha_1, ..., \alpha_n) \geq 1/n$ (see [10]). From this definition it follows that the inequality (2) is satisfied for infinitely many natural numbers q when $u < 1/n$. When $u(\alpha_1, ..., \alpha_n) = 1/n$ for almost all points of the variety $(\alpha_1, ..., \alpha_n) \in R^n$ of less dimension, then we call this manifold as an *extremal manifold*. By Khintchine's Transference Principle (see [5, 9]), mentioned above Mahler hypothesis is equivalent to the hypothesis on extremality of the variety $(x, x^2, \ldots, x^n)$.

In 1993 Karatsuba A. A. advanced an opinion that the question on extremality of some algebraic varieties could be investigated by using of results on convergence exponent in the Tarry's problem (about the problem see [1]). This hypothesis was proven in [7].

Let we are given with some continuously differentiable n-dimensional manifold $\Gamma = (f_1(\bar{x}), ..., f_N(\bar{x}))$, $\bar{x} \in \Omega \subset \mathbb{R}^n, n < N$. In this work we continue consideration of conditions supplying the extremality of the manifold. Consider the integral (for some integral $h > 0$)

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left|\int_{\Omega} e^{2\pi i(\alpha_1 f_1(\bar{x})+\alpha_2 f_2(\bar{x})\cdots+\alpha_n f_n(\bar{x}))} d\bar{x}\right|^{2h} d\alpha_1 d\alpha_2\cdots d\alpha_n.$$

The number γ is called to be the convergence exponent for the integral

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left|\int_{\Omega} e^{2\pi i(\alpha_1 f_1(\bar{x})+\alpha_2 f_2(\bar{x})\cdots+\alpha_n f_1(\bar{x}))} dx\right|^{2h} d\alpha_1 d\alpha_2\cdots d\alpha_n,$$

if this integral is convergent when $2h > \gamma$ and divergent when $2h < \gamma$. In the section 3 we prove the extremality of above manifold if the last integral has finite exponent of convergence.

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2. Auxiliary statements

Following lemma is known as Borel-Kantelly's lemma and plays an important role in the questions concerning extremality of manifolds (see[10]).

Lemma 1. Let $A_q\,(q = 1, 2, ...)$ be a sequence of measurable sets in $\mathbb{R}^n$, and

$$\sum_{q=1}^{\infty} \mathrm{m}esA_q < \infty.$$

Then the measure of a set E of such points $x \in R^n$ which fall into infinite number of sets A_q equals to zero.

Proof. For every $x \in E \subset R^n$ and natural n there is a natural number $m > n$ for which $x \in A_m$. Then for any $x \in E$ and natural number $n \in N$

$$x \in \bigcup_{k=n}^{\infty} A_k.$$

So,

$$E \subset \bigcup_{k=n}^{\infty} A_k.$$

Since the series of measures is convergent, then for arbitrary $\varepsilon > 0$ there exist a number n such that

$$mes \bigcup_{k=n}^{\infty} A_k \leq \sum_{k=n}^{\infty} mesA_k < \varepsilon.$$

From the said above we deduce that $mesE = 0$. Lemma 1 is proven.

Below we will use the symbol $<<$ introduced by Vinogradov I. M. For two quantities A and B we write $A << B$ if one can find a fixed contant c such that $A \leq cB$.

Following lemma belongs to Kavalevskaja E. I. (see [4,8,10]).

Lemma 2. Let *m, n, q* be natural numbers, $f_j(\bar{x}), j = 1, ..., N$ be a real measurable functions defined in the cube $\Omega = [0, 1]^r, 1 \leq r \leq N$. Denote by $\mu(q)$ the measure of a set of that $\bar{x} \in \Omega = [0, 1]^r$ for which

$$\|f_j(\bar{x})\| < q^{-r_j}\,(1 \leq j \leq N).$$

Then,

$$\mu(q) << q^{-r} \sum_{|c_1|<q^{r_1}} \cdots \sum_{|c_N|<q^{r_N}} \left| \int_{\Omega} e^{2\pi i(c_1 f_1(\bar{x})+\cdots+c_N f_N(\bar{x}))} d\bar{x} \right|;$$

here $r = r_1 + \cdots + r_N$, and the constant in the symbol $<<$ depends on N only.

Let we are given with some continuously differentiable n-dimensional manifold $\Gamma = (f_1(\bar{x}), ..., f_N(\bar{x}))$, $\bar{x} \in \Omega = [0,1]^n, n < N$. Taking natural number h such that $nh > N$ consider the map

$$\varphi_j : \Omega^h \to R^N$$

defined by the equalities

$$\varphi_j(\bar{x}) = \varphi_j(\bar{x}_1, ..., \bar{x}_h) = f_j(\bar{x}_1) + \cdots + f_j(\bar{x}_h);\ \bar{x}_s = (x_{s1}, ..., x_{sm}).$$

Let the Jacoby matrix of the map $(\bar{x}_1, ..., \bar{x}) \mapsto (\varphi_1(\bar{x}), ..., \varphi_h(\bar{x}))$, i. e. the matrix composed of the gradients of the functions $\varphi_1(\bar{x}), ..., \varphi_h(\bar{x})$, be the matrix of maximal rank. It is easy to see that the Jacoby matrix has a view

$$\begin{pmatrix} \frac{\partial \varphi_1}{\partial x_{11}} & \cdots & \frac{\partial \varphi_1}{\partial x_{hn}} \\ \vdots & \ddots & \vdots \\ \frac{\partial \varphi_N}{\partial x_{11}} & \cdots & \frac{\partial \varphi_N}{\partial x_{hn}} \end{pmatrix}.$$

In the work [3] there was proven the following result.

Lemma. If the Jacoby matrix of the map $(\bar{x}_1, ..., \bar{x}_h) \mapsto (\varphi_1(\bar{x}), ..., \varphi_h(\bar{x}))$ has a maximal rank for some natural h then the differentiable manifold Γ is extremal.

3. Main results.

Theorem 1. Let $\mathrm{g}(\bar{x}) = \sum_{i=1}^N \alpha_i f_i(\bar{x})$. Then, in the conditions of the lemma the following formula is fair

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left(\int_\Omega e^{2\pi i g(\bar{x})} d\bar{x} \right)^{2h} d\alpha_1 \cdots d\alpha_N = \int_\Pi \frac{ds}{\sqrt{G_0}},$$

where the surface integral at the right side of the equality is taken over the surface defined by system of the equations

$$\begin{gathered} f_1(\bar{x}_1) + f_1(\bar{x}_2) + \cdots + f_1(\bar{x}_h) - f_1(\bar{x}_{h+1}) - f_1(\bar{x}_{h+2}) - \cdots - f_1(\bar{x}_{2h}) = 0, \\ f_2(\bar{x}_1) + f_2(\bar{x}_2) + \cdots + f_2(\bar{x}_h) - f_2(\bar{x}_{h+1}) - f_2(\bar{x}_{h+2}) - \cdots - f_2(\bar{x}_{2h}) = 0, \\ \cdots \quad \cdots \quad \cdots \\ f_j(\bar{x}_1) + f_j(\bar{x}_2) + \cdots + f_j(\bar{x}_h) - f_j(\bar{x}_{h+1}) - f_j(\bar{x}_{h+2}) - \cdots - f_j(\bar{x}_{2h}) = 0, \qquad (3) \\ \cdots \quad \cdots \quad \cdots \\ f_N(\bar{x}_1) + f_N(\bar{x}_2) + \cdots + f_N(\bar{x}_h) - f_N(\bar{x}_{h+1}) - f_N(\bar{x}_{h+2}) - \cdots - f_N(\bar{x}_{2h}) = 0 \end{gathered}$$

in Ω^{2h}, G_0 is a Gram determinant of gradients of functions standing on the left parts of equations from the system (3).

Remark. We can describe G_0 more explicitly. Let's designate

$$F_j(\bar{x}) = f_j(\bar{x}_1) + f_j(\bar{x}_2) + \cdots + f_j(\bar{x}_h) -$$
$$- f_j(\bar{x}_{h+1}) - f_j(\bar{x}_{h+2}) - \cdots - f_j(\bar{x}_{2h})$$

with $\bar{x} = (\bar{x}_1, \bar{x}_2, ..., \bar{x}_{2h}) \in R^{2hn}$. It is easy to see that the gradient vector for the function $F_j(\bar{x})$ has a view

$$\nabla F_j(\bar{x}) = (\nabla f_j(\bar{x}_1), \nabla f_j(\bar{x}_2), ..., \nabla f_j(\bar{x}_h),$$
$$-\nabla f_j(\bar{x}_{h+1}), -\nabla f_j(\bar{x}_{h+2}), ..., -\nabla f_j(\bar{x}_{2h})).$$

Now we put

$$A_0 = \begin{pmatrix} \nabla F_1(\bar{x}) \\ \vdots \\ \nabla F_N(\bar{x}) \end{pmatrix}.$$

Then one can write $G_0 = \det(A_0 A_0^T)$.

Proof of the theorem 1. Performing easy calculations we get

$$\left(\int_\Omega e^{2\pi i g(\bar{x})} d\bar{x} \right)^h = \int_\Omega \cdots \int_\Omega e^{2\pi i (g(\bar{x}_1) + \cdots + g(\bar{x}_h))} d\bar{x}_1 \cdots d\bar{x}_h, \tag{4}$$

where the function $g(\bar{x})$ stands for a linear combination of the functions $f_1(\bar{x}), ..., f_1(\bar{x})$:

$$g(\bar{x}) = \alpha_1 f_1(\bar{x}) + \cdots + \alpha_N f_N(\bar{x})$$

with real coefficients. Consider now the functions

$$\varphi_j(\bar{x}) = u_j = f_j(\bar{x}_1) + \cdots + f_j(\bar{x}_h),\ j = 1, ..., N,$$

with $\bar{x}_s = (x_{s1}, ..., x_{sn})$. Since the considered functions are continuous and the domain Ω is closed, then there exists a positive number $\eta > 0$ such that $G \geq \eta$. Applying the consequence of the lemma 1 from the work [2,6], we can represent the integral (4) as below

$$\int_\Omega \cdots \int_\Omega e^{2\pi i (\alpha_1 (f_1(\bar{x}_1) + \cdots + f_1(\bar{x}_h)) + \cdots + \alpha_N (f_N(\bar{x}_1) + \cdots + f_N(\bar{x}_h)))} d\bar{x}_1 \cdots d\bar{x}_h =$$
$$= \int_{m_1}^{M_1} \cdots \int_{m_n}^{M_n} \left(\int_\Pi \frac{ds}{\sqrt{G}} \right) e^{2\pi i (\alpha_1 u_1 + \cdots + \alpha_n u_n)} du_1 \cdots du_n, \tag{5}$$

designating by $\Pi = \Pi(\bar{u})$ the surface defined by the system of equations

$$f_j(\bar{x}_1) + \cdots + f_j(\bar{x}_h) = u_j,\ j = 1, ..., N,$$

and here the numbers m_j, M_j stand for the minimal and maximal values of the function $\varphi_j(\bar{x})$. Then, considering the last integral as a Fourier transformation, we will have by Parseval identity:

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left| \int_\Omega e^{2\pi i (\alpha_1 f_1(\bar{x}) + \cdots + \alpha_N f_N(\bar{x}))} d\bar{x} \right|^{2h} d\alpha_1 \cdots d\alpha_N =$$

$$= (2\pi)^N \int_{m_1}^{M_1} \cdots \int_{m_n}^{M_n} \left(\int_\Pi \frac{ds}{\sqrt{G}} \right)^2 du_1 \cdots du_n, \tag{6}$$

and the equality is understood in the sense that from the convergence of one of its two parts the convergence of other part follows, and the corresponding values are equal.

Now we will use (6) to prove the statement of the main theorem. Let's assume that the right side part of the equality (6) is convergent. Applying the lemma 1, we have:

$$\int_{\Pi(\bar{u})} \frac{ds}{\sqrt{G}} = \lim_{h\to 0} \frac{1}{(2\delta)^N} \int_{u_j-\delta<\varphi_j<u_j+\delta} d\bar{x}. \tag{7}$$

Therefore, designating the left part of (6) $\varphi_{\mathrm{D}}(\bar{u})$, we can, represent the last integral by the lemma 1 and its corollary write

$$\int_{m_1}^{M_1} \cdots \int_{m_N}^{M_N} \left(\int_{\Pi(\bar{u})} \frac{ds}{\sqrt{G}} \right)^2 d\bar{u} = \int_{m_1}^{M_1} \cdots \int_{m_N}^{M_N} \varphi_D(\bar{u})\varphi_D(\bar{u}) d\bar{u} =$$

$$= \int_{m_1}^{M_1} \cdots \int_{m_N}^{M_N} \varphi_D(\bar{u}) \lim_{h\to 0} \frac{1}{(2\delta)^N} \int_{u_j-\delta<\varphi_j<u_j+\delta} d\bar{x} d\bar{u}.$$

Applying the lemma 3 under integral on the right part it is possible to rearrange the orders of integration and passing to the limit. For this purpose we put $\delta = \delta_n$ with $\delta_n \to 0$ and apply the specified lemma to our integral, when $\delta = \delta_n$:

$$\int_{m_1}^{M_1} \cdots \int_{m_N}^{M_N} \varphi_D(\bar{u})\varphi_D(\bar{u}) d\bar{u} =$$

$$= \int_{m_1}^{M_1} \cdots \int_{m_N}^{M_N} \varphi_D(\bar{u}) \lim_{n\to\infty} \frac{1}{(2\delta_n)^N} \int_{u_j-\delta_\mathrm{n}<\varphi_j<u_j+\delta_\mathrm{n}} d\bar{x} d\bar{u} =$$

$$= \lim_{n\to 0} \frac{1}{(2\delta_n)^N} \int_{m_1}^{M_1} \cdots \int_{m_N}^{M_N} \varphi_D(\bar{u}) \int_{u_j-\delta_\mathrm{n}<\varphi_j<u_j+\delta_\mathrm{n}} d\bar{x} d\bar{u} =$$

$$= \lim_{h\to 0} \frac{1}{(2\delta)^N} \int_{m_1}^{M_1} \cdots \int_{m_N}^{M_N} \left(\int_{\Pi'(\bar{u})} \frac{ds'}{\sqrt{G'}} \right) \int_{\substack{u_j - \delta < \varphi_j < u_j + \delta \\ j=1,\ldots,N}} d\bar{x} d\bar{u}, \tag{8}$$

where ds' means an element of the area of the surface defined in Ω by the system of equations $f_j(\bar{x}'_1) + \cdots + f_j(\bar{x}'_h) = u_j$, $j = 1, \ldots, N$, and is a Gram determinant for the system of functions standing at the left side of this system of equations. For the points $\bar{x}' \in \Omega^h$ we introduce the function $f(\bar{x}')$ defining its value at $\bar{x}' \in \Pi'(\bar{u})$ to be equal to the inner integral:

$$f(\bar{x}') = \int_{\substack{u_j - \delta < \varphi_j < u_j + \delta \\ j=1,\ldots,N}} d\bar{x}.$$

Let's consider, at fixed δ, inner integral in the last chain of equalities (8), i.e. the integral

$$\int_{\Pi'(\bar{u})}\left(\int_{\substack{u_j-\delta<\varphi_j<u_j+\delta\\ j=1,...,N}} d\bar{x}\right)\frac{ds'}{\sqrt{G'}}=\int_{\Pi'(\bar{u})} f(\bar{x}')\frac{ds'}{\sqrt{G'}}.$$

Let's prove that the function $f(\bar{x}')$ is continuous in Ω^h. Let $\bar{x}'_1$, $\bar{x}'_2\in\Omega^h$,$\bar{x}'_1=(\bar{x}_{11},...,\bar{x}_{1h})$, $\bar{x}'_2=(\bar{x}_{21},...,\bar{x}_{2h})$; $\bar{x}_{ij}=(x^1_{ij},...,x^n_{ij})\in R^n, i=1,2$, and

$$\sum_j\sum_s(x^s_{1j}-x^s_{2j})^2\le\varepsilon$$

for given $\varepsilon>0$. Then, denoting $u^1_j=\varphi_j(\bar{x}'_1)$, $u^2_j=\varphi_j(\bar{x}'_2)$ (here we use top indexing) we in accordance with the formula on finite increments have:

$$\left|u^1_j-u^2_j\right|=\left|\sum_{1\le s\le n}\sum_{1\le i\le h}\left(\frac{\partial f_j(\bar{x}'_i+\bar{\theta})}{\partial x_i{}^s}({x'_{1i}}^s-{x'_{2i}}^s)\right)\right|\le M\sqrt{nh\varepsilon}$$

for some $\bar{\theta}$, if $\sum_s\sum_i({x'_{1i}}^s-{x'_{2i}}^s)^2\le\varepsilon$, and M stands for maximal value of partial derivatives of the functions $f_j(\bar{x})$ in the considered domain. Therefore, recalling the definition of the function $f(\bar{x}')$, we find:

$$\left|f(\bar{x}'_1))-f(\bar{x}'_2)\right|=\Big|\int_{\substack{u^1_j-\delta<\varphi_j<u^1_j+\delta\\ j=1,...,N}} d\bar{x}-$$

$$-\int_{\substack{u^2_j-\delta<\varphi_j<u^{21}_j+\delta\\ j=1,...,N}} d\bar{x}\ \Big|.$$

The integrals at the right side of this equality express volumes of pre-images of two cubes with sufficiently close centers, when ε is small enough. From geometric representations it is clear that the difference between these volumes coincides with the sum of volumes of pre-images of parallelepipeds including lateral sides of the two initial cubes. Since the number of lateral sides is not exceeding $2N$, then we have

$$\left|f(\bar{x}'_1)-f(\bar{x}'_2)\right|\le 2N\max_j\int_{u^1_j-\delta-M\sqrt{nh\varepsilon}<\varphi_j<u^1_j-\delta+M\sqrt{nh\varepsilon}} d\bar{x}+$$

$$+2N\max_j\int_{u^2_j+\delta-M\sqrt{nh\varepsilon}<\varphi_j<u^2_j+\delta+M\sqrt{nh\varepsilon}} d\bar{x}.$$

These integrals can be bounded by a similar way. Estimate first of them. We have

$$\int_{u^2_j+\delta-M\sqrt{nh\varepsilon}<\varphi_j<u^2_j+\delta+M\sqrt{nh\varepsilon}} d\bar{x}=$$

$$\int_{u_j^2+\delta-M\sqrt{nh\varepsilon}}^{u_j^2+\delta+M\sqrt{nh\varepsilon}} du_1 \int_{m_2}^{M_2} du_2 \cdots \int_{m_N}^{M_N} du_N \int_{\Pi(\bar{u})} \frac{ds}{\sqrt{G}} \leq$$

$$\leq 2\prod_{k=2}^{N}(M_k-m_k)\sqrt{\frac{nh\varepsilon}{\eta}}\Pi_0;\ \Pi_0 = \max_{\bar{u}} \int_{\Pi(\bar{u})} \frac{ds}{\sqrt{G}}$$

Since the domain Ω is bounded and the functions are continuous the last expression tends to 0 as $\varepsilon \to 0$. Therefore, the function $f(\bar{x}')$ is continuous. Applying the consequence to the lemma 1 of the work [6], we find:

$$\int_{m_1}^{M_1} \cdots \int_{m_N}^{M_N} \int_{\Pi(\bar{u})} d\bar{u} \int_{\substack{u_j-\delta<\varphi_j<u_j+\delta \\ j=1,\dots,N}} d\bar{x}\frac{ds'}{\sqrt{G'}} =$$

$$\int_{m_1}^{M_1} \cdots \int_{m_N}^{M_N} d\bar{u} \int_{\Pi(\bar{u})} f(\bar{x}')\frac{ds'}{\sqrt{G'}} = \int_{\substack{-\delta<\varphi_j-\varphi_j'<\delta \\ j=1,\dots,N}} d\bar{x}d\bar{x}'.$$

So, from the equality (8) we derive

$$\int_{m_1}^{M_1} \cdots \int_{m_N}^{M_N} \varphi_D(\bar{u})\varphi_D(\bar{u})d\bar{u} =$$

$$= \lim_{h\to 0}\frac{1}{(2\delta)^N} \int_{\substack{-\delta<\varphi_j-\varphi_j'<\delta \\ j=1,\dots,N}} d\bar{x}d\bar{x}' = \int_{\Pi_0} \frac{ds}{\sqrt{G_0}},$$

where G_0 is defined above.

The left part of the received equality under condition of existence of the right or left part of (8) coincides with the integral on the right part (8). It is clear that the all of reasonings performed above can be made in opposite direction. So, the theorem 1 is proven.

Theorem 2. Let the conditions of the theorem 1 be satisfied. If the integral

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left| \int_{\Omega} e^{2\pi i(\alpha_1 f_1(\bar{x})+\alpha_2 f_2(\bar{x})\cdots+\alpha_n f_n(\bar{x}))} dx \right|^{2h} d\alpha_1 d\alpha_2 \cdots d\alpha_n,$$

has finite exponent of convergence, then the manifold $\Gamma = (f_1(\bar{x}), f_2(\bar{x}), \cdots, f_n(\bar{x}))$ is extremal.

This theorem is an easy consequence of the theorem 1.

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