Caspian Journal of Applied Mathematics, Ecology and Economics
V. 7, No 1, 2019, July
ISSN 1560-4055

On Strong Law of Large Numbers for the Family of First Passage Times for the Level in Random Walk Described by a Non-Linear Function of Autoregression Process of Order One $(AR(1))$

I.A. Ibadova*, A.D. Farhadova

Abstract. In the paper we prove strong law of large numbers for the family of first passage times for the level in random walk described by a non-linear function of autoregression process of order one $(AR(1))$.

Key Words and Phrases: strong law of large numbers, autoregression process, first passage times, random walk.

2010 Mathematics Subject Classifications: 60F05

---

1. Introduction

Let on some probability space (Ω, F, P) we are given the sequence of independent identically distributed random variables $\xi_n = \xi_n(\omega), n \geq 1, \omega \in \Omega$.

As is known ([1]-[9]), autoregression process of order one is determined as the solution of the equation

$$X_n = \beta X_{n-1} + \xi_n, \quad n \geq 1$$

where β is some fixed number and the initial value of the process X_0 is independent of the innovation $\{\xi_n\}$.

Assume

$$T_n = \sum_{k=1}^{n} X_n X_{k-1} \quad \text{and} \quad \overline{T}_n = \frac{T_n}{n}, \quad n \geq 1.$$

A number of asymptotic properties of distributed sums T_n, $n \geq 1$ were studied in the paper [1].

Let us consider the family of the first exit times

$$t_a = \inf\left\{n \geq 1 : n\Delta\left(\overline{T}_n\right) > a\right\} \tag{1}$$

*Corresponding author.

for the level $a \geq 0$, where $\Delta(x)$, $x \in R = (-\infty, \infty)$ is some Borel function.

The family of the stoppage time t_a, $a \geq 0$ of the form (1) play a significant roll in applied fields of theory of probability and mathematical statistics ([1-6]). Note that the boundary value problems related to the family of the first passage time

$$\tau_a = \inf \left\{ n \geq 1 : n\Delta \left(\frac{S_n}{n} \right) > a \right\},$$

where

$$S_n = \sum_{k=1}^{n} \xi_k, \qquad n \geq 1$$

(see [7], [10]) are on the base of classic theory of nonlinear renewal.

In the case $\Delta(x) = x$ the limit theorems for the family of the first exit times t_a of the form (1) were studied in the monograph [10].

In the present paper we prove a theorem on strong law of large numbers for the family τ_a, $a \geq 0$.

2. Formulation and proof of the main result

For the function $\Delta(x)$ we will suppose that it is positive and twice continuously-differentiable in R.

In the paper [1] (see also [9], it was proved that under the continuous $E\xi_1 = 0$, $D\xi_1 = 1$, $|\beta| < 1$ and $EX_0^2 < \infty$ it holds the strong law of large numbers for the sequence of the sums $T_n, n \geq 1$:

$$\frac{T_n}{n} \overset{a.s.}{\to} \frac{\beta}{1-\beta^2} = \lambda \quad \text{as} \quad n \to \infty. \tag{2}$$

By the made assumptions for the function $\Delta(x)$ we have

$$n\Delta\left(\overline{T}_n\right) = n\Delta(\lambda) + u\Delta'(\lambda)\left(\overline{T}_n - \lambda\right) +$$

$$+\frac{n}{2}\Delta''(\lambda_n)\left(\overline{T}_n - \lambda\right)^2 = n\Delta(\lambda) + \Delta'(\lambda)(T_n - n\lambda) +$$

$$+\frac{1}{2}\Delta''(\lambda)\left(\frac{T_n - n\lambda}{\sqrt{n}}\right)^2,$$

where λ_n is an intermediate point between λ and $\overline{T}_n$, $n \geq 1$.

Assume

$$Z_n = n\Delta(\lambda) + n\Delta(\lambda) + \Delta'(\lambda)(T_n - n\lambda) = \sum_{k=1}^{n} \eta_k,$$

$$\eta_k = \Delta(\lambda) + \Delta'(\lambda)(X_k X_{k-1} - \lambda)$$

and

$$\varepsilon_n = \frac{1}{2}\Delta''(\lambda_n)\left(\frac{T_n - n\lambda}{\sqrt{n}}\right)^2$$

$$H_n = n\Delta\left(\overline{T}_n\right).$$

Then we have

$$H_n = Z_n + \varepsilon_n. \tag{3}$$

By (2),

$$\frac{Z_n}{n} \overset{a.s.}{\to} \Delta(\lambda) \quad \text{and} \quad \frac{\varepsilon_n}{n} \overset{a.s.}{\to} o \tag{4}$$

as $n \to \infty$, by continuity

$$\Delta''(\lambda_n) \overset{a.s.}{\to} \Delta''(\lambda), \quad n \to \infty.$$

Then from (3) and (4) it follows that

$$\frac{H_n}{n} \overset{a.s.}{\to} \Delta(\nu) \quad \text{as} \quad n \to \infty. \tag{5}$$

It holds

Theorem 1. *Let* $|\beta| < 1$, $E\xi_1 = 0$, $D\xi_1 = 1$ *and* $EX_0^2 < \infty$. *Assume that the above mentioned conditions are fulfilled for the functions* $\Delta(x)$, *moreover* $\Delta(\lambda) > 0$.

Then

$$\frac{t_a}{a} \overset{a.s.}{\to} \frac{1}{\Delta(\lambda)}, \quad a \to \infty.$$

Proof. From (5) it follows that $\sup_n H_n = \infty$. Hence, by definition of the variable t_a it follows that $P(t_a < \infty) = P\left(\sup_n H_n > a\right) = 1$ for all $a \geq 0$. Show that

$$t_a \overset{a.s.}{\to} \infty \quad \text{as} \quad a \to \infty$$

Indeed, by definition of the variable t_a it increases as a function of a. Therefore

$$P\left(t_\infty = \lim_{a\to\infty} t_a \leq \infty\right) = 1.$$

We have

$$P(t_\infty \leq n) = P\left(\lim_{a\to\infty} t_a \leq n\right) =$$

$$= \lim_{a\to\infty} P(t_a \leq n) = \lim_{a\to\infty} P\left(\max_{k\leq n} H_k > a\right) = o$$

for all $n \geq 1$.

This means that for all $n \geq 1$

$$P(t_\infty > n) = 1.$$

Hence it follows that $P(t_\infty = \infty) = 1$.

Thus, we have

$$P\left(\lim_{a\to\infty} t_a = \infty\right) = 1. \tag{6}$$

Prove that from (5) and (6) it follows that

$$\frac{Ht_a}{t_a} \overset{a.s.}{\to} \Delta(\nu) \quad \text{as} \quad a \to \infty. \tag{7}$$

Denote

$$A = \left\{\omega : \frac{H_n}{n} \to \Delta(\nu)\right\}$$

$$B = \{\omega : t_a \to \infty\}$$

$$C = \left\{\omega : \frac{H_{t_a}}{t_a} \to \Delta(\nu)\right\}.$$

It is clear that

$$A \cap B \subset C. \tag{8}$$

Taking into account $P(A) = P(B) = 1$, we have

$$P(A \cap B) = P(A) + P(B) - P(A \cup B) = 1$$

hence

$$P(A \cup B) = 1.$$

Then from (8) it follows that $P(C) = 1$. Thus, (7) is proved. By (7) the statement of the theorem follows from the following two-sided inequality

$$\frac{H_{t_a - 1}}{t_a} \le \frac{a}{t_a} < \frac{H_{t_a}}{t_a},$$

whose validity follows from the definition of the first exit time t_a of the form (1) .

From the proved theorem and the well known theorem on convergence of a sequence of identically integrable random variables (see e.i. [10]) it follows the following result.

Corollary 1. *Let the theorem conditions be fulfilled and the family* $\frac{t_a}{a}$, $a > 0$ *be identically integrable. Then*

$$\frac{Et_a}{a} \to \frac{1}{\Delta(\nu)}, a \to \infty.$$

Remark 1. *Note that the statement of the Corollary in the case* $\Delta(x) = x$ *was proved in the paper [4], where the sufficient condition was found for identically integrable family* $\frac{t_a}{a}$, $a > 0$.

References

[1] V.F. Melfi, *Nonlinear Markov renewal theory with statistical applications,* The Annals of Probability, **20(2)**, 1992, 753-771.

[2] V.F. Melfi, *Nonlinear renewal theory for Markov random walks,* Stochastic Processes and their Applications, **54**, 1994, 71-93.

[3] A.A. Novikov, *Some remarks on distribution of the first passage time and optimal stop of* $AR(1)$*-sequences,* Theoriya veroyatn i ee primen, **53(3)**, 2008, 458-471.

[4] F.H. Rahimov, V.A. Abdurakhmanov, T.E. Hashimova, *On the asymptotics of the mean value of the moment of first level-crossing by the first order autoregression process* $AR(1)$*,* Transaction of NAS of Azerbaijan, **XXXIV(4)**, 2014, 93-96.

[5] F.H. Rahimov, F.J. Azizov, V.S. Khalilov, *Integral limit theorems for the first passage time for the level of random walk, described by a nonlinear function of the sequence autoregression* $AR(1)$*,* Transactions of NAS of Azerbaijan, **XXXIV(1)**, 2014, 99-104.

[6] F.H. Rahimov, F.J. Azizov, V.S. Khalilov, *Integral limit theorems for the first passage time for the level of random walk, described with* $AR(1)$ *sequences,* Transactions of NAS of Azerbaijan, **XXXII(4)**, 2013, 95-100.

[7] F.N. Rahimov, I.A. Ibadova, A.D. Farkhadova, T.E. Hashimova, *Limit theorems for the random walk the autoregression process of order one,* Proceedings of IAM, **5(1)**, 2016, 25-33.

[8] I.A. Ibadova, V.S. Khalilov, A.D. Farkhadova, *Strong law of large numbers for a class of first passage time in the first order autoregression* $(AR(1))$ *scheme,* Transactions of NAS of Azerbbaijan, **XXXVII(1)**, 2017, 121-125.

[9] Pollard, D. Convergence of Stochastic Processes, *Springer, New-York,* 1984.

[10] Woodroofe M. Nonlinear renewal theory in sequential analysis, *SIAM. Philadelphia*, 1982.

I.A. Ibadova
Institute of Mathematics and Mechanics of NAS of Azerbaijan, Baku, Azerbaijan
E-mail: ibadovairade@yandex.ru

A.D. Farhadova
Baku State University, Baku, Azerbaijan
E-mail: farxadovaaynura@gmail.com

Received: 18 December 2018
Accepted: 29 May 2019