

## Calculating Steady-state Probabilities of the G/M/n/m Queueing Systems

S.A. Aliyev\*, Y.I. Yeleyko, Y.V. Zhernovyi

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**Abstract.** This article proposes a method for calculating steady-state probabilities of the  $G/M/n/m$  queueing systems. The approach based on the use of fictitious phases and hyperexponential approximations with parameters of the paradoxical and complex type by method of moments. The obtained results are verified using simulation models.

**Key Words and Phrases:** non-Markovian queueing system, hyperexponential approximation, fictitious phases, method of moments.

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### 1. Introduction

For the study of non-Markovian process in queueing systems, phase-type distributions are used with exponential distributions of delays in the phases [2, 5, 8]. In the case of fixing the number of the phase, the states of the system has a Markov property that makes it possible to represent the transitions between them in the form of a discrete Markov process with continuous time. The order of approximation is the number of retained initial moments of the original distribution.

Recently, interest in the hyperexponential distribution has increased since its use showed its high performance in solving problems of summation of recurrent flows [4], in computing characteristics of queueing systems with impatient customers [3] and Jackson's networks of queueing [6], and also in analyzing stock management systems [1].

Article [8] shows that the use of hyperexponential approximation ( $H_l$ ) makes it possible to determine with high accuracy the steady-state probabilities of non-Markovian single-channel queueing systems. These probabilities are determined using solutions of a system of linear algebraic equations obtained by the method of fictitious phases. To find parameters of the  $H_l$ -approximation of a certain distribution it is sufficient to solve the system of equations of the moments method. For the values  $V < 1$  of the variation coefficient, roots of this system are complex-valued or paradoxical (i.e., negative or with probabilities that exceed the boundaries of the interval  $[0, 1]$ ) but in most cases as a result of summation of probabilities of microstates, their complex-valued and paradoxical parts are annihilated.

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\*Corresponding author.

The purpose of the paper is to use of the hyperexponential approximation method for calculating steady-state probabilities of the  $G/M/n/m$  queueing systems. The obtained results are verified using simulation models. We also indicate ways to evaluate the accuracy of approach the obtained steady-state distribution to the true distribution without the need to use simulation models.

## 2. Equations for steady-state probabilities of the $H_l/M/n/m$ system

The hyperexponential distribution of order  $l$  is a phase-type distribution and provides for choosing one of  $l$  alternative phases by a random process. With probability  $\alpha_s$ , the process is at the  $s$ th phase and is in it during an exponentially distributed time with a parameter  $\lambda_s$ .

Suppose that the times elapsed between two consecutive arrivals are independent random variables distributed according to the hyperexponential law  $H_l$  ( $l \geq 2$ ) with probabilities  $\alpha_s$  and parameters  $\lambda_s$  ( $1 \leq s \leq l$ ) and the service time of each customer is distributed exponentially with parameter  $\mu$ . Let  $n$  and  $m$  denote the number of channels in the system and limit on the queue length respectively.

Let us enumerate the  $H_l/M/n/m$  system's states as follows:  $x_{0(s)}$  corresponds to the empty system and the time interval until the arrival of the first customer is in the phase  $s$  ( $1 \leq s \leq l$ );  $x_{k(s)}$  is the state, when there are  $k$  customers in the system ( $1 \leq k \leq n+m$ ), the time interval until the arrival of the next customer is in the phase  $s$  ( $1 \leq s \leq l$ ). We denote by  $p_{0(s)}$  and  $p_{k(s)}$  respectively, steady-state probabilities that the system is in the each of these states. To calculate  $p_{0(s)}$  and  $p_{k(s)}$  we obtain the system of linear equations:

$$\begin{aligned}
 & -\lambda_s p_{0(s)} + \mu p_{1(s)} = 0, \quad 1 \leq s \leq l; \\
 & -(\lambda_s + k\mu)p_{k(s)} + \alpha_s \sum_{u=1}^l \lambda_u p_{k-1(u)} + (k+1)\mu p_{k+1(s)} = 0, \quad 1 \leq k \leq n-1, \quad 1 \leq s \leq l; \\
 & -(\lambda_s + n\mu)p_{k(s)} + \alpha_s \sum_{u=1}^l \lambda_u p_{k-1(u)} + n\mu p_{k+1(s)} = 0, \quad n \leq k \leq n+m-1, \quad 1 \leq s \leq l; \\
 & -(\lambda_s + n\mu)p_{n+m(s)} + \alpha_s \sum_{u=1}^l \lambda_u (p_{n+m-1(u)} + p_{n+m(u)}) = 0, \quad 1 \leq s \leq l; \\
 & \sum_{k=0}^{n+m} \sum_{u=1}^l p_{k(u)} = 1.
 \end{aligned} \tag{1}$$

Solving the system (1), we find the steady-state probabilities  $p_k$  of the presence in the queueing system of  $k$  customers using the formulas

$$p_k = \sum_{u=1}^l p_{k(u)}, \quad 0 \leq k \leq n+m. \tag{2}$$

### 3. Features of finding probabilities $p_k$ in the case of complex-valued or paradoxical parameters of $H_l$ -approximation

We calculate the approximate steady-state probabilities  $p_k$  for the  $G/M/n/m$  system using solutions of equations (1), written for the  $H_l/M/n/m$  system, considering the order of approximation  $l$  from 2 to 6.

To find parameters of  $H_l$ -approximation of a certain distribution with a given coefficient of variation it is sufficient to solve the system of equations of the moments method only for the case of any one given mean value of this distribution since roots of the equations of the moments method are invariant with respect to the scale transformation.

The system of equations of the moments method for approximating the distribution of some random variable  $X$  using a random variable  $Y_l$  distributed by law of  $H_l$  is of the form

$$\sum_{s=1}^l \frac{\alpha_s}{\lambda_s^i} = \frac{m_i}{i!}, \quad 0 \leq i \leq 2l - 1; \quad \sum_{s=1}^l \alpha_s = 1, \quad (3)$$

where  $m_i = E(X^i)$  is the initial moment of order  $i$  of the random variable  $X$ . The dependence of the nature of the roots of system (3) on values of the variation coefficient  $V$  for the original gamma distributions and Weibull distributions is described in [8]. For the values  $V < 1$  of the variation coefficient, some of the roots of system (3) are complex-valued but in most cases as a result of summation of probabilities of microstates the steady-state probabilities  $p_k$  are real-valued.

To illustrate this fact, we consider the solutions of system (1) for complex-valued parameters  $\alpha_s$  and  $\lambda_s$ , limited to the case when  $l = 2$ ,  $n = 1$  and  $m = 1$ . In this case, using the solutions of system (1) and formula (2), we obtain

$$\begin{aligned} p_0 &= \frac{\mu^2}{\Delta} ((\alpha_2\lambda_1 + \alpha_1\lambda_2)\mu^2 + \alpha_2\lambda_1^3 + \alpha_1\lambda_2^3 + \\ &\quad + (\alpha_2(\alpha_1 + 2\alpha_2)\lambda_1^2 + 2\alpha_1\alpha_2\lambda_1\lambda_2 + \alpha_1(2\alpha_1 + \alpha_2)\lambda_2^2) \mu), \\ p_1 &= \frac{\lambda_1\lambda_2\mu}{\Delta} (\mu^2 + 2(\alpha_2\lambda_1 + \alpha_1\lambda_2)\mu + \alpha_2\lambda_1^2 + \alpha_1\lambda_2^2), \quad p_2 = 1 - p_0 - p_1, \\ \Delta &= (\alpha_2\lambda_1 + \alpha_1\lambda_2) (\mu^4 + ((\alpha_1 + 2\alpha_2)\lambda_1 + (2\alpha_1 + \alpha_2)\lambda_2) \mu^3 + (\lambda_1 + \lambda_2)^2 \mu^2 + \\ &\quad + ((2\alpha_1 + \alpha_2)\lambda_1 + (\alpha_1 + 2\alpha_2)\lambda_2) \lambda_1\lambda_2\mu + \lambda_1^2\lambda_2^2). \end{aligned} \quad (4)$$

If parameters  $\alpha_s$  and  $\lambda_s$  ( $s = 1, 2$ ) are complex-valued, then they can only be complex conjugate, and all possible cases of alternation of signs before the imaginary unit can be reduced to such two:

$$\begin{aligned} 1) \quad &\alpha_1 = a + ib, \quad \lambda_1 = c + id; \quad \alpha_2 = a - ib, \quad \lambda_2 = c - id; \\ 2) \quad &\alpha_1 = a + ib, \quad \lambda_1 = c - id; \quad \alpha_2 = a - ib, \quad \lambda_2 = c + id. \end{aligned} \quad (5)$$

In each of these cases, the imaginary parts in expressions (4) for  $p_k$  ( $k = 0, 1, 2$ ) are reduced, because the expressions

$$\begin{aligned} &\lambda_1 + \lambda_2, \quad \lambda_1\lambda_2, \quad \alpha_1\alpha_2, \quad \lambda_1^2 + \lambda_2^2, \quad \alpha_2\lambda_1 + \alpha_1\lambda_2, \quad \alpha_1\lambda_1 + \alpha_2\lambda_2, \\ &\alpha_2\lambda_1^2 + \alpha_1\lambda_2^2, \quad \alpha_2\lambda_1^3 + \alpha_1\lambda_2^3, \quad \alpha_2^2\lambda_1^2 + \alpha_1^2\lambda_2^2 \end{aligned}$$

of which consist  $p_k$ , are real-valued.

In the case of complex-valued or paradoxical roots  $\alpha_s$  and  $\lambda_s$  of system (3), let us name the function  $F_{H_l}(t) = 1 - \sum_{s=1}^l \alpha_s e^{-\lambda_s t}$  ( $t \geq 0$ ) the distribution pseudo-function by law of  $H_l$ . Let us show that the function  $F_{H_l}(t)$  is a real-valued function if  $\alpha_s$  and  $\lambda_s$  ( $1 \leq s \leq l$ ) are roots of system (3).

In fact, if some of the roots of system (3) are complex-valued, then they can only be complex conjugate, and all possible cases of alternation of signs before the imaginary unit can be reduced to two cases presented in (5). In each of these cases, the imaginary parts in the expression for  $F_{H_l}(t)$  are reduced, so the result is the real-valued function:

- 1)  $\alpha_1 e^{-\lambda_1 t} + \alpha_2 e^{-\lambda_2 t} = 2 e^{-ct} (a \cdot \cos(dt) + b \cdot \sin(dt))$ ;
- 2)  $\alpha_1 e^{-\lambda_1 t} + \alpha_2 e^{-\lambda_2 t} = 2 e^{-ct} (a \cdot \cos(dt) - b \cdot \sin(dt))$ .

The absolute deviation of the function of distribution by law  $G$  from a function  $F_{H_l}(t)$  which parameters are roots of system (3), we will evaluate with the help of integral

$$\Delta_l(F) = \int_0^{\infty} |F_{H_l}(t) - F_G(t)| dt,$$

where  $F_G(t)$  is the probability distribution function by law  $G$ .

Let  $\Gamma(V)$ ,  $W(V)$  and  $U[a, b]$  denote the gamma distribution, Weibull distribution with coefficients of variation  $V$ , and uniform distribution on the interval  $[a, b]$  respectively.

Table 1 gives deviation values of  $\Delta_l(F)$  for  $l = 2, \dots, 6$ , calculated by results of approximation of different distributions with means 1. With increasing order of  $H_l$ -distribution, the value of deviation  $\Delta_l(F)$  decreases, and with the increase of the coefficient of variation for  $V > 1$  the deviation increases, much faster for the Weibull distribution compared with the gamma distribution. For distributions  $W(0.7)$ ,  $W(0.8)$ ,  $W(0.9)$  and  $W(0.95)$  for some values of  $l$  the deviation  $\Delta_l(F) = \infty$ . In each of these cases, one of roots  $\lambda_s$  of system (3) is real, but negative. Therefore, for the corresponding distribution pseudo-function, the limit relation  $\lim_{t \rightarrow \infty} F_{H_l}(t) = \infty$  is valid. For these values of  $l$ , the steady-state probabilities  $p_k$ , obtained using solutions of equations (1), written for the  $H_l/M/n/m$  system, can be paradoxical.

Table 1: Values of the absolute deviation  $\Delta_l(F)$  for different distributions

Distribution name	$\Delta_2(F)$	$\Delta_3(F)$	$\Delta_4(F)$	$\Delta_5(F)$	$\Delta_6(F)$
$\Gamma(0.001)$	0.3629	0.2605	0.2092	0.1773	0.1549
$U[0, 2]$	0.1139	0.0632	0.0411	0.0295	0.0224
$\Gamma(0.7)$	0.0007	$7.2 \cdot 10^{-5}$	$1.4 \cdot 10^{-5}$	$3.7 \cdot 10^{-6}$	$1.2 \cdot 10^{-6}$
$W(0.7)$	0.0071	0.0026	0.0006	$\infty$	$6.1 \cdot 10^{-5}$
$W(0.8)$	0.0043	$\infty$	0.0004	0.0001	$\infty$
$W(0.9)$	0.0049	0.0005	$\infty$	0.0001	$4.8 \cdot 10^{-5}$
$W(0.95)$	0.0031	0.0005	0.0001	$\infty$	$3.5 \cdot 10^{-5}$
$\Gamma(4)$	0.3146	0.1412	0.0787	0.0497	0.0340
$W(3)$	0.3973	0.2790	0.2170	0.1786	0.1524

Calculations show that properties of the solutions of system (1) almost repeats the form of the roots  $\alpha_s$  ( $1 \leq s \leq l$ ) of system (3). Let's show it on examples of  $U[0, 0.25]/M/n/m$  and  $\Gamma(0.7)/M/n/m$  queueing systems.

For the order of approximation  $l$  from 2 to 6 the roots of system (3) for uniform distribution on the interval  $[0, 0.25]$  are as follows:

$$l = 2 : \alpha_{1,2} = 0.5 \pm 0.86603i, \quad \lambda_{1,2} = 12 \pm 6.92820i;$$

$$l = 3 : \alpha_1 = 2.65193, \quad \alpha_{2,3} = -0.82596 \pm 0.60435i, \\ \lambda_1 = 18.57748, \quad \lambda_{2,3} = 14.7113 \pm 14.03505i;$$

$$l = 4 : \alpha_{1,2} = -0.58906 \mp 0.89679i, \quad \alpha_{3,4} = 1.08906 \pm 4.95602i, \\ \lambda_{1,2} = 16.83032 \pm 21.25934i, \quad \lambda_{3,4} = 23.16968 \pm 6.93787i;$$

$$l = 5 : \alpha_1 = 15.24547, \quad \alpha_{2,3} = 1.02783 \mp 0.49426i, \quad \alpha_{4,5} = -8.15056 \pm 2.37119i, \\ \lambda_1 = 29.17391, \quad \lambda_{2,3} = 18.59739 \pm 28.56818i, \quad \lambda_{4,5} = 26.81565 \pm 13.94129i;$$

$$l = 6 : \alpha_{1,2} = 0.31983 \pm 1.17903i, \quad \alpha_{3,4} = -3.40926 \mp 12.71978i, \\ \alpha_{5,6} = 3.58943 \pm 36.22605i, \\ \lambda_{1,2} = 20.12746 \pm 35.94138i, \quad \lambda_{3,4} = 29.88567 \pm 21.01018i; \\ \lambda_{5,6} = 33.98688 \pm 6.94008i.$$

For  $l = 2$  solutions of the corresponding system (1) are complex conjugate with positive real parts; for  $l = 3$   $p_{k(1)} > 0 \forall k$ ,  $p_{k(2)}$  and  $p_{k(3)}$  are complex conjugate with negative real parts for most values of  $k$ . For  $l = 4$  we have two pairs of complex conjugate solutions with negative real parts for most values of  $k$  in the first pair and with positive real parts  $\forall k$  in the second pair. For  $l = 5$   $p_{k(1)} > 0 \forall k$ , and for  $s = 2, 3$  and  $s = 4, 5$  we have two pairs of complex conjugate solutions  $p_{k(s)}$  with positive real parts in the first pair and with negative real parts in the second pair for most values of  $k$ . For  $l = 6$  we have three pairs of complex conjugate solutions  $p_{k(s)}$  with negative real parts in the second pair and with positive real parts in the first and third pairs.

For the order of approximation  $l$  from 2 to 6 the roots of system (3) for  $\Gamma(0.7)$  distribution with mean 0.125 are as follows:

$$\begin{aligned}
 l = 2: & \quad \alpha_{1,2} = 0.5 \pm 6.18520i, \quad \lambda_{1,2} = 16 \pm 1.31077i; \\
 l = 3: & \quad \alpha_1 = 0.02814, \quad \alpha_{2,3} = 0.48592 \pm 15.70863i, \\
 & \quad \lambda_1 = 39.57700, \quad \lambda_{2,3} = 16.21150 \pm 0.53937i; \\
 l = 4: & \quad \alpha_1 = 0.00548, \quad \alpha_2 = 0.08597, \quad \alpha_{3,4} = 0.45428 \pm 29.37436i, \\
 & \quad \lambda_1 = 69.97401, \quad \lambda_2 = 25.49286, \quad \lambda_{3,4} = 16.26656 \pm 0.29607i; \\
 l = 5: & \quad \alpha_1 = 0.00186, \quad \alpha_2 = 0.01855, \quad \alpha_3 = 0.16685, \quad \alpha_{4,5} = 0.40637 \pm 47.30434i, \\
 & \quad \lambda_1 = 108.92822, \quad \lambda_2 = 37.04067, \quad \lambda_3 = 21.45221, \\
 & \quad \lambda_{4,5} = 16.28945 \pm 0.18743i; \\
 l = 6: & \quad \alpha_1 = 0.00081, \quad \alpha_2 = 0.00679, \quad \alpha_3 = 0.03738, \quad \alpha_4 = 0.26943, \\
 & \quad \alpha_{5,6} = 0.34279 \pm 69.58145i, \\
 & \quad \lambda_1 = 156.50494, \quad \lambda_2 = 51.28750, \quad \lambda_3 = 27.95129, \quad \lambda_4 = 19.65369, \\
 & \quad \lambda_{5,6} = 16.30125 \pm 0.12940i.
 \end{aligned}$$

For  $l$  from 2 to 6 properties of solutions  $p_{k(s)}$  of system (1) in the sense of their signs and whether they are real or complex, completely coincide with the properties of the roots  $\alpha_s$  ( $1 \leq s \leq l$ ) of system (3) with the same numbers.

#### 4. Numerical results

Let us present the results of calculating steady-state probabilities on examples of the  $U[0, 0.25]/M/10/15$ ,  $U[0, 0.125]/M/20/15$  systems and  $\Gamma(V)/M/n/15$ ,  $W(0.9)/M/n/15$  systems for  $n = 10, 20$  and  $V = 0.001, 0.7, 4$ .

Let  $E(T_\lambda)$  denote the mean of the times elapsed between two consecutive arrivals. We take  $E(T_\lambda) = 0.125$  and  $E(T_\lambda) = 0.0625$  for  $n = 10$  and  $n = 20$  respectively, and  $\mu = 1$  is the parameter of exponential distribution of service times.

The obtained results are verified using simulation models constructed with the help of the GPSS World tools [7]. The results obtained using GPSS World slightly differ from one another for different numbers of library random-number generators used for simulating random variables, i.e., times elapsed between two consecutive arrivals and service times. Therefore, we use averaged results obtained using simulation models with different values of random-numbers generators that take on values of natural numbers from 6 to 10.

**Table 2. Results of the calculation of steady-state characteristics of the  $G/M/10/15$  and  $G/M/20/15$  systems with different  $G$ -distributions**

G-distribuiou name, value of $n$	Characte- ristic name	Method of calculation and values of characteristics					
		$H_2$	$H_3$	$H_4$	$H_5$	$H_6$	GPSS World
$\Gamma(0.001)$ , $n = 10$	$N$	8.4783	8.4809	8.4809	8.4809	8.4809	8.4759
	$\Delta_{l(sim)}$	0.0036	0.0026	0.0026	0.0026	0.0026	—
	$\Delta_{l,l-1}$	—	$3.38 \cdot 10^{-3}$	$3.02 \cdot 10^{-5}$	$3.64 \cdot 10^{-7}$	$3.49 \cdot 10^{-9}$	—
$\Gamma(0.001)$ , $n = 20$	$N$	16.2230	16.2245	16.2245	16.2245	16.2245	16.1875
	$\Delta_{l(sim)}$	0.0102	0.0095	0.0095	0.0095	0.0095	—
	$\Delta_{l,l-1}$	—	$2.21 \cdot 10^{-3}$	$1.07 \cdot 10^{-5}$	$9.55 \cdot 10^{-8}$	$1.24 \cdot 10^{-9}$	—
$U[0, 1/8]$ , $n = 10$	$N$	8.8186	8.8206	8.8206	8.8206	8.8206	8.8138
	$\Delta_{l(sim)}$	0.0034	0.0023	0.0023	0.0023	0.0023	—
	$\Delta_{l,l-1}$	—	$2.73 \cdot 10^{-3}$	$4.64 \cdot 10^{-5}$	$1.16 \cdot 10^{-6}$	$3.57 \cdot 10^{-8}$	—
$U[0, 1/16]$ , $n = 20$	$N$	16.4409	16.4422	16.4422	16.4422	16.4422	16.4356
	$\Delta_{l(sim)}$	0.0038	0.0038	0.0038	0.0038	0.0038	—
	$\Delta_{l,l-1}$	—	$1.81 \cdot 10^{-3}$	$1.93 \cdot 10^{-5}$	$4.05 \cdot 10^{-7}$	$1.36 \cdot 10^{-8}$	—
$\Gamma(0.7)$ , $n = 10$	$N$	8.9531	8.9531	8.9531	8.9531	8.9531	8.9605
	$\Delta_{l(sim)}$	0.0017	0.0017	0.0017	0.0017	0.0017	—
	$\Delta_{l,l-1}$	—	$2.11 \cdot 10^{-5}$	$9.39 \cdot 10^{-8}$	$1.21 \cdot 10^{-9}$	$2.55 \cdot 10^{-11}$	—
$\Gamma(0.7)$ , $n = 20$	$N$	16.5339	16.5339	16.5339	16.5339	16.5339	16.5306
	$\Delta_{l(sim)}$	0.0027	0.0027	0.0027	0.0027	0.0027	—
	$\Delta_{l,l-1}$	—	$1.41 \cdot 10^{-5}$	$4.20 \cdot 10^{-8}$	$4.46 \cdot 10^{-10}$	$1.08 \cdot 10^{-11}$	—
$W(0.9)$ , $n = 10$	$N$	9.2464	9.2463	—	9.2463	9.2463	9.2428
	$\Delta_{l(sim)}$	0.0019	0.0019	—	0.0019	0.0019	—
	$\Delta_{l,l-1}$	—	$1.97 \cdot 10^{-4}$	—	—	$4.55 \cdot 10^{-9}$	—
$W(0.9)$ , $n = 20$	$N$	16.7440	16.7440	—	16.7440	16.7440	16.7426
	$\Delta_{l(sim)}$	0.0027	0.0027	—	0.0027	0.0027	—
	$\Delta_{l,l-1}$	—	$1.40 \cdot 10^{-4}$	—	—	$2.04 \cdot 10^{-9}$	—
$\Gamma(4)$ , $n = 10$	$N$	10.1667	9.7778	9.7522	9.7532	9.7536	9.7521
	$\Delta_{l(sim)}$	0.0784	0.0216	0.0094	0.0052	0.0032	—
	$\Delta_{l,l-1}$	—	0.0680	0.0146	$5.41 \cdot 10^{-3}$	$2.25 \cdot 10^{-3}$	—
$\Gamma(4)$ , $n = 20$	$N$	16.5677	16.2055	16.1864	16.1874	16.1876	16.1773
	$\Delta_{l(sim)}$	0.0652	0.0183	0.0092	0.0062	0.0048	—
	$\Delta_{l,l-1}$	—	0.0576	0.0117	$4.41 \cdot 10^{-3}$	$1.83 \cdot 10^{-3}$	—

Let us introduce the designation:  $N$  is the average number of customers in a queueing system, and

$$\Delta_{(l,l-1)} = \sum_{k=0}^{n+15} |p_{k(l)} - p_{k(l-1)}|, \quad \Delta_{l(sim)} = \sum_{k=0}^{n+15} |p_{k(l)} - p_{k(sim)}|,$$

$$p_{k(sim)} = \frac{1}{5} \sum_{i=6}^{10} p_{k(sim,i)}, \quad 0 \leq k \leq n+15, \quad 2 \leq l \leq 6.$$

Here  $p_{k(l)}$  are values of probabilities  $p_k$  obtained using the  $H_l$ -approximation,  $p_{k(sim)}$  is the average value of probabilities  $p_{k(sim,i)}$ , obtained by means of the simulation model using the number  $i$  of random-numbers generator for  $6 \leq i \leq 10$ . Thus, the quantities  $\Delta_{l(sim)}$  are measures of deviations of the distributions  $\{p_{k(l)}\}$  from distribution  $\{p_{k(sim)}\}$ , and the quantities  $\Delta_{(l,l-1)}$  give an opportunity to estimate the deviation of distributions  $\{p_{k(l)}\}$  from distributions  $\{p_{k(l-1)}\}$ .

In Table 2 we present the results of calculation of steady-state characteristics of the  $G/M/10/15$  and  $G/M/20/15$  systems with the considered gamma, Weibull and uniform distributions. The values of deviations  $\Delta_{l(sim)}$  and  $\Delta_{(l,l-1)}$  decrease with increasing order of  $H_l$ -distributions in approximations, and it means that the values of distribution  $\{p_{k(l)}\}$  with each step getting closer to a true distribution  $\{p_k\}$ . With the growth of the variation coefficient of distributions after the value of  $V > 1$ , as expected taking into account the behavior of deviations  $\Delta_l(F)$ , the values of the absolute deviations  $\Delta_{l(sim)}$  and  $\Delta_{(l,l-1)}$  also increase. For the distribution  $W(0.9)$  the deviation  $\Delta_4(F) = \infty$  and, consequently, some values of "probabilities" of the distribution  $\{p_{k(4)}\}$  go beyond the interval  $[0, 1]$ .

Presented results show that increasing the number of channels of the  $G/M/n/m$  system has no significant effect on accuracy of calculating the steady-state probabilities.

Testing the proposed method on the  $M/G/1/m$  systems, for which we can find exact values of the steady-state distribution  $\{p_k\}$ , shows that in cases where the deviation  $\Delta_{(6,5)}$  is less than  $10^{-2}$ , the deviation of the distribution  $\{p_{k(l)}\}$  from the true distribution  $\{p_k\}$  and deviation  $\Delta_{(l+1,l)}$  are numbers of the same order, and at the same time the deviations of distribution  $\{p_{k(sim)}\}$  from the distribution  $\{p_k\}$  usually no less than  $10^{-4}$ . Thus, in most cases we can use values  $\Delta_{(l,l-1)}$  to evaluate accuracy of the approximation of the distribution  $\{p_{k(l-1)}\}$  to the true  $\{p_k\}$  for  $3 \leq k \leq 6$ . In cases where  $\Delta_{(l,l-1)} < 10^{-4}$ , we can argue that the distribution  $\{p_{k(l-1)}\}$  is more accurate approximation than  $\{p_{k(sim)}\}$ .

## 5. Conclusions

This paper shows that the application of hyperexponential approximation of distributions the times elapsed between two consecutive arrivals allows us to calculate steady-state probabilities of the  $G/M/n/m$  queueing systems with high accuracy (higher than in the case of using simulation models). We find these probabilities using solutions of a system of linear algebraic equations obtained by the method of fictitious phases.

To obtain parameters of  $H_l$ -approximation of a certain distribution it is necessary to solve the system of equations of the moments method. For the values  $V < 1$  of the



variation coefficient, some of the roots of this system are complex-valued or, having a sense of probabilities, go beyond the interval  $[0, 1]$ , but in most cases the final result is close to the desired distribution  $\{p_k\}$ .

Computing deviations  $\Delta_{(l,l-1)}$  allows us to track the accuracy of approaching distributions  $\{p_{k(l-1)}\}$  to the true distribution  $\{p_k\}$  without the need to use simulation models.

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Soltan A. Aliyev

*Institute of Mathematics and Mechanics of NAS of Azerbaijan, AZ 1141, Baku, Azerbaijan*  
 Email: soltanaliyev@yahoo.com

Yaroslav I. Yeleyko

*Ivan Franko National University of Lviv, Lviv, Ukraine*  
*Jan Kochanowski University in Kielce, Kielce, Poland*  
 Email: yikts@yahoo.com

Yuriy V. Zhernovyi

*Ivan Franko National University of Lviv, Lviv, Ukraine*

*Email: yu.zhernovyi@lnu.edu.ua*

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