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On the 3D Dynamic Normal Stress Field on the Interface of the Bi-layered Hollow Cylinder Under Action a Moving Load in the Interior of That

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Abstract. The paper studies normal stress field on the interface surface of the bi-layered hollow cylinder under action on the interior of that the moving load in the 3D state with utilizing the exact equations and relations of the elastodynamics. It is assumed that in the interior of the cylinder the point located with respect to the cylinder axis moving forces act and the distribution of these forces is non-axisymmetric and is located within a certain central angle. To solve the corresponding mathematical problem the moving coordinate system is used and the Fourier transform of with respect to the axial coordinate is employed. These transforms are presented in the Fourier series form with respect to the circumferential coordinate and the coefficients of these series are found analytically from the corresponding field equations and relations. The inverses of the mentioned transforms are determined numerically as a result of which normal radial stress acting on the interface surface between the layers of the cylinder is analyzed. It is examined the influence of the problem parameters such as moving load velocity, the thicknesses ration of the cylinder's layers, the ration of the inner layer thickness to the external radius of the cross-section of this layer and material properties of the layers to the stress response to the moving load.

1. Introduction

In the paper [1, 2] studied the corresponding 3D dynamic problem for the system consisting of the hollow cylinder and surrounding elastic medium and the review of the related other investigations were considered in the papers [1 − 4]. Consequently, the present paper attempt to develop the investigations started in the paper [1] for the bi-layered hollow cylinder.

Note that detailed consideration of the dynamics of the bi-material elastic systems has been made in the monograph [5] from which and from the other reviews made in the papers [1- 4] follows that up to now the regarding investigations have been made mainly for axisymmetric cases (except the study carried out in the papers [1, 2]). Therefore, each investigation on the 3D dynamics of the cylindrical bi-material systems can be taken as new knowledge in this field which has not only theoretical and application sense.

Taking the foregoing discussion into consideration, in the present paper it is made the attempt to investigate, within the scope of the 3D elastodynamics, normal interface stress on the interface surface of the bi-layered hollow cylinder in the case wherein the interior of the cylinder the moving load acting within a certain arc and point located with respect to the axial coordinate moving load acts.

2. Formulation of the problem

We introduce to the consideration a bi-layered hollow cylinder the sketch of which is illustrated in Fig. 1 and assume that the thicknesses of the walls of the inner and outer cylindersare$h^{(2)}$and $h^{(1)}$respectively, and the external radius of the cross section of the inner cylinder is$R$. We denote by the upper index (2) (by the upper index (1) the values related to the inner (outer) layer of the cylinder and associate the cylindrical system of coordinates $Orz\theta$(Fig. 1a) with the axis of the cylinder. Moreover, we assume that in the interior of the inner hollow cylinder a point located with respect to the cylinder axis and that non-uniformly distributed in the circumferential direction (Fig. 1b) moving normal forces act and these forces move with constant velocity V in the Oz axis direction. Thus, within these framework we attempt to investigate the non-axisymmetric dynamic response of the bi-layered hollow cylinder to the moving forces and analyze the response of the interface normal stress to these forces.

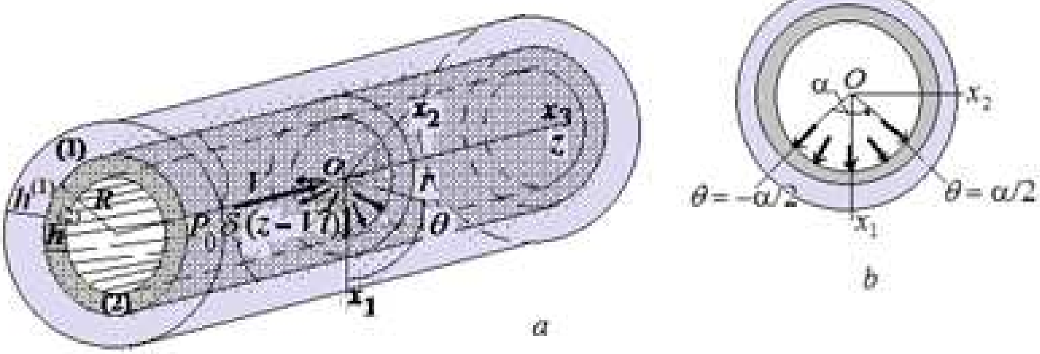


Fig. 1.The sketch of the considered system (a) and the sketch of the distribution of the non-axisymmetric normal forces (b)

We write the following complete system of field equations of the 3D elastodynamics, as well as the corresponding boundary and contact conditions within the framework of which the present investigation will be made.

Equations of motion:

$$\frac{\partial\sigma_{rr}^{(m)}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{r\theta}^{(m)}}{\partial\theta}+\frac{\partial\sigma_{rz}^{(m)}}{\partial z}+\frac{1}{r}(\sigma_{rr}^{(m)}-\sigma_{\theta\theta}^{(m)})=\rho^{(m)}\frac{\partial^2u_r^{(m)}}{\partial t^2}$$

$$\frac{\partial\sigma_{r\theta}^{(m)}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{\theta\theta}^{(m)}}{\partial\theta}+\frac{\partial\sigma_{z\theta}^{(m)}}{\partial z}+\frac{2}{r}\sigma_{r\theta}^{(m)}=\rho^{(m)}\frac{\partial^2u_\theta^{(m)}}{\partial t^2}$$

$$\frac{\partial\sigma_{rz}^{(m)}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{z\theta}^{(m)}}{\partial\theta}+\frac{\partial\sigma_{zz}^{(m)}}{\partial z}+\frac{1}{r}\sigma_{rz}^{(m)}=\rho^{(m)}\frac{\partial^2 u_z^{(m)}}{\partial t^2}. \tag{1}$$

Elasticity relations:

$$\sigma_{rr}^{(m)}=(\lambda^{(m)}+2\mu^{(m)})\frac{\partial u_r^{(m)}}{\partial r}+\lambda^{(m)}\frac{1}{r}(\frac{\partial u_\theta^{(m)}}{\partial r}+u_r^{(m)})+\lambda^{(m)}\frac{\partial u_z^{(m)}}{\partial z},$$

$$\sigma_{\theta\theta}^{(m)}=\lambda^{(m)}\frac{\partial u_r^{(m)}}{\partial r}+(\lambda^{(m)}+2\mu^{(m)})\frac{1}{r}(\frac{\partial u_\theta^{(m)}}{\partial r}+u_r^{(m)})+\lambda^{(m)}\frac{\partial u_z^{(m)}}{\partial z},$$

$$\sigma_{zz}^{(m)}=\lambda^{(m)}\frac{\partial u_r^{(m)}}{\partial r}+\lambda^{(m)}\frac{1}{r}(\frac{\partial u_\theta^{(m)}}{\partial r}+u_r^{(m)})+(\lambda^{(m)}+2\mu^{(m)})\frac{\partial u_z^{(m)}}{\partial z},$$

$$\sigma_{r\theta}^{(m)}=\mu^{(m)}\frac{\partial u_\theta^{(m)}}{\partial r}+\mu^{(m)}(\frac{1}{r}\frac{\partial u_r^{(m)}}{\partial\theta}-\frac{1}{r}u_\theta^{(m)}),$$

$$\sigma_{z\theta}^{(m)}=\mu^{(m)}\frac{\partial u_\theta^{(m)}}{\partial z}+\mu^{(k)}\frac{\partial u_z^{(m)}}{r\partial\theta},\sigma_{zr}^{(k)}=\mu^{(k)}\frac{\partial u_r^{(k)}}{\partial z}+\mu^{(k)}\frac{\partial u_z^{(k)}}{\partial r}. \tag{2}$$

The conventional notation is use in equations (1) and (2).
The corresponding boundary and contact conditions for the case under consideration can be formulated as follows.

$$\sigma_{rr}^{(2)}\Big|_{r=R-h^{(2)}}=\begin{cases}-P_\alpha\delta(z-Vt) \text{ for } -\alpha/2\le\theta\le\alpha/2\\ 0 \text{ for } \theta\in([-\pi,+\pi]-[-\alpha/2,\alpha/2])\end{cases},$$

$$\sigma_{r\theta}^{(2)}\Big|_{r=R-h^{(2)}}=0,\quad \sigma_{rz}^{(2)}\Big|_{r=R-h^{(2)}}=0,$$

$$\sigma_{rr}^{(1)}\Big|_{r=R+h^{(1)}}=0,\quad \sigma_{r\theta}^{(1)}\Big|_{r=R+h^{(1)}}=0,\quad \sigma_{rz}^{(1)}\Big|_{r=R+h^{(1)}}=0, \tag{3}$$

$$\sigma_{rr}^{(1)}\Big|_{r=R}=\sigma_{rr}^{(2)}\Big|_{r=R},\quad \sigma_{r\theta}^{(1)}\Big|_{r=R}=\sigma_{r\theta}^{(2)}\Big|_{r=R},\quad \sigma_{rz}^{(1)}\Big|_{r=R}=\sigma_{rz}^{(2)}\Big|_{r=R},$$

$$u_r^{(1)}\Big|_{r=R}=u_r^{(2)}\Big|_{r=R},\quad u_\theta^{(1)}\Big|_{r=R}=u_\theta^{(2)}\Big|_{r=R},\quad u_z^{(1)}\Big|_{r=R}=u_z^{(2)}\Big|_{r=R}, \tag{4}$$

$$\left|\sigma_{rr}^{(1)}\right|;\ \left|\sigma_{\theta\theta}^{(1)}\right|;\ \left|\sigma_{zz}^{(1)}\right|;\ \left|\sigma_{r\theta}^{(1)}\right|;\ \left|\sigma_{rz}^{(1)}\right|;\ \left|\sigma_{\theta z}^{(1)}\right|;$$

$$\left|u_r^{(1)}\right|;\ \left|u_\theta^{(1)}\right|;\ \left|u_z^{(1)}\right|\to 0 \text{ as } \sqrt{(z-Vt)^2}\to+\infty, \tag{5}$$

where in (3)P_α is determined from the following relation

$$\int_{-\alpha/2}^{+\alpha/2}P_\alpha(R-h)\cos\theta d\theta=(R-h)P_0=const\Rightarrow\ P_\alpha=P_0/(2\sin(\alpha/2)). \tag{6}$$

Thus, the investigation of the response of the interface normal stress to the moving load is reduced to the boundary-contact problem (1) – (5) for solution to which the method developed in the papers [1,2] is employed. Now we consider some fragments of the application of the mentioned method for the problem under consideration.

3. Method of solution

As in the papers [1, 2] for solution to the foregoing mathematical problem, according to [6], we use the following representation:

$$u_r^{(m)} = \frac{1}{r}\frac{\partial}{\partial\theta}\Psi^{(m)} - \frac{\partial^2}{\partial r\partial z}\mathrm{X}^{(m)}, \quad u_\theta^{(m)} = -\frac{\partial}{\partial r}\Psi^{(m)} - \frac{1}{r}\frac{\partial^2}{\partial\theta\partial z}\mathrm{X}^{(m)},$$

$$u_z^{(m)} = (\lambda^{(m)} + \mu^{(m)})^{-1}\left((\lambda^{(m)} + 2\mu^{(m)})\Delta_1 + \mu^{(m)}\frac{\partial^2}{\partial z^2} - \rho^{(m)}\frac{\partial^2}{\partial t^2}\right)\mathrm{X}^{(m)},$$

$$\Delta_1 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}, m = 1, 2 \tag{7}$$

In (7) the functions $\Psi^{(m)}$ and $\mathrm{X}^{(m)}$ are the solutions of the equations

$$\left(\Delta_1 + \frac{\partial^2}{\partial z^2} - \frac{\rho^{(k)}}{\mu^{(k)}}\frac{\partial^2}{\partial t^2}\right)\Psi^{(m)} = 0, \left[\left(\Delta_1 + \frac{\partial^2}{\partial z^2}\right)\left(\Delta_1 + \frac{\partial^2}{\partial z^2}\right) - \right.$$

$$-\rho^{(m)}\frac{\lambda^{(m)} + 3\mu^{(m)}}{\mu^{(m)}(\lambda^{(m)} + 2\mu^{(m)})}\times$$

$$\left.\times(\Delta_1 + \frac{\partial^2}{\partial z^2})\right)\frac{\partial^2}{\partial t^2} + \frac{(\rho^{(m)})^2}{\mu^{(m)}(\lambda^{(m)} + 2\mu^{(m)})}\frac{\partial^4}{\partial t^4}\Bigg]\mathrm{X}^{(m)} = 0. \tag{8}$$

We introduce a moving cylindrical coordinate system $O'r'\theta'z'$ which is connected with the reference cylindrical coordinate system $Or\theta z$ through the following relations:

$$r' = r, \quad \theta' = \theta, \quad z' = z - Vt. \tag{9}$$

As a result of the employing of the moving coordinate system (9), the operators $\partial^2/\partial t^2$ and $\partial^4/\partial t^4$ in the foregoing equations are replaced with the operators $V^2\partial^2/\partial z'^2$ and $V^4\partial^4/\partial z'^4$, respectively, and in this way, equations rewritten in the moving coordinate system, are obtained. The exponential Fourier transform $f_F = \int_{-\infty}^{+\infty} f(z')e^{isz'}dz'$ with respect to the moving coordinate z' (where s is a transformation parameter) is applied to all the equations and relations rewritten with the moving coordinates.

Below, we will make all mathematical operations with the moving coordinates and will omit the upper primes over them.

According to the problem statement,we use the following presentations for the originals of the sought values.

$$\left\{\sigma_{rr}^{(m)};\ \sigma_{\theta\theta}^{(m)};\sigma_{zz}^{(m)};\sigma_{r\theta}^{(m)};u_r^{(m)};u_\theta^{(m)};\ \Psi^{(m)}\right\} =$$

$$\frac{1}{\pi}\int_0^{+\infty}\left\{\sigma_{rrF}^{(m)};\ \sigma_{\theta\theta F}^{(m)};\sigma_{zzF}^{(m)};\sigma_{r\theta F}^{(m)};u_{rF}^{(m)};u_{\theta F}^{(m)};\ \Psi_F^{(m)}\right\}\cos(sz)ds,$$

$$\left\{\sigma_{\theta z}^{(m)};\ \sigma_{rz}^{(m)};u_z^{(m)};\mathrm{X}^{(m)}\right\} = \frac{1}{\pi}\int_0^{+\infty}\left\{\sigma_{\theta zF}^{(m)};\ \sigma_{rzF}^{(m)};u_{zF}^{(m)};\ \mathrm{X}_F^{(m)}\right\}\sin(sz)ds. \tag{10}$$

Substituting the expressions in Eq. (10) into the equations in (8) and into the rewritten relations in the moving coordinate system, it is obtained the following equations for the functions $\Psi_F^{(m)}$ and $\mathrm{X}_F^{(m)}$:

$$\left(\Delta_1 - s^2(1-\frac{\rho^{(k)}}{\mu^{(k)}}V^2)\right)\Psi_F^{(m)} = 0,$$

$$\left[\left(\Delta_1 - s^2\right)\left(\Delta_1 - s^2\right) - \rho^{(m)}\frac{\lambda^{(m)}+3\mu^{(m)}}{\mu^{(m)}(\lambda^{(m)}+2\mu^{(m)})}\times\right.$$

$$\left.(\Delta_1 - s^2))(-s^2V^2) + \frac{(\rho^{(m)})^2}{\mu^{(m)}(\lambda^{(m)}+2\mu^{(m)})}s^4V^4\right]\mathrm{X}_F^{(m)} = 0. \qquad (11)$$

According to the periodicity of the problem under consideration with respect to the circumferential coordinate θ, the Fourier transform of the functions $\Psi_F^{(m)}$ and $\mathrm{X}_F^{(m)}$can be presented in the Fourier series form as follows.

$$\Psi_F^{(m)}(r,s,\theta) = \sum_{n=1}^{\infty}\Psi_{Fn}^{(m)}(r,s)\sin n\theta, \quad \mathrm{X}_F^{(m)}(r,s,\theta) = \frac{1}{2}\mathrm{X}_{F0}^{(m)}(r,s) + \sum_{n=1}^{\infty}\mathrm{X}_{Fn}^{(m)}(r,s)\cos n\theta. \qquad (12)$$

In this way, we obtain from expressions in (12) andequations in (11) the following equation:

$$\left(\Delta_{1n} - (\zeta_1^{(m)})^2\right)\psi_{Fn}^{(m)} = 0, \quad \left(\Delta_{1n} - (\zeta_2^{(m)})^2\right)\left(\Delta_{1n} - (\zeta_3^{(m)})^2\right)\mathrm{X}_{Fn}^{(m)} = 0,$$

$$\Delta_{1n} = \frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{\mathrm{d}}{r\mathrm{d}r} - \frac{n^2}{r^2}, \qquad (13)$$

where

$$(\zeta_1^{(m)})^2 = s^2\left(1 - \frac{\rho^{(m)}V^2}{\mu^{(m)}}\right) \qquad (14)$$

$(\zeta_2^{(m)})^2$ and $(\zeta_3^{(m)})^2$ in (13) are determined from the solutions of the following equation.

$$\mu^{(m)}(\zeta^{(m)})^4 - s^2(\zeta^{(m)})^2\left[-\rho^{(m)}V^2 - (\lambda^{(m)}+2\mu^{(m)}) +\right.$$

$$\left.+\frac{\mu^{(m)}}{\lambda^{(m)}+2\mu^{(m)}}\left(-\rho^{(m)}V^2 - \mu^{(m)}\right) + \frac{(\lambda^{(m)}+\mu^{(m)})^2}{\lambda^{(m)}+2\mu^{(m)}}\right] +$$

$$s^4\left(\frac{-\rho^{(m)}V^2}{\lambda^{(m)}+2\mu^{(m)}} - 1\right)\left(-\rho^{(m)}V^2 - \mu^{(m)}\right) = 0. \qquad (15)$$

The solutions to equations in (13) are determined as follows:

$$\psi_{Fn}^{(m)} = A_{1n}^{(m)}I_n(\zeta_1^{(m)}r) + B_{1n}^{(m)}K_n(\zeta_1^{(m)}r), \quad \chi_{Fn}^{(m)} = A_{2n}^{(m)}I_n(\zeta_2^{(m)}r) + A_{3n}^{(m)}I_n(\zeta_3^{(m)}r) +$$

$$B_{2n}^{(m)} K_n(\zeta_2^{(m)} r) + B_{3n}^{(m)} K_n(\zeta_3^{(m)} r), \quad m = 1, 2. \tag{16}$$

Using (16), (12), (7) and (2) it is completely determined the Fourier transforms of the sought values.Finally, using the algorithm developed and applied in the papers [1-4] the originals of these values are determined. Note that one of the main procedures of this algorithm is the determination of the unknown constants $A_{1n}^{(m)}$, $B_{1n}^{(m)}$, $A_{2n}^{(m)}$, $B_{2n}^{(m)}$, $A_{3n}^{(m)}$ and $B_{3n}^{(m)}$ for which it is obtained a complete system of algebraic equations from the boundary and contact conditions in (3) and (4) respectively.

This completes the consideration of the solution method more detail version of which is given in the papers [1, 2].

4. Numerical results

In the present paper, we will consider numerical results related to the interface normal stress acting on the interface surface between the layers of the cylinder. The algorithm for obtaining numerical results are detailed in the works [1-5] and therefore do not consider here again that. Nevertheless, we note that under obtaining numerical results we take twenty terms in the series in (12). Moreover, we note that these results are obtained for the following two cases:

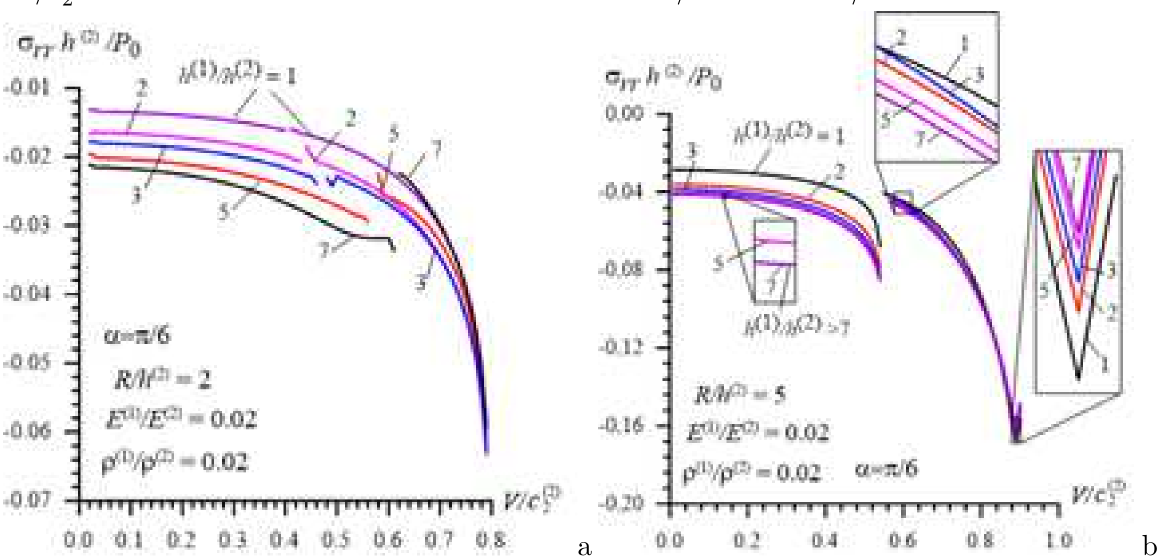
Case 1. $E^{(1)}/E^{(2)} = 0.02, \quad \rho^{(1)}/\rho^{(2)} = 0.01, \quad \nu^{(1)} = \nu^{(2)} = 0.25,$

Case 2. $E^{(1)}/E^{(2)} = 0.5$, $\rho^{(2)}/\rho^{(2)} = 0.5$, $\nu^{(1)} = \nu^{(2)} = 0.3$.

Assume that θ= 0, z/h = 0 and α=$\pi/6$, and consider the graphs of the dependencies between

$$\sigma_{rr} = \sigma_{rr}^{(1)}\Big|_{r=R} = \sigma_{rr}^{(2)}\Big|_{r=R} \tag{17}$$

and $V/c_2^{(2)}$ constructed for various values of the ratios $R/h^{(2)}$ and $h^{(1)}/h^{(2)}$.



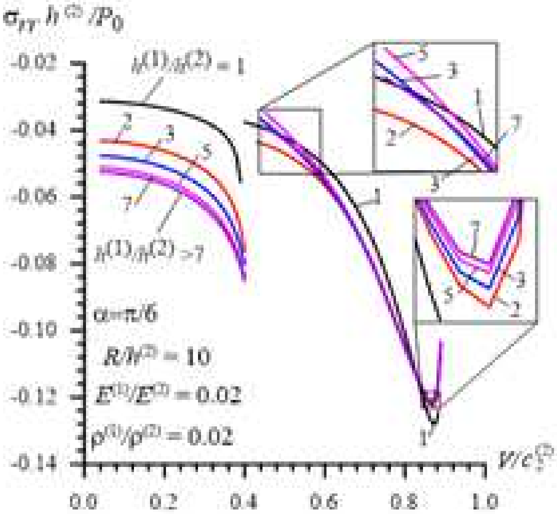


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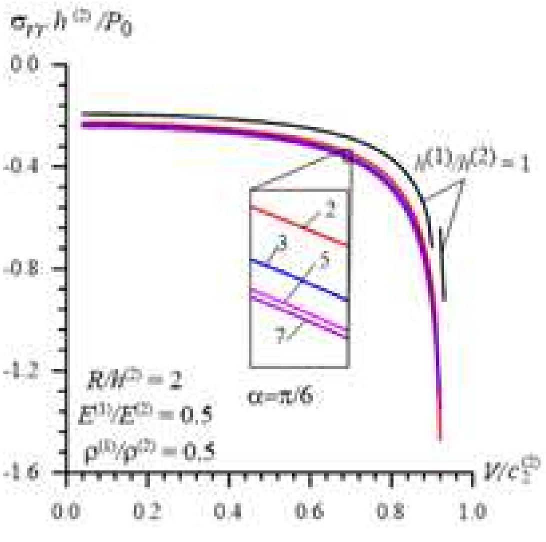
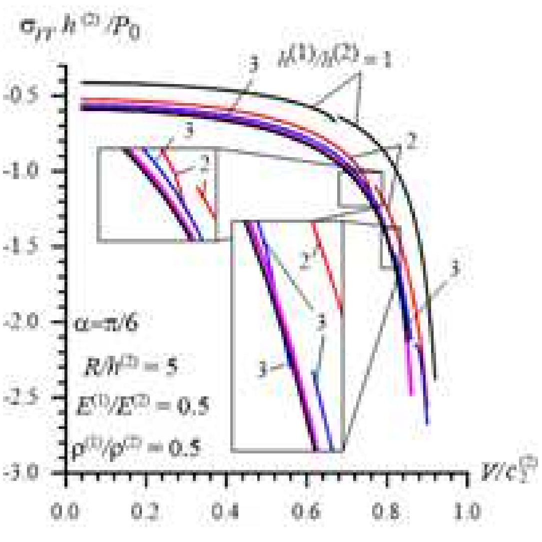
Fig.2. Response of the interface normal stress to the moving load velocity in Case 1 under $R/h^{(2)} = 2$(a), 5 (b) and 10 (c)



a



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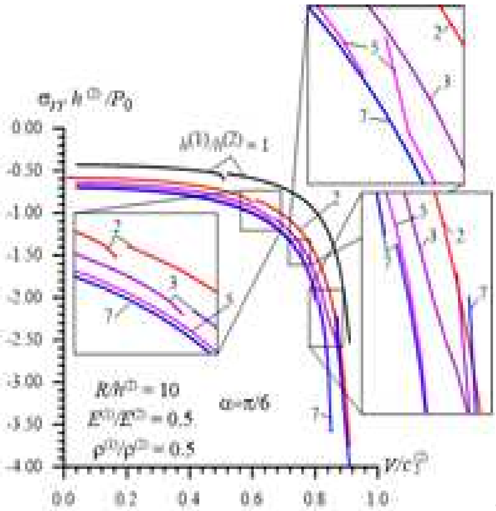

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Fig.3. Response of the interface normal stress to the moving load velocity in Case 2 under $R/h^{(2)} = 2$(a), 5 (b) and 10 (c)

The mentioned graphs are presented in Figs. 2 and 3 for Case 1 and Case 2 respectively for various values of the ratio $h^{(1)}/h^{(2)}$ under $R/h^{(2)} = 2$ (a), 5 (b) and 10 (c). Note that these graphs have a discontinuity at certain values of the dimensionless moving velocity $V/c_2^{(2)}$ which indicates the corresponding critical velocities. Moreover note that, in general, in 3D moving load problems in the subsonic regime there exist two critical velocities, however, in the axisymmetric moving load problems one.

Thus, it follows from the graphs that before the first critical velocity the absolute values of the interface dimensionless normal stress $\sigma_{rr}h^{(2)}/P_0$ increase monotonically with $V/c_2^{(2)}$. At the same time, an increase in the values of the ratio $h^{(1)}/h^{(2)}$ also causes to increase the absolute values of the stress and in the cases under consideration for $h^{(1)}/h^{(2)} \geq 7$ coincide with the corresponding ones obtained in the paper [1], i.e. with the corresponding results which were obtained for the "hollow cylinder + surrounding medium" system. This statement confirms the validity of the calculation algorithm and PC programs used under obtaining of the present results. Moreover, this statement agrees with the well-known physicomechanical and engineering considerations.

Comparison of the results obtained for Case 1 (Fig. 2) with corresponding ones obtained for Case 2 (Fig. 3) shows that the absolute values of the stress obtained in Case 2 is greater significantly than those obtained in Case 1. This situation can be established with the relation $\left(E^{(1)}/E^{(2)}\right)_{Case1} \ll \left(E^{(1)}/E^{(2)}\right)_{Case2}$which also agrees with the engineering consideration.

With this, we restrict ourselves to consideration of the numerical results related to the interface normal stresses obtained for problem under consideration and note that this consideration will be continued in the further works by the author.

5. Conclusions

Thus, in the present paper, the 3D dynamic problem of the moving load acting in the interior of the bi-layered hollow cylinder is studied with employing 3D exact equations of elastodynamics and the numerical results on the response of the interface normal stress to the moving load velocity are presented and discussed. It is assumed that the forces acting in the interior of the inner layer of the cylinder is point located with respect to the axial coordinate and is distributed along a certain arc within the corresponding central angle.

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