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The System of Convolution Equations in Concrete Banach Space

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Abstract. The regularity properties of degenerate abstract convolution-elliptic equations are investigated. We prove that the corresponding convolution-elliptic operator is $R-$sectorial and is also a negative generator of an analytic semigroup. These results permit us to, show the separability of the differential operators in a $E-$valued weighted spaces. By using these results integro-differential equations in concrete weighted Banach space $L_{p,\gamma}\left(R^n;l_q\right)$ are obtained.

Key Words and Phrases: $R-$sectorial operators, abstract weighted spaces, operator-valued multipliers, convolution equations, integro-differential equations.
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1. Introduction, notations and background

Regularity properties of differential operator equations, especially elliptic and parabolic type have been studied extensively e.g in [1], [2], [4], [7-8], [12] , [16-18] , [21-22] and the references therein. Moreover, convolution-differential equations (CDEs) have been treated e.g. in [4] , [15]. Convolution operators in Banach-valued functıon spaces studied e.g. in [3] , [10] , [13], [16] , [17] , [18]. However, the convolution-differential operator equations (CDOEs) are relatively less investigated subject. In [4] the parabolic type CDEs with bounded operator coefficient was investigated. In [18] regularity properties of degenerate CDOEs are studied. The main aim of the present paper is to study the maximal L_p-regularity properties of the following degenerate integro-differential equations

$$\sum_{|\alpha|\leq l} a_\alpha * D^{[\alpha]}u_m + \sum_{m=1}^{\infty} d_m * u_m = f_m, \tag{1.1}$$

in concrete weighted Banach space $L_{p,\gamma}\left(R^n;l_q\right)$, where l is a natural number, $a_\alpha = a_\alpha\left(x\right)$ are complex-valued functions, $d_j = d_j\left(x\right)$, $u_j = u_j(x)$, $f_m = f_m(x)$, $\alpha = \left(\alpha_1, \alpha_2, ..., \alpha_n\right)$, α_k are nonnegative integers, $|\alpha| = \sum\limits_{k=1}^{n} \alpha_k$, λ is a complex parameter and $A = A\left(x\right)$ is a linear operator in a Banach space E for $x \in R^n$.

In this paper, first we establish the uniform separability properties of the linear CDOEs and the uniform maximal regularity of the infinite system of degenrate integro-differential equations (1.1). Moreover, we prove that the operator generated by problem linear CDOEs is $R-$sectorial. Since the equation (1.1) has an unbounded operator coefficient, some difficulties occur. This fact is derived by using the representation formula for the solution of corresponding convolution equation and operator valued multipliers in $E-$ valued weighted L_p- spaces.

We start by giving the notation and definitions to be used in paper.

Let E be a Banach space and $\gamma=\gamma(x)$, $x=(x_1,x_2,...,x_n)$ be a positive measurable weighted function on a measurable subset $\Omega\subset R^n$. Let $L_{p,\gamma}(\Omega;E)$ denote the space of strongly $E-$valued functions that are defined on Ω with the norm

$$\|f\|_{L_{p,\gamma}}=\|f\|_{L_{p,\gamma}(\Omega;E)}=\left(\int\limits_{\Omega}\|f(x)\|_E^p\,\gamma(x)dx\right)^{1/p},\ 1\le p<\infty,$$

$$\|f\|_{L_{\infty,\gamma}(\Omega;E)}=ess\sup_{x\in\Omega}\left[\gamma(x)\,\|f(x)\|_E\right].$$

The weight $\gamma=\gamma\left(x\right)$ satisfy an A_p condition, i.e., $\gamma\in A_p$, $p\in(1,\infty)$ if there is a positive constant C such that

$$\sup_Q\left(\frac{1}{|Q|}\int\limits_Q\gamma(x)dx\right)\left(\frac{1}{|Q|}\int\limits_Q\gamma^{-\dfrac{1}{p-1}}(x)dx\right)^{p-1}\le C$$

for all cubes $Q\subset R^n$ (see e.g [11, Ch.9]).

The result [20] implies that the space l_q for $q\in(1,\infty)$ satisfies multiplier condition with respect to $p\in(1,\infty)$ and the weight functions $\gamma\left(x\right)=\prod\limits_{k=1}^{n}|x_k|^{\nu}$ for $-\frac{1}{n}<\nu<\frac{1}{n}\left(p-1\right)$.

Here, $\mathbb{N}$ denotes the set of natural numbers. $\mathbb{R}$ denotes the set of real numbers. Let $\mathbb{C}$ be the set of complex numbers and

$$S_\varphi=\{\lambda\in\mathbb{C},\quad|\arg\lambda|\le\varphi\}\cup\{0\}\,,\quad 0\le\varphi<\pi.$$

Let E_1 and E_2 be two Banach spaces and let $B\left(E_1,E_2\right)$ denote the space of bounded linear operators from E_1 to E_2. For $E_1=E_2=E$ we denote $B\left(E,E\right)$ by $B\left(E\right)$.

Let $D\left(A\right)$, $R\left(A\right)$ denote the domain and range of the linear operator in E, respectively. Let Ker A denote a null space of A.

A closed linear operator A is said to be $\varphi-$ sectorial (or sectorial for $\varphi=0$) in a Banach space E with bound $M>0$ if Ker $A=\{0\}$, $D\left(A\right)$ and $R\left(A\right)$ are dense on E, and $\left\|\left(A+\lambda I\right)^{-1}\right\|_{B(E)}\le M\left|\lambda\right|^{-1}$ for all $\lambda\in S_\varphi$, $\varphi\in[0,\ \pi)$, where I is an identity operator in E. Sometimes $A+\lambda I$ will be written as $A+\lambda$ and will be denoted by A_λ. It is known (see e.g. [19, §1.15.1]) that the fractional powers of the operator A are well defined.

Let $E(A^\theta)$ denote the space $D(A^\theta)$ with the graph norm

$$\|u\|_{E(A^\theta)} = \left(\|u\|_E^p + \left\|A^\theta u\right\|_E^p\right)^{\frac{1}{p}}, \quad 1 \leq p < \infty, \quad -\infty < \theta < \infty.$$

Note that the above norms are equivalent for $p \in [1, \infty)$.

Here, $S = S(R^n; E)$ denotes the $E-$valued Schwartz class, i.e. the space of $E-$valued rapidly decreasing smooth functions on R^n, equipped with its usual topology generated by seminorms. $S(R^n; \mathbb{C})$ will be denoted by just S.

Let $S'(R^n; E)$ denote the space of all continuous linear operators, $L : S \to E$, equipped with topology of bounded convergence. Recall $S(R^n; E)$ is norm dense in $L_{p,\gamma}(R^n; E)$ when $1 < p < \infty$, $\gamma \in A_p$.

Let Ω be a domain in R^n. $C(\Omega, E)$ and $C^{(m)}(\Omega; E)$ will denote the spaces of $E-$valued uniformly bounded strongly continuous and $m-$times continuously differentiable functions on Ω, respectively.

Here, $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$, where α_i are integers. An $E-$valued generalized function $D^\alpha f$ is called a generalized derivative in the sense of Schwartz distributions of the function $f \in S(R^n; E)$ if

$$\langle D^\alpha f, \varphi \rangle = (-1)^{|\alpha|} \langle f, D^\alpha \varphi \rangle$$

holds for all $\varphi \in S$.

Let F denote the Fourier transform. Throughout this section the Fourier transformation of a function f will be denoted by $\widehat{f}$ and $F^{-1} f = \check{f}$. It is known that

$$F(D_x^\alpha f) = (i\xi_1)^{\alpha_1} ... (i\xi_n)^{\alpha_n} \widehat{f}, \quad D_\xi^\alpha (F(f)) = F\left[(-ix_1)^{\alpha_1} ... (-ix_n)^{\alpha_n} f\right]$$

for all $f \in S'(R^n; E)$.

Suppose E_1 and E_2 are two Banach spaces. A function $\Psi \in L_\infty(R^n; B(E_1, E_2))$ is called a Fourier multiplier from $L_{p,\gamma}(R^n; E_1)$ to $L_{p,\gamma}(R^n; E_2)$ for $p \in (1, \infty)$ if the map $u \to Tu = F^{-1}\Psi(\xi) F u$, $u \in S(R^n; E_1)$ is well defined and extends to a bounded linear operator

$$T : L_{p,\gamma}(R^n; E_1) \to L_{p,\gamma}(R^n; E_2).$$

A Banach space E is called a UMD space (see e.g [5], [6]) if the Hilbert operator

$$(Hf)(x) = \lim_{\varepsilon \to 0} \int\limits_{|x-y| > \varepsilon} \frac{f(y)}{x - y} dy$$

is initially defined on $S(R; E)$ and is bounded in $L_p(R; E)$, $p \in (1, \infty)$ (see e.g. [6], [8]). UMD spaces include e.g. L_p, l_p spaces and Lorentz spaces L_{pq}, $p, q \in (1, \infty)$.

A set $K \subset B(E_1, E_2)$ is called $R-$bounded (see e.g [7], [21]) if there is a constant $C > 0$ such that for all $T_1, T_2, ..., T_m \in K$ and $u_1, u_2, ..., u_m \in E_1$, $m \in \mathbb{N}$

$$\int_0^1 \left\| \sum_{j=1}^m r_j(y) T_j u_j \right\|_{E_2} dy \leq C \int_0^1 \left\| \sum_{j=1}^m r_j(y) u_j \right\|_{E_1} dy,$$

where $\{r_j\}$ is a sequence of independent symmetric $\{-1;1\}$ −valued random variables on $[0,1]$. The smallest C for which the above estimate holds is called the $R-$bound of K and denoted by $R(K)$.

A Banach space E is said to be a space satisfying the multiplier condition with respect to weighted function γ and $p \in (1,\infty)$ (or multiplier condition with respect to $p \in (1,\infty)$ when $\gamma(x) \equiv 1$) if for any $\Psi \in C^{(n)}(R^n \setminus \{0\}; B(E))$ the $R-$boundedness of the set

$$\left\{ |\xi|^{|\beta|} D_\xi^\beta \Psi(\xi) : \xi \in R^n \setminus \{0\},\ \beta = (\beta_1, \beta_2, ..., \beta_n),\ \beta_k \in \{0,1\} \right\}$$

implies that Ψ is a Fourier multiplier in $L_{p,\gamma}(R^n; E)$.

Note that, if E is UMD space then it satisfies the multiplier condition with respect to $p \in (1,\infty)$ (see e.g. [7], [10], [21]).

A sectorial operator $A(x)$, $x \in R^n$ is said to be uniformly $R-$sectorial in a Banach space E if there exists a $\varphi \in [0, \pi)$ such that

$$\sup_{x \in R^n} R\left(\left\{ \left[A(x)(A(x) + \xi I)^{-1} \right] : \xi \in S_\varphi \right\} \right) \leq M.$$

Note that, in Hilbert spaces every norm bounded set is $R-$bounded. Therefore, in Hilbert spaces all sectorial operators are $R-$sectorial.

Let $A = A(x)$, $x \in R^n$ be closed linear operator in E with domain $D(A)$ independent of x. The Fourier transformation of $A(x)$ is a linear operator with the domain $D(A)$ defined as

$$\hat{A}(\xi) u(\varphi) = A(x) u(\hat{\varphi}) \text{ for } u \in S'(R^n; D(A)),\ \varphi \in S(R^n).$$

(For details see e.g [2, Section 3]).

Let E_0 and E be two Banach spaces, where E_0 is continuously and densely embedded into E. Let l be a natural number. $W_{p,\gamma}^l(R^n; E_0, E)$ denotes the space of all functions from $S'(R^n; E_0)$ such that $u \in L_{p,\gamma}(R^n; E_0)$ and the generalized derivatives $D_k^l u = \frac{\partial^l u}{\partial x_k^l} \in L_{p,\gamma}(R^n; E)$ with the norm

$$\|u\|_{W_{p,\gamma}^l(R^n;E_0,E)} = \|u\|_{L_{p,\gamma}(R^n;E_0)} + \sum_{k=1}^n \left\| D_k^l u \right\|_{L_{p,\gamma}(R^n;E)} < \infty.$$

It is clear that

$$W_{p,\gamma}^l(R^n; E_0, E) = W_{p,\gamma}^l(R^n; E) \cap L_{p,\gamma}(R^n; E_0).$$

$W_{p,\gamma}^{[l]}(R^n;E_0,E)$ denotes the space of all functions from $S'(R^n;E_0)$ such that $u\in L_p(R^n;E_0)$ and $D_k^{[l]}u\in L_p(R^n;E)$ with the norm

$$\|u\|_{W_{p,\gamma}^{[l]}(R^n;E_0,E)}=\|u\|_{L_p(R^n;E_0)}+\sum_{k=1}^n\left\|D_k^{[l]}u\right\|_{L_p(R^n;E)}<\infty.$$

Note that if $l\geq 2$, E is a space satisfying the multiplier condition with respect to weighted function γ and $p\in(1,\infty)$, then the above definitions are equivalent with usual definitions, i.e.

$$\|u\|_{W_{p,\gamma}^{l}(R^n;E_0,E)}\simeq\|u\|_{L_{p,\gamma}(R^n;E_0)}+\sum_{|\alpha|\leq l}\|D^\alpha u\|_{L_{p,\gamma}(R^n;E)},$$

$$\|u\|_{W_{p,\gamma}^{[l]}(R^n;E_0,E)}\simeq\|u\|_{L_p(R^n;E_0)}+\sum_{|\alpha|\leq l}\left\|D^{[\alpha]}u\right\|_{L_p(R^n;E)}.$$

In a similar way as [7, Theorem 3.25] we obtain:

Proposition A. Let E be a UMD space and $\gamma\in A_p$. Assume Ψ_h is a set of operator functions in $C^{(n)}(R^n\setminus\{0\};B(E))$ depending on the parameter $h\in Q\in\mathbb{R}$ and there exists a positive constant K such that

$$\sup_{h\in Q}R\left(\left\{|\xi|^{|\beta|}D^\beta\Psi_h(\xi):\ \xi\in R^n\setminus\{0\},\ \beta_k\in\{0,1\}\right\}\right)\leq K.$$

Then the set Ψ_h is a uniformly bounded collection of Fourier multipliers in $L_{p,\gamma}(R^n;E)$.

2. Convolution-elliptic equations

The main aim of the present section is to study the maximal L_p-regularity properties of the degenerate linear CDOEs

$$\sum_{|\alpha|\leq l}a_\alpha * D^{[\alpha]}u+(A+\lambda)*u=f(x),\ x\in R^n, \tag{2.1}$$

in $E-$ valued weighted L_p- spaces, where l is a natural number, $a_\alpha=a_\alpha(x)$ are complex-valued functions, $\alpha=(\alpha_1,\alpha_2,...,\alpha_n)$, α_k are nonnegative integers, $|\alpha|=\sum_{k=1}^n\alpha_k$, λ is a complex parameter and $A=A(x)$ is a linear operator in a Banach space E for $x\in R^n$.

Here, the convolutions $a_\alpha * D^{[\alpha]}u,\ A*u$ are defined in the distribution sense (see e.g. [2]). $\gamma=\gamma(x)$ is a positive measurable function on $\Omega\subset R^n$ and

$$D^{[\alpha]}=D_{x_1}^{[\alpha_1]}D_{x_2}^{[\alpha_2]}...D_{x_n}^{[\alpha_n]},\ D_{x_i}^{[\alpha_i]}=\left(\gamma(x)\frac{\partial}{\partial x_i}\right)^{\alpha_i}.$$

First we consider the following nondegenerate CDOE

$$\sum_{|\alpha|\le l} a_\alpha * D^\alpha u + (A+\lambda) * u = f, \tag{2.2}$$

where λ are parameters, a_α are complex-valued functions defined in (2.1) and A is a linear operator in a Banach space E.

Condition 2.1. Suppose the following are satisfied:

(1) $L(\xi) = \sum\limits_{|\alpha|\le l} \widehat{a}_\alpha(\xi)(i\xi)^\alpha \in S_{\varphi_1}$, $\varphi_1 \in [0,\pi)$ for $\xi \in R^n$,

$|L(\xi)| \ge C \sum\limits_{k=1}^{n} \left|\widehat{a}_{\alpha(l,k)}\right| |\xi_k|^l$, $\alpha(l,k) = (0,0,...,l,0,0,...,0)$ i.e.$\alpha_i = 0$, $i \ne k$, $\alpha_k = l$;

(2) $\widehat{a}_\alpha \in C^{(n)}(R^n)$ and $|\xi|^{|\beta|} \left|D^\beta \widehat{a}_\alpha(\xi)\right| \le C_1$, $\beta_k \in \{0,1\}$, $0 \le |\beta| \le n$;

(3) for $0 \le |\beta| \le n$, $\xi, \xi_0 \in R^n \setminus \{0\}$, $\left[D^\beta \widehat{A}(\xi)\right] \widehat{A}^{-1}(\xi_0) \in C(R^n; B(E))$,

$|\xi|^{|\beta|} \left\| \left[D^\beta \widehat{A}(\xi)\right] \widehat{A}^{-1}(\xi_0) \right\| \le C_2$.

In a similar way as [16, Theorem 2.7] we obtain:

Theorem 2.1. Assume that Condition 2.1 holds and E is a Banach space satisfying the multiplier condition with respect to weighted function $\gamma \in A_p$ and $p \in (1,\infty)$. Let $\hat{A}$ be a uniformly $R-$sectorial operator in E with $\varphi \in [0,\ \pi)$, $\lambda \in S_{\varphi_2}$ and $0 \le \varphi + \varphi_1 + \varphi_2 < \pi$. Then, problem (2.2) has a unique solution u and the coercive uniform estimate holds

$$\sum_{|\alpha|\le l} |\lambda|^{1-\frac{|\alpha|}{l}} \|a_\alpha * D^\alpha u\|_{L_{p,\gamma}(R^n;E)} + \|A*u\|_{L_{p,\gamma}(R^n;E)} + |\lambda| \|u\|_{L_{p,\gamma}(R^n;E)} \le C \|f\|_{L_{p,\gamma}(R^n;E)} \tag{2.3}$$

for all $f \in L_{p,\gamma}(R^n; E)$.

Let O be an operator in $L_{p,\gamma}(R^n;E)$ generated by problem (2.2) for $\lambda = 0$, i.e.

$$D(O) \subset W^l_{p,\gamma}(R^n; E(A), E),\ Ou = \sum_{|\alpha|\le l} a_\alpha * D^\alpha u + A*u.$$

From Theorem 2.1 we have:

Result 2.1. Assume that the all conditions of Theorem 2.1 hold. Then, for all $\lambda \in S_{\varphi_2}$ the following uniform coercive estimate holds

$$\sum_{|\alpha|\le l} |\lambda|^{1-\frac{|\alpha|}{l}} \left\|a_\alpha * D^\alpha (O+\lambda)^{-1}\right\|_{B(L_{p,\gamma}(R^n;E))} +$$

$$\left\|A*(O+\lambda)^{-1}\right\|_{B(L_{p,\gamma}(R^n;E))} + \left\|\lambda (O+\lambda)^{-1}\right\|_{B(L_{p,\gamma}(R^n;E))} \le C.$$

Result 2.2. Theorem 2.1, particularly implies that the operator O is uniformly sectorial in $L_{p,\gamma}(R^n;E)$; moreover, if $\hat{A}$ is uniformly $R-$sectorial for $\varphi \in \left(\frac{\pi}{2}, \pi\right)$, then the operator O is a negative generator of an analytic semigroup in $L_{p,\gamma}(R^n;E)$ (see e.g. [19, §1.14.5]).

From Theorem 2.1 and Proposition A we obtain:

Result 2.3. Let conditions of Theorem 2.1 hold for $E \in UMD$. Then the assertions of Theorem 2.1 are valid.

We find sufficient conditions that guarantee the separability of the problem (2.1). For this purpose we need the following

Remark 2.1. Consider the following substitution

$$y_k = \int_0^{x_k} \gamma^{-1}(z)dz, \; k = 1, 2, ..., n. \tag{2.4}$$

It is clear that, under the substitution (2.4), $D^{[\alpha]}u$ transforms to $D^{\alpha}u$. Moreover, the spaces $L_p(R^n; E)$, $W^{[l]}_{p,\gamma}(R^n; E(A), E)$ are mapped isomorphically onto the weighted spaces $L_{p,\gamma}(R^n; E)$ and $W^l_{p,\gamma}(R^n; E(A), E)$ respectively where,

$$\gamma = \widetilde{\gamma}(y) = \gamma(x(y)) = \gamma(x_1(y_1), x_2(y_2), ..., x_n(y_n)).$$

Moreover, under (2.4) the degenerate problem (2.1) considered in $L_p(R^n; E)$ is transformed into the non degenerate problem (2.2) in $L_{p,\gamma}(R^n; E)$, where

$$a_\alpha = a_\alpha(y) = a_\alpha(\widetilde{\gamma}(y)), \; u = u(y) = \tilde{u}(y) = u(\widetilde{\gamma}(y)),$$

$$A = A(y) = \tilde{A}(y) = A(\widetilde{\gamma}(y)), \; f = f(y) = \widetilde{f}(y) = f(\widetilde{\gamma}(y)).$$

Let

$$\tilde{X} = L_p(R^n; E), \; \tilde{Y} = W^{[l]}_{p,\gamma}(R^n; E(A), E), \; p \in (1, \infty).$$

In this section we show the following result:

Theorem 2.2. Assume that Condition 2.1 holds for $a_\alpha = a_\alpha(y)$ and E is a Banach space satisfying the multiplier condition with respect to weighted function $\gamma \in A_p$ and $p \in (1, \infty)$. Let $\hat{A}$ be a uniformly $R-$sectorial operator in E with $\varphi \in [0, \; \pi)$, $\lambda \in S_{\varphi_2}$ and $0 \leq \varphi + \varphi_1 + \varphi_2 < \pi$ for $A = A(y)$. Then for all $f \in \tilde{X}$ there is a unique solution of the problem (2.1) and the following coercive uniform estimate holds:

$$\sum_{|\alpha| \leq l} |\lambda|^{1-\frac{|\alpha|}{l}} \left\| a_\alpha * D^{[\alpha]}u \right\|_{\tilde{X}} + \|A * u\|_{\tilde{X}} + |\lambda| \|u\|_{\tilde{X}} \leq C \|f\|_{\tilde{X}}. \tag{2.5}$$

Proof. By Remark 2.1, the degenerate problem (2.1) is transformed into the non degenerate problem (2.2) considered in the weighted space $L_{p,\gamma}(R^n; E)$. Then in view of Theorem 2.1 we obtain the estimate (2.5).

3. Degenerate convolution equations in the space $L_{p,\gamma}(R^n; l_q)$

Consider the following system of convolution equations

$$\sum_{|\alpha|\leq l} a_\alpha * D^{[\alpha]}u_m + \sum_{m=1}^{\infty} d_m * u_m = f_m, \tag{3.1}$$

in the concrete Banach space $L_{p,\gamma}(R^n; l_q)$, where l is a natural number, $a_\alpha = a_\alpha(x)$ are complex-valued functions, $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$, α_k are nonnegative integers, $d_m = d_m(x)$, $u_m = u_m(x)$, $f_m = f_m(x)$, $x \in R^n$. The convolutions $a_\alpha * D^{[\alpha]}u$, $d_m * u_m$ are defined in the distribution sense and

$$D^{[\alpha]} = D_{x_1}^{[\alpha_1]} D_{x_2}^{[\alpha_2]} ... D_{x_n}^{[\alpha_n]},\ D_{x_k}^{[\alpha_k]} = \left(\gamma(x)\frac{\partial}{\partial x_k}\right)^{\alpha_k}.$$

$$\gamma(x) = \prod_{k=1}^{n} |x_k|^{\gamma},\quad -\frac{1}{n} < \gamma < \frac{p-1}{n},$$

is a positive measurable weighted function.

For $1 < q < \infty$ we set

$$l_q = \left\{\xi; \xi = \{\xi_i\}_{i=1}^{\infty};\ \|\xi\|_{l_q} = \left(\sum_{i=1}^{\infty} |\xi_i|^q\right)^{1/q} < \infty,\ \xi_i - \text{complex numbers}\right\}.$$

Moreover, if $\gamma(x)$ is a positive measurable function, and if $1 < p < \infty$, then

$$L_{p,\gamma}(R^n; l_q) = \left\{f; f = \{f_i(x)\}_{i=1}^{\infty}, \|f\|_{L_{p,\gamma}(R^n;l_q)} = \left(\int_{R^n} \|\{f_i(x)\}\|_{l_q}^p \gamma(x)dx\right)^{1/p} < \infty\right\}.$$

Clearly, $L_{p,\gamma}(R^n; l_q)$ is a Banach space. It is known that

$$\|f\|_{L_{p,\gamma}(R^n;l_q)} = \left(\int_{R^n} \left(\sum_{i=1}^{\infty} |f_i(x)|^q\right)^{\frac{p}{q}} \gamma(x)dx\right)^{\frac{1}{p}}.$$

Let $d(x) = \{d_m(x)\}$, $d_m > 0$, $u = \{u_m\}$, $d * u = \{d_m * u_m\}$, $l_q(d) =$

$$\left\{u \in l_q,\ \|u\|_{l_q(d)} = \left(\sum_{m=1}^{\infty} |d_m(x) * u_m|^q\right)^{\frac{1}{q}} < \infty\right\},\ 1 < q < \infty,$$

$$X = L_p(R^n; l_q),\ Y = W_{p,\gamma}^{[l]}(R^n; l_q(d), l_q),\ B = B(X),$$

and Q denote the differential operator in $L_p(R^n; l_q)$ generated by (3.1), i.e., $D(Q) =$

$W_{p,\gamma}^{[l]}(R^n; l_q(d), l_q)$, $Qu = \sum_{|\alpha|\leq l} a_\alpha * D^{[\alpha]}u + d * u$

Condition 3.1. Assume that there exist positive constants C_1 and C_2 such that for $\{d_m(x)\}_1^{\infty} \in l_q$ for all $x \in R^n$ and some $x_0 \in R^n$,

$$C_1 |d_m(x_0)| \leq |d_m(x)| \leq C_2 |d_m(x_0)|.$$

Suppose $\hat{a}_\alpha$, $\hat{d}_m \in C^{(n)}(R^n)$ and there exist positive constants M_1 and M_2 such that

$$|\xi|^{|\beta|}\left|D^\beta \hat{a}_\alpha(\xi)\right| \leq M_1,\ |\xi|^{|\beta|}\left|D^\beta \hat{d}_m(\xi)\right| \leq M_2,$$

$$\xi \in R^n \setminus \{0\},\ \beta_k \in \{0,1\},\ 0 \leq |\beta| \leq n.$$

Applying Theorem 2.2. we have:

Theorem 3.1. Suppose Condition 3.1 and the (1) part of Condition 2.1 are satisfied. Then:

(a) for all $f(x) = \{f_m(x)\}_1^\infty \in L_p(R^n; l_q(d))$, for $\lambda \in S_\varphi$, $\varphi \in [0,\pi)$ problem (3.1) has a unique solution $u = \{u_m(x)\}_1^\infty$ that belongs to Y and the following coercive estimate holds

$$\sum_{|\alpha|\leq l}\left(\int_{R^n}\left(\sum_{m=1}^{\infty}\left|a_\alpha * D^{[\alpha]}u_m\right|^q\right)^{\frac{p}{q}}dx\right)^{\frac{1}{p}} + \left(\int_{R^n}\left(\sum_{m=1}^{\infty}|d_m * u_m|^q\right)^{\frac{p}{q}}dx\right)^{\frac{1}{p}}$$

$$\leq C\left(\int_{R^n}\left(\sum_{m=1}^{\infty}|f_m|^q\right)^{\frac{p}{q}}dx\right)^{\frac{1}{p}}.$$

(b) For $\lambda \in S_\varphi$ there exists a resolvent $(Q+\lambda)^{-1}$ and

$$\sum_{|\alpha|\leq l}|\lambda|^{1-\frac{|\alpha|}{l}}\left\|a_\alpha * \left[D^{[\alpha]}(Q+\lambda)^{-1}\right]\right\|_B +$$

$$\left\|d * (Q+\lambda)^{-1}\right\|_B + \left\|\lambda(Q+\lambda)^{-1}\right\|_B \leq C.$$

Proof. In fact, let $E = l_q$ and $A = [d_m(x)\delta_{jm}]$, $m, j = 1, 2, ...\infty$, where δ_{jm} is the Kronecker symbol ($\delta_{jm} = 1$ for $j = m$, $\delta_{jm} = 0$ for $j \neq m$). Then it is easy to see that $\hat{A}(\xi) = \left[\hat{d}_m(\xi)\delta_{jm}\right]$ is uniformly $R-$sectorial in l_q and the all conditions of Theorem 2.2 hold. Moreover, by [20] we get that the space l_q satisfies the multiplier condition with respect to power weighted function $\gamma(x) = |x|^\gamma$, $-\frac{1}{n} < \gamma < \frac{p-1}{n}$ and $p \in (1,\infty)$. Therefore, by virtue of Theorem 2.2 we obtain the $\sum\limits_{|\alpha|\leq l}\left\|a_\alpha * D^{[\alpha]}u\right\|_X + \|d * u\|_X \leq C\|f\|_X$. From this we get that assertion (a). Taking into account Theorem 2.2 and Remark 2.1 we have for all $\lambda \in S_\varphi$ there exist the resolvent of operator Q and has the estimate

$$\sum_{|\alpha|\leq l}|\lambda|^{1-\frac{|\alpha|}{l}}\left\|a_\alpha * D^{[\alpha]}(Q+\lambda)^{-1}\right\|_{B(X)} + \left\|d * (Q+\lambda)^{-1}\right\|_{B(X)} + \left\|\lambda(Q+\lambda)^{-1}\right\|_{B(X)} \leq C.$$

This means that the assertion (b) is obtained.

Remark 3.1. There are a lot of sectorial operators in concrete Banach spaces. Therefore, putting in (2.1) concrete Banach spaces instead of E and concrete sectorial differential, pseudo differential operators, or finite, infinite matrices, etc. instead of A, by virtue of Theorem 2.2 we can obtain the maximal regularity properties of different class of convolution equations.

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