Caspian Journal of Applied Mathematics, Ecology and Economics
V. 7, No 1, 2019, July
ISSN 1560-4055

Investigation of Laplace Transforms for Distribution of the First Passage of Zero Level of the Semi-Markov Random Process with Positive Tendency and Negative Jump

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Abstract. One of the important problems of stochastic process theory is to define the Laplace transforms for the distribution of semi-markov random processes. With this purpose, we will investigate the semi-markovrandom processes with positive tendency and negative jump in this article. The first passage of the zero level of the process will be included as a random variable. The Laplace transforms for the distribution of this random variable is defined. The parameters of the distribution will be calculated on the basis of the final results.
Key Words and Phrases: Laplace transforms, semi-Markov random process, random variable, process with positive tendency and negative jumps.
2010 Mathematics Subject Classifications: 60A10, 60J25

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1. Introduction

There are number of works devoted to definition of the Laplace transforms for the distribution of the first pas- sage of the zero level. (Borovkov 2004) [1] defined the explicit form of the distribution, while (Klimov 1996) [3] and (Lotov V. I.) [2] indicated implicit form of the distribution of the first passage of zero level. The presented work explicitly defines the Laplace transforms for the unconditional and conditional distribution of the semi-markov random processes with positive tendency and negative jump.

2. Mathematical Statement of the problem

Let a sequence of independent and identically distributed pairs of random variables $\{\xi_k, \zeta_k\}_{k\geq 1}$, $k = \overline{1,\infty}$ defined on a probability space (Ω, F, P) such that ξ_k and ζ_k are independent random variables and $\xi_k > 0$,$\zeta_k > 0$. Using these random variables we will derive the following step processes of semi-Markov random walk:

$$X_z(t) = z + t - \sum_{i=1}^{k-1} \zeta_i \text{ if } \sum_{i=1}^{k-1} \xi_i \leq t < \sum_{i=1}^{k} \xi_i,\ k = \overline{1,\infty}\ z \geq 0$$

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$X_z(t)$ process is the (asymptotic) semi-Markov random processes with positive tendency and negative jump.One of the realizations of the process $X_z(t)$ will be in the following form:

a) $X_z(t) = z + t$ if $t < \xi_1$ (see, Figure 1),

b) $X_z(t) = z + t - \zeta_1$, if $\xi_1 \geq t < \xi_1 + \xi_2$ (see, Figure 2)

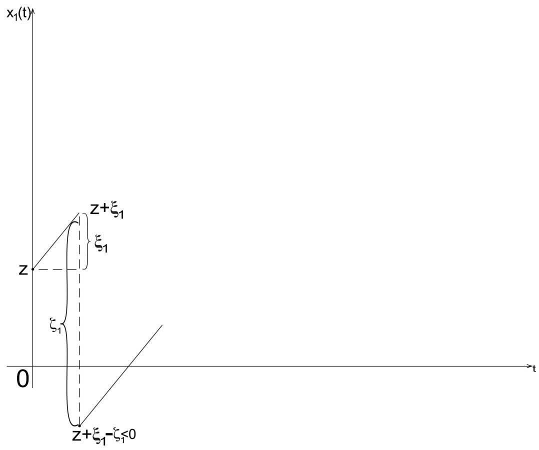

Fig. 1.

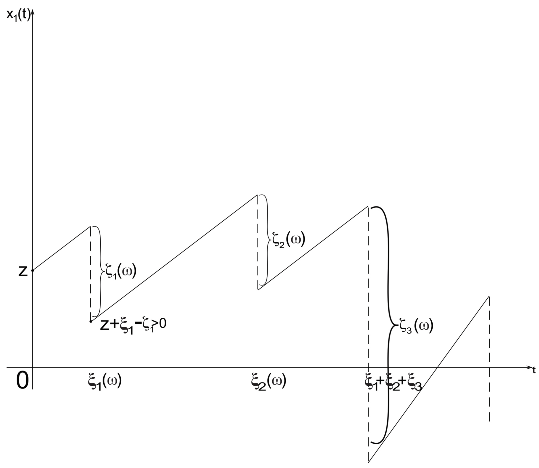

Fig. 2.

Let's include the τ_z^0 random variable defined as below:

$$\tau_z^0 = min\{t : X_z(t) \leq 0\}$$

where τ_z^0, is the time of the first passage of $X(t)$ process. We need to find Laplace transform for distribution of τ_z^0 random variable. Let us set Laplace transform for the distribution of τ_z^0 random variable as $L(\theta)$

$$L(\theta) = Ee^{-\theta\tau_z^0},\ \theta > 0;$$

$$L(\theta|z) = E(e^{-\theta\tau_z^0}|X_z(t) = z),,\ z \geq 0$$

In this case we can express the equation as

$$\tau_z^0 = \begin{cases} \xi_1, & z + \xi_1 - \zeta_1 < 0 \\ \xi_1 + T^0_{z+\xi_1-\zeta_1}, & z + \xi_1 - \zeta_1 > 0 \end{cases}$$

Thus, T and τ_z^0 are evenly distributed random variables.Our goal is to find Laplace transform of relative and non-relative distribution of τ_z^0 random variable.

Theorem 1. *Let a sequence of independent and identically distributed pairs of random variables $\{\xi_k, \zeta_k\}_{k\geq 1}$, $k = \overline{1,\infty}$, defined on a probability space (Ω, F, P) such that ξ_k and ζ_k are independent random variables and $\xi_k > 0, \zeta_k > 0$. Then an integral equation of Laplace transform of distribution of τ_z^0 random variable will be as follows:*

$$L(\theta|z) = \int_{s=0}^{\infty} e^{-\theta s} P\{\zeta_1 > z + s\} P\{\xi_1 \in ds\} -$$

$$- \int_{s=0}^{\infty} e^{-\theta s} \int_{\alpha=0}^{z+s} L(\theta|\alpha) d_\alpha P\{\zeta_1 < z + s - \alpha\} dP\{\xi_1 < s\} \qquad (1)$$

Proof: According to the formula of total probability, we can put it as

$$E\left(e^{-\theta\tau_z^0}|X_z(0) = z\right) = \int_\Omega e^{-\theta\tau_z^0} P(d\omega) =$$

$$= \int_{\{\omega: z+\xi_1-\zeta_1<0\}} e^{-\theta\xi_1} P(d\omega) + \int_{\{\omega: z+\xi_1-\zeta_1>0\}} e^{-\theta(\xi_1+T)} P(d\omega)$$

If to consider the following substitution

$$\xi_1 = s;\ \ \varsigma_1 = y : T = \beta$$

we derive

$$E\left(e^{-\theta s\tau_z^0}|X_1(0) = z\right) = \int_{s=0}^{\infty} \int_{y=z+s}^{\infty} e^{\theta s} P\{\xi_1 \in ds; \zeta_1 \in dy\} +$$

$$+ \int_{s=0}^{\infty} \int_{y=0}^{z+s} \int_{\beta=0}^{\infty} e^{-\theta(s+\beta)} P\{\xi_1 \in ds; \zeta_1 \in dy; T \in d\beta\} =$$

$$= \int\limits_{s=0}^{\infty} e^{-\theta s} P\{\xi_1 \in ds\} \int\limits_{y=z+s}^{\infty} P\{\zeta_1 \in dy\} +$$

$$+ \int\limits_{s=0}^{\infty} e^{-\theta s} \int\limits_{y=0}^{z+s} dP\{\zeta_1 \in y\} dP\{\xi_1 < s\} L(\theta | z+s-y) =$$

$$= \int\limits_{s=0}^{\infty} e^{-\theta s} P\{\xi_1 \in ds\} P\{\zeta_1 > z+s\} +$$

$$+ \int\limits_{s=0}^{\infty} e^{-\theta s} \int\limits_{\beta=z+s}^{0} L(\theta|\beta) dP\{\zeta_1 < z+s-\beta\} dP\{\xi_1 < s\}$$

or

$$L(\theta|z) = \int\limits_{s=0}^{\infty} e^{-\theta s} P\{\zeta_1 > z+s\} P\{\xi_1 \in ds\} +$$

$$+ \int\limits_{s=0}^{\infty} e^{-\theta s} \int\limits_{y=0}^{z+s} L(\theta|z+s-y) P\{\zeta_1 \in ds\} P\{\xi_1 \in ds\}$$

Let's assume that $z+s-y = \alpha$. In this case we will receive the following integral equation:

$$L(\theta|z) = \int\limits_{s=0}^{\infty} e^{-\theta s} P\{\zeta_1 > z+s\} P\{\xi_1 \in ds\} -$$

$$- \int\limits_{s=0}^{\infty} e^{-\theta s} \int\limits_{\alpha=0}^{z+s} L(\theta|\alpha) d_\alpha P\{\zeta_1 < z+s-\alpha\} dP\{\xi_1 < s\}$$

The theorem 1 is proved.

We will solve this integral equation in special case. Let's assume that $\xi_1(\omega)$ random variable has the Erlangian distribution of third construction, while $\zeta_1(\omega)$ random variable has the single construction Erlangian distribution:

$$P\{\xi_1(\omega) < t\} = \left[1 - \left(1 + \lambda t + \frac{\lambda^2 t^2}{2}\right) e^{-\lambda t}\right] \varepsilon(t), \quad \lambda > 0,$$

$$P\{\zeta_1(\omega) < t\} = \left[1 - e^{-\mu t}\right] \varepsilon(t), \quad \mu > 0$$

where $\varepsilon(t) = \begin{cases} 0, & t < 0, \\ 1, & t > 0. \end{cases}$

In this case Equation (1) will be as follows:

$$L(\theta|z)=\frac{\lambda^3e^{-\mu z}}{(\lambda+\mu+\theta)^3}+\frac{\lambda^3\mu e^{-\mu z}}{2}\int\limits_{s=0}^{\infty}S^2e^{-(\lambda+\mu+\theta)s}\int\limits_{\alpha=0}^{z+s}e^{\mu\alpha}L(\theta|\alpha)d\alpha ds \qquad (2)$$

We can derive differential equation from this integral equation. For this purpose, we will multiply both sides of equation (2) by $e^{\mu z}$

$$\mu L(\theta|z)+L'(\theta|z)=\frac{\lambda^3\mu}{2}\int\limits_{s=0}^{\infty}S^2e^{-(\lambda+\theta)s}L(\theta|z+s)ds$$

If to consider the following substitution $x=z+s$, multiply both sides of last equation by $e^{-(\lambda+\theta)z}$ and derive on z we can find the following differential equation:

$$L^{IV}(\theta|z)-[3(\lambda+\theta)-\mu]L'''(\theta|z)+3(\lambda+\theta)(\lambda+\theta-\mu)L''(\theta|z)-$$

$$-(\lambda+\theta)^2(\lambda+\theta-3\mu)L'(\theta|z)-\mu[(\lambda+\theta)^2-\lambda^3]L(\theta|z)=0$$

The general solution of this differential equation will be as follows :

$$L(\theta|z)=C_1(\theta)e^{k_{1(\theta)z}}=\frac{\lambda^3}{[\lambda+\theta-k_1(\theta)]^3}e^{k_{1(\theta)z}}.$$

This expression is the Laplace transform of the conditional distribution of τ_z^0 random variable. Then, we will need to find Laplace transform for the unconditional distribution of τ_z^0 random variable. In accordance with formula of total probability

$$L(\theta)=\int\limits_{z=0}^{\infty}L(\theta|z)\lambda^3z^2e^{-\lambda z}dz=\int\limits_{z=0}^{\infty}c_1(\theta)e^{k_{1(\theta)}z}\lambda^3z^2e^{-\lambda z}dz=$$

$$=c_1(\theta)\lambda^3\int\limits_{z=0}^{\infty}z^2e^{[k_1(\theta-\lambda)]z}dz=\frac{\lambda^3}{[\lambda-k_1(\theta)]^3}C_1(\theta)$$

Therefore

$$L(\theta)=\frac{\lambda^3}{[\lambda-k_1(\theta)]^3}C_1(\theta)$$

Respectively, we will get the following characteristics:

$$E\tau_1^0=-L'(0)=\frac{3(\lambda+\mu)}{\lambda(\lambda-3\mu)}$$

$$E(\tau_z^0|z)=\frac{3(1+z\mu)}{\lambda-3\mu}$$

$$E(\tau_z^0|z)=\frac{3}{(\lambda-3\mu)^2}+\frac{12(3+z\lambda)\mu}{(\lambda-3\mu)^3}$$

3. Conclusions

In this article we have defined Laplace transforms for the unconditional and conditional distribution of the first passage of zero level of semimarkov random processes with positive tendency and negative jump.

Acknowledgements

The research of Ulviyya Y. Karimova and Konul K. Omarova was supported by the Science Development Foundation under the President of the Republic of Azerbaijan-Grant N EIF/MQM/Elm-Tehsil-1- 2016-1(26)-71/19/1.

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Received: 21 December 2018
Accepted: 07 June 2019