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Asymptotic Behavior of the Distribution Function of the Ahlfors-Beurling Transform

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Abstract. In the present paper, we study the asymptotic behavior of the distribution function of the Ahlfors-Beurling transform of a Lebesgue integrable function as $\lambda \to +\infty$ and as $\lambda \to 0+$.

Key Words and Phrases: Ahlfors–Beurling transform, distribution function, asymptotic behavior.

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1. Introduction

The Ahlfors-Beurling transform of a function $f \in L_p(C)$, $1 \le p < \infty$ is defined as the following singular integral:

$$(Bf)(z) = -\frac{1}{\pi} \lim_{\varepsilon \to 0} \int_{\{w \in C:\, |z-w| > \varepsilon\}} \frac{f(w)}{(z-w)^2} dm(w).$$

The Ahlfors–Beurling transform is one of the important operators in complex analysis. It is the "Hilbert transform" on complex plane. It has been shown in [1,3,6,9,11,15,17] that this transform plays an essential role in applications to the theory of quasiconformal mappings and to the Beltrami equation with discontinuous coefficients.

From the theory of singular integrals (see [13]) it is known that the Ahlfors–Beurling transform is a bounded operator in the space $L_p(C)$, $1 < p < \infty$, that is, if $f \in L_p(C)$, then $B(f) \in L_p(C)$ and

$$\|Bf\|_{L_p} \le C_p \|f\|_{L_p}. \tag{1}$$

In the case $f \in L_1(C)$ only the weak inequality holds,

$$m\{z \in C: \ |(Bf)(z)| > \lambda\} \le \frac{C_1}{\lambda} \|f\|_{L_1}, \tag{2}$$

where m stands for the Lebesgue measure, C_p, C_1 are constants independent of f and $m\{z \in C: \ |(Bf)(z)| > \lambda\}$ - the distribution function of the Ahlfors-Beurling transform of the function f.

In [2,4,5,7,8,10,12,14,16] the boundedness of the operator B in other function spaces (in the spaces of Sobolev, Besov, Campanato, Morrey, etc.) was studied.

In the present paper, we study the asymptotic behavior of the distribution function of the Ahlfors-Beurling transform of a Lebesgue integrable function as $\lambda \to +\infty$ and as $\lambda \to 0+$.

2. Asymptotic behavior of the distribution function of the Ahlfors-Beurling transform as $\lambda \to +\infty$

In this section we studying the asymptotic behavior of the distribution function of the Ahlfors-Beurling transform as $\lambda \to +\infty$.

Theorem 1. *Let $f \in L_1(C)$. Then the equation*

$$\lim_{\lambda\to+\infty} \lambda m\left\{z \in C : \ |(Bf)(z)| > \lambda\right\} = 0 \tag{3}$$

holds.

Proof. Since $f \in L_1(C)$, then for every $\varepsilon > 0$ there exists $n \in N$ and $R > 0$ such that

$$\|f - [f]_R^n\|_{L_1} \le \frac{\varepsilon}{4C_1}, \tag{4}$$

where $[f]_R^n(z) = [f]^n \chi(B(0;R))(z)$, $[f(z)]^n = f(z)$ for $|f(z)| \le n$, $[f(z)]^n = 0$ for $|f(z)| > n$, $\chi(B(0;R))(z)$- characteristic function of the circle $B(0;R) = \{z \in C : \ |z| < R\}$. It follows from (1) and (4) that for every $\lambda > 0$ the inequality

$$m\left\{z \in C : \ |B(f - [f]_R^n)(z)| > \frac{\lambda}{2}\right\} \le \frac{2C_1}{\lambda}\|f - [f]_R^n\|_{L_1} \le \frac{\varepsilon}{2\lambda} \tag{5}$$

holds. Since the function $[f]_R^n(z)$ is bounded, then we get that $[f]_R^n \in L_p(C)$ for each $p \ge 1$. It follows that $B[f]_R^n \in L_p(C)$ for each $p > 1$. Denote

$$F_1(z) = B[f]_R^n(z) \cdot \chi(B(0;\,2R)), F_2(z) = B[f]_R^n(z) \cdot \chi(C \backslash B(0;\,2R)).$$

Then

$$B[f]_R^n(z) = F_1(z) + F_2(z),$$

The function $F_1(z)$ is concentrated on the closed circle $\overline{B(0;\,2R)}$, and the function $F_2(z)$ is concentrated on the set $C\backslash B(0;\,2R)$. For every $p > 1$ from the inclusion $B[f]_R^n \in L_p(C)$ it follows that $F_1(z) \in L_p(C)$. Since the function $F_1(z)$ is concentrated on the bounded set, then we have that $F_1(z) \in L_1(C)$. Then for sufficiently large values of $\lambda > 0$

$$\frac{\lambda}{2} m\left\{z \in C : \ |F_1(z)| > \lambda/2\right\} \le \int_{\{z\in C:\ |F_1(z)|>\lambda/2\}} |F_1(z)|\, dm(z) < \frac{\varepsilon}{4}. \tag{6}$$

On the other hand, for any $z \in C \backslash B(0;\, 2r)$ we have

$$|B([f]_R^n)(z)| = \frac{1}{\pi}\int_{B(0;\,R)} \frac{|[f]_R^n(w)|}{|z-w|^2} dm(w) \le$$

$$\le \frac{1}{\pi R^2}\int_{B(0;\,R)} |[f]_R^n(w)|\, dm(w) = \frac{1}{\pi R^2}\,\|[f]_R^n\|_{L_1} \le \frac{1}{\pi R^2}\,\|f\|_{L_1}.$$

This shows that the function $F_2(z)$ is bounded. Then it follows from (6) that for sufficiently large values of $\lambda > 0$

$$m\{z \in C:\ |B[f]_R^n(z)| > \lambda/2\} \le m\{z \in C:\ |F_1(z)| > \lambda/2\} < \frac{\varepsilon}{2\lambda}. \tag{7}$$

It follows from (5) and (7) that for sufficiently large values of $\lambda > 0$

$$m\{z \in C:\ |(Bf)(z)| > \lambda/2\} \le$$

$$\le m\{z \in C:\ |B[f]_R^n(z)| > \lambda/2\} + m\left\{z \in C:\ |B(f - [f]_R^n)(z)| > \frac{\lambda}{2}\right\} < \frac{\varepsilon}{2\lambda} + \frac{\varepsilon}{2\lambda} = \frac{\varepsilon}{\lambda}.$$

This shows that the equation (3) holds. Theorem 1 is proved. ◄

3. Asymptotic behavior of the distribution function of the Ahlfors-Beurling transform as $\lambda \to 0+$

In this section we studying the asymptotic behavior of the distribution function of the Ahlfors-Beurling transform as $\lambda \to 0+$.

Theorem 2. *Let $f \in L_1(C)$. Then the equation*

$$\lim_{\lambda \to 0+} \lambda m\{z \in C:\ |(Bf)(z)| > \lambda\} = \left|\int_C f(z)\, dm(z)\right| \tag{8}$$

holds.

At first we prove the auxiliary lemma.

Lemma 1. *If $f \in L_1(C)$ and $\int_C f(z)\, dm(z) = 0$, then the equation*

$$m\{z \in C:\ |(Bf)(z)| > \lambda\} = o(1/\lambda),\ \lambda \to 0+ \tag{9}$$

holds.

Proof of Lemma 1. At first assume that the function f is concentrated on some circle $B(0;\, R) \subset C$. In this case, from the equality

$$(Bf)(z) = -\frac{1}{\pi}\lim_{\varepsilon \to 0}\int_{\{w \in B(0;R):\,|z-w|>\varepsilon\}} \frac{f(w)}{(z-w)^2} dm(w) =$$

$$= -\frac{1}{\pi}\lim_{\varepsilon\to 0}\int_{\{w\in B(0;\,R)\,:\,|z-w|>\varepsilon\}}\frac{f(w)}{(z-w)^2}dm(w)+\frac{1}{\pi}\int_{B(0;\,R)}\frac{f(w)}{(z-z_0)^2}dm(w)=$$

$$=\frac{1}{\pi}\lim_{\varepsilon\to 0}\int_{\{w\in B(0;\,R)\,:\,|z-w|>\varepsilon\}}(z_0-w)\times$$

$$\times\left[\frac{1}{(z-w)^2(z-z_0)}+\frac{1}{(z-w)(z-z_0)^2}\right]f(w)\,dm(w)\,,z\neq z_0,$$

where $z_0\in C$, we get that

$$|(Bf)(z)|\leq\frac{16}{\pi\,|z|^3}\int_{B(0;\,R)}|z_0-w|\,|f(w)|\,dm(w)=\frac{k_0}{|z|^3},$$

for values of $|z|>R_0$, where

$$k_0=\frac{16}{\pi}\int_{B(0;\,R)}|z_0-w|\,|f(w)|\,dm(w)\,,R_0=2\max\{R,\ |z_0|\}\,.$$

Then it follows that

$$m\{z\in C:\ |(Bf)(z)|>\lambda\}\leq m\{z\in C:\ |z|\leq R_0\}+m\left\{z\in C:\ \frac{k_0}{|z|^3}>\lambda\right\}=$$

$$=m\{z\in C:\ |z|\leq R_0\}+m\left\{z\in C:\ |z|<\sqrt[3]{\frac{k_0}{\lambda}}\right\}=\pi R_0^2+\pi\left(\frac{k_0}{\lambda}\right)^{2/3},$$

whence it follows asymptotic equality (9).

Now let's consider the general case. From the condition $\int_C f(z)\,dm(z)=0$ it follows that for any $\varepsilon>0$ there exist the functions f_1 and f_2 satisfying the condition: $f=f_1+f_2$, the function f_1 is concentrated on some circle $B(0;\,R)\subset C$ and $\int_C f_1(z)\,dm(z)=0$, the function f_2 satisfies the inequality $\|f_2\|_{L_1}<\frac{\varepsilon}{4C_1}$, where C_1 is a constant in estimation (1). Since the function f_1 is concentrated on the circle $B(0;\,R)\subset C$ and $\int_C f_1(z)\,dm(z)=0$, then for the function f_1 equality (9) is satisfied, and therefore there exists $\lambda(\varepsilon)>0$ such that for $0<\lambda<\lambda(\varepsilon)$ the inequality

$$\lambda m\left\{z\in C:\ |(Bf_1)(z)|>\frac{\lambda}{2}\right\}<\frac{\varepsilon}{2} \tag{10}$$

holds. On the other hand, from the inequality (1) it follows that

$$\lambda m\left\{z\in C:\ |(Bf_2)(z)|>\frac{\lambda}{2}\right\}\leq 2C_1\,\|f_2\|_{L_2}<\frac{\varepsilon}{2} \tag{11}$$

for any $\lambda>0$. From inequalities (10), (11) and the inclusion

$$\{z\in C:\ |(Bf)(z)|>\lambda\}\subset\left\{z\in C:\ |(Bf_1)(z)|>\frac{\lambda}{2}\right\}\bigcup\left\{z\in C:\ |(Bf_2)(z)|>\frac{\lambda}{2}\right\}$$

we get

$$\lambda m\left\{z\in C:\ \left|\left(Bf\right)(z)\right|>\lambda\right\}<\varepsilon$$

for $0<\lambda<\lambda(\varepsilon)$. This shows that equality (9) was satisfied for all functions $f\in L_1(C)$, satisfying the condition $\int_C f(z)\,dm(z)=0$. This completes the Proof of the Lemma 1 ◀

Proof of Theorem 2. In the case $\int_C f(z)\,dm(z)=0$ the assertion of the Theorem follows from Lemma 1. Let's consider the case $\int_C f(z)\,dm(z)=\eta\neq 0$. Denote by $f_1(z)=\frac{\eta}{\pi}\chi\left(B(0;1)\right)(z)$, where $\chi\left(B(0;1)\right)$ is a characteristic function on the unit circle $B(0;1)$ and $f_2(z)=f(z)-f_1(z)$. Then $\int_C f_2(z)\,dm(z)=0$, and from Lemma 1

$$m\left\{z\in C:\ \left|\left(Bf_2\right)(z)\right|>\lambda\right\}=o\left(\frac{1}{\lambda}\right),\lambda\to 0+\,. \tag{12}$$

Since for any $|z|>2$

$$\left|\left(Bf_1\right)(z)\right|=\frac{|\eta|}{\pi^2}\left|\int_{B(0;\,1)}\frac{dm(w)}{(z-w)^2}\right|\leq\frac{|\eta|}{\pi}\cdot\frac{1}{\left(|z|-1\right)^2},$$

$$\left|\left(Bf_1\right)(z)\right|=\frac{|\eta|}{\pi^2}\left|\int_{B(0;\,1)}\frac{dm(w)}{(z-w)^2}\right|=\frac{|\eta|}{\pi^2}\left|\int_{B(0;\,1)}\frac{dm(w)}{\left(|z|-w\right)^2}\right|\geq$$

$$\geq\frac{|\eta|}{\pi^2}Re\left(\int_{B(0;\,1)}\frac{dm(w)}{\left(|z|-w\right)^2}\right)\geq\frac{|\eta|}{\pi}\cdot\frac{\left(|z|-1\right)^2}{\left(|z|+1\right)^4},$$

then for any $0<\lambda<\frac{|\eta|}{49\pi}$

$$m\left\{z\in C:\ \left|\left(Bf_1\right)(z)\right|>\lambda\right\}\leq m\left\{z\in C:\ |z|\leq 2\right\}+m\left\{z\in C:\ \frac{|\eta|}{\pi}\cdot\frac{1}{\left(|z|-1\right)^2}>\lambda\right\}=$$

$$=4\pi+m\left\{z\in C:\ |z|<1+\sqrt{\frac{|\eta|}{\pi\lambda}}\right\}=4\pi+\pi\left(1+\sqrt{\frac{|\eta|}{\pi\lambda}}\right)^2, \tag{13}$$

$$m\left\{z\in C:\ \left|\left(Bf_1\right)(z)\right|>\lambda\right\}\geq m\left\{|z|\geq 2:\ \ \frac{|\eta|}{\pi}\cdot\frac{\left(|z|-1\right)^2}{\left(|z|+1\right)^4}>\lambda\right\}=$$

$$=m\left\{|z|\geq 2:\ \ \frac{\left(|z|+1\right)^2}{|z|-1}<\sqrt{\frac{|\eta|}{\pi\lambda}}\right\}=m\left\{|z|\geq 2:\ \ |z|+3+\frac{4}{|z|-1}<\sqrt{\frac{|\eta|}{\pi\lambda}}\right\}\geq$$

$$\geq m\left\{|z|\geq 2:\ \ |z|+7<\sqrt{\frac{|\eta|}{\pi\lambda}}\right\}\geq\pi\left(\sqrt{\frac{|\eta|}{\pi\lambda}}-7\right)^2-4\pi. \tag{14}$$

It follows from (13) and (14) that

$$\lim_{\lambda\to 0+}\lambda m\left\{z\in C:\ \left|\left(Bf_1\right)(z)\right|>\lambda\right\}=|\eta|\,. \tag{15}$$

For any $0 < \varepsilon < 1$, by the inclusions

$$\{z \in C: \; |(Bf_1)(z)| > (1+\varepsilon)\lambda\} \setminus \{z \in C: \\ |(Bf_2)(z)| > \varepsilon\lambda\} \subset \{z \in C: \; |(Bf)(z)| > \lambda\} \subset \\ \subset \{z \in C: \; |(Bf_2)(z)| > \varepsilon\lambda\} \bigcup \{z \in C: \; |(Bf_1)(z)| > (1-\varepsilon)\lambda\}$$

and equalities (12), (15) we have

$$\frac{|\eta|}{1+\varepsilon} \le \liminf_{\lambda\to 0+} \lambda \cdot m\{z \in C: \; |(Bf)(z)| > \lambda\} \le$$

$$\le \limsup_{\lambda\to 0+} \lambda \cdot m\{z \in C: \; |(Bf)(z)| > \lambda\} \le \frac{|\eta|}{1-\varepsilon}.$$

This implies the equation (8) and completes the proof of the Theorem 2. ◀

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