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The Resolvent of the Discrete Dirac Operator

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Abstract. The discrete Dirac operator is considered whose coefficients tend to different limits on $\pm\infty$. An explicit form of the resolvent of this operator is found.

Key Words and Phrases: discrete Dirac operator, resolvent, Yost solution.

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1. Introduction and main result

We consider the system of difference equations

$$\begin{cases} a_{1,n}y_{2,n+1} + a_{2,n}y_{2,n} = \lambda y_{1,n}, \\ a_{1,n-1}y_{1,n-1} + a_{2,n}y_{1,n} = \lambda y_{2,n}, \ \ n = 0, \pm 1, \pm 2, ..., \end{cases} \tag{1}$$

where $a_{j,n}, , j = 1, 2,$ are real coefficients and satisfy the conditions

$$(-1)^{j-1} a_{j,n} > 0, n = 0, \pm 1, \pm 2, ..., \ a_{j,n} \to 0, n \to +\infty, j = 1, 2, \tag{2}$$

$$\sum_{n<0} |n| \left| (-1)^{j-1} a_{j,n} - 1 \right| < \infty, j = 1, 2. \tag{3}$$

Note that the system of difference equations (1) is a discrete analogue of the one-dimensional Dirac system. In this regard, the operator will be called the discrete Dirac operator. Various questions of the spectral theory of the Dirac operator were studied in [1, 2, 3]. We note that the direct and inverse problems of spectral analysis for the system (1) in various statements and in different classes were considered in [4, 5, 6, 7, 8, 9].

Let $\ell_2 ((-\infty, \infty), C)$ denote the Hilbert space of all complex vector sequences $y = \begin{pmatrix} y_{1,n} \\ y_{2,n} \end{pmatrix}_{n=-\infty}^{\infty}$ with the norm $\|y\| = \sum_{n=-\infty}^{\infty} \left\{ |y_{1,n}|^2 + |y_{2,n}|^2 \right\}$. We also define the operator L generated in $\ell_2 ((-\infty, \infty), C)$ by (1). By virtue of (2), (3), the operator L is bounded and self-adjoint.

It is known that in studying various problems of the spectral theory of linear operators, of particular interest are formulas for the expansion in eigenfunctions. In the present paper, an explicit form of the operator L resolvent is found. Similar questions for the one-dimensional Dirac system, the Schrödinger equation, and its difference analogue were investigated in the works [2, 5, 6, 7, 8, 9].

We denote the operator defined in $\ell_2([0,\infty), C)$ by system of equations (1) for $n \geq 0$ and the boundary condition $y_{1,0} = 0$ by L_0. It follows from the condition (2) that L_0 is a completely continuous self-adjoint operator. Since the eigenvalues of the operator L_0 are simple and L_0 is completely continuous, its spectrum consists of simple eigenvalues$\lambda_n = \pm\mu_n, n = 1, 2, ...$, where $\lambda_n \to 0$ as $n \to \infty$, and the point $\lambda = 0$. The latter is either a simple eigenvalue of the operator L_0 or the only point of its continuous spectrum. It is known (see, for example, [10], Ch. 7, § 4) that the eigenvectors of a completely continuous self-adjoint operator form an orthogonal basis in the corresponding space. Consequently, the spectral function of the operatorL_0, which we denote by$\rho(\lambda)$, is a step function concentrated at the points $\lambda_n, n = 1, 2, ...$ For the sake of simplicity, in what follows we assume that the spectrum of the operator L_0 lies in the interval $(-\ 2,\ 2)$. Denote by $P_{j,n}(\lambda)$, $Q_{j,n}(\lambda)$ the solutions of the system of equations (1), defined by the initial conditions $P_{1,0}(\lambda) = Q_{2,1}(\lambda) = 0$, $P_{2,1}(\lambda) = 1$, $Q_{1,1}(\lambda) = a_{2,1}^{-1}$.
Consider the spectral function

$$\rho(\lambda) = \sum_{\lambda_n < \lambda} \alpha_n^{-1},$$

where

$$\alpha_n = \sum_{k=1}^{\infty} \left\{ P_{1,k}^2(\lambda_n) + P_{2,k}^2(\lambda_n) \right\}, \sum_{n=1}^{\infty} \alpha_n^{-1} = 1.$$

Following [9], we introduce the Weyl function $m(\lambda) = \langle R_\lambda \delta, \delta \rangle$ of the operator L_0, where R_λ is the resolvent of the operator L_0 and $\delta = \begin{pmatrix} 0,0,0,... \\ 1,0,0,.. \end{pmatrix} \in \ell_2([0,\infty), C)$.
The Weyl function is related to the spectral function (see [11, 12]) by the equality

$$m(\lambda) = \int_{-\infty}^{\infty} \frac{d\rho(t)}{t - \lambda},$$

which implies that

$$m(\lambda) = \sum_{n=1}^{\infty} \frac{1}{\alpha_n(\lambda_n - \lambda)}. \tag{4}$$

We also introduce the Weyl solution

$$f_{j,n}^{+}(\lambda) = Q_{j,n}(\lambda) + m(\lambda) P_{j,n}(\lambda) \tag{5}$$

of the system of equations (1). By (4), the Weyl solution is analytic on the whole complex λ-plane except for the simple poles$\lambda_k, k = 1, 2, ...$ (The point $\lambda = 0$ is a nonisolated singularity of the Weyl solution). In addition, it is known (see, for instance, [11, 12]) that for $n > 0$ the equality $f_{j,n}^{+}(\lambda) = (R_\lambda \delta)_n$ is valid. Consequently, for every $N > -\infty$ the Weyl solution belongs to $\ell_2([N,\infty), C)$ with respect to the variable n.

We denote by Γ the complex $\lambda-$plane with a cut along the segment $[-2,2]$. In the plane we consider the function

$$z = z(\lambda) = -\frac{\lambda^2-2}{2} + \frac{\lambda}{2}\sqrt{\lambda^2-4},$$

choosing a regular branch of the radical such that $\sqrt{\lambda^2-4} > 0$ with $\lambda > 2$. It is known that the system of equation (1) has solution $\left\{ f_{j,n}^{-}(\lambda) \right\}$, $j = 1,2$, representable in the form [9]

$$\left. f_{j,n}^{-}(\lambda) = \alpha_j^{-}(n) \left(\frac{z^{-1}-1}{\lambda} \right)^{2-j} z^{-n} \left(1 + \sum_{m \le 1} K_j^{-}(n,m)\, z^{-m} \right),\ n = 0, \pm 1, \pm 2, ..., \right\} \tag{6}$$

and the quantities $\alpha_1^{\pm}(n)$, $\alpha_2^{\pm}(n)$, $K_1^{\pm}(n,m)$, $K_2^{\pm}(n,m)$ satisfy the relations

$$\left. \begin{array}{l} \alpha_j^{-}(n) = 1 + o(1), \quad n \to -\infty,\ j = 1,2, \\ K_j^{-}(n,m) = O\left(\sigma^{-}\left(n + \left[\frac{m}{2} \right] + 1 \right) \right), n + m \to -\infty \end{array} \right\} \tag{7}$$

where $\sigma^{-}(n) = \sum_{m \le n} \{ |a_{1,m} - 1| + |a_{2,m} + 1| \}$, by $[x]$ denote the integer part x . According to (6), (7) for each functions $\left\{ f_{j,n}^{-}(\lambda) \right\}$, $j = 1,2$, are regular in the plane Γ and continuous up to its boundary $\partial\Gamma$.

Let $u_{j,n}$ and $v_{j,n}$ be two solutions of the system of equations (1). We call them the Wronskian quantity $\{u_{j,n}, v_{j,n}\} = a_{1,n-1}(u_{1,n-1}v_{2,n} - u_{2,n}v_{1,n-1})$. Put $w(\lambda) = \left\{ f_{j,n}^{+}(\lambda), f_{j,n}^{-}(\lambda) \right\}$. Let us state the main result of this paper.

Theorem 1. *The functions*

$$R_{nm}(\lambda) = \begin{pmatrix} R_{nm}^{11} & R_{nm}^{12} \\ R_{nm}^{21} & R_{nm}^{22} \end{pmatrix}, R_{nm}^{ij} = -w^{-1}(\lambda) \begin{cases} f_{i,n}^{+}(\lambda)\, f_{j,m}^{-}(\lambda), m \le n, \\ f_{j,m}^{+}(\lambda)\, f_{i,n}^{-}(\lambda), m > n, \end{cases} \tag{8}$$

are elements of the operator L resolvent matrix and satisfy the equations

$$\begin{array}{l} a_{1,n}R_{n+1,m}^{22} + a_{2,n}R_{nm}^{22} - \lambda R_{nm}^{12} = 0, \\ a_{1,n}R_{n+1,m}^{21} + a_{2,n}R_{nm}^{21} - \lambda R_{nm}^{11} = \delta_{nm}, \\ a_{1,n-1}R_{n-1,m}^{11} + a_{2,n}R_{nm}^{11} - \lambda R_{nm}^{21} = 0, \\ a_{1,n-1}R_{n-1,m}^{12} + a_{2,n}R_{nm}^{12} - \lambda R_{nm}^{22} = \delta_{nm}, \end{array} \tag{9}$$

where δ_{nm} is the Kronecker symbol.

Proof. Let $h = \{h_{1,n}, h_{2,n}\} \in \ell^2((-\infty,\infty); C)$ be an arbitrary finite sequence. In order to construct the resolvent of the operator L, we need to solve the equation

$$Ly = \lambda y + h.$$

We rewrite the last equation in the form

$$\begin{cases} a_{1,n}y_{2,n+1} + a_{2,n}y_{2,n} = \lambda y_{1,n} + h_{1,n}, \\ a_{1,n-1}y_{1,n-1} + a_{2,n}y_{1,n} = \lambda y_{2,n} + h_{2,n}. \end{cases} \tag{10}$$

We are looking for a solution to the system of equations in the form

$$y_{j,n} = C_n f^{+}_{j,n}(\lambda) + D_n f^{-}_{j,n}(\lambda)\, j = 1, 2, \tag{11}$$

where C_n and D_n are the quantities to be determined. Substituting representation (11) into the system of equations (10) after simple transformations, we obtain

$$\begin{cases} a_{1,n-1}\left(C_{n-1} - C_n\right) f^{+}_{1,n-1}(\lambda) + a_{1,n-1}\left(D_{n-1} - D_n\right) f^{-}_{1,n-1}(\lambda) = h_{2,n}, \\ a_{1,n-1}\left(C_{n-1} - C_n\right) f^{+}_{2,n}(\lambda) + a_{1,n-1}\left(D_{n-1} - D_n\right) f^{-}_{2,n}(\lambda) = -h_{1,n-1}. \end{cases}$$

Solving the last system of equations with respect to $C_{n-1} - C_n$ and $D_{n-1} - D_n$, we find that

$$C_{n-1} - C_n = w^{-1}(\lambda)\left[f^{-}_{1,n-1}(\lambda)\, h_{1,n-1} + f^{-}_{2,n}(\lambda)\, h_{2,n}\right], \tag{12}$$

$$D_{n-1} - D_n = w^{-1}(\lambda)\left[f^{+}_{1,n-1}(\lambda)\, h_{1,n-1} + f^{+}_{2,n}(\lambda)\, h_{2,n}\right]. \tag{13}$$

Note that to fulfil the conditions $y \in \ell^2\left((-\infty, \infty); C\right)$ you need to take $C_{-\infty} = 0$, $D_{\infty} = 0$. Adding then equalities (12) for $n = n, n-1, n-2, ...,$ and equalities (13) for $n = n + 1, , n+2, n+3, ...,$ we have

$$\begin{aligned} C_n &= -w^{-1}(\lambda) \textstyle\sum_{k=-\infty}^{n-1} \left[f^{-}_{1,k}(\lambda)\, h_{1,k} + f^{-}_{2,k+1}(\lambda)\, h_{2,k+1}\right], \\ D_n &= -w^{-1}(\lambda) \textstyle\sum_{k=n}^{\infty} \left[f^{+}_{1,k}(\lambda)\, h_{1,k} + f^{+}_{2,k+1}(\lambda)\, h_{2,k+1}\right]. \end{aligned}$$

Substituting the last equalities into representation (11), we obtain

$$\begin{aligned} y_{j,n} = &-w^{-1}(\lambda)\left[\textstyle\sum_{k=-\infty}^{n-1} f^{+}_{j,n}(\lambda)\, f^{-}_{1,k}(\lambda)\, h_{1,k} + \sum_{k=n}^{\infty} f^{-}_{j,n}(\lambda)\, f^{+}_{1,k}(\lambda)\, h_{1,k}\right] - \\ &-w^{-1}(\lambda)\left[\textstyle\sum_{k=-\infty}^{n-1} f^{+}_{j,n}(\lambda)\, f^{-}_{2,k}(\lambda)\, h_{2,k} + \sum_{k=n}^{\infty} f^{-}_{j,n}(\lambda)\, f^{+}_{2,k}(\lambda)\, h_{2,k}\right]. \end{aligned}$$

On the other hand, by the definition of the resolvent, we have

$$y_{j,n} = \sum_{k=-\infty}^{\infty} \left[R^{j1}_{nk} h_{1,k} + R^{j2}_{nk} h_{2,k}\right]. \tag{14}$$

Comparison of the last equalities leads us to formulas (8). Using (8), it is directly verified that equations (9) are valid, and it follows from (9) that the vector $y = \{y_{1,n}, y_{2,n}\}_{-\infty}^{\infty}$, defined by formula (14) is a solution to the system of equations (10). Thus, the theorem is proved. ◄

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