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One Remark on the Eigenvalues of the Schrodinger Operator with Growing Potential

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Abstract. The Schrodinger operator $L = -\frac{d^2}{dx^2} + |x|$ on the whole axis is considered. The spectrum of the operator is investigated. An asymptotic formula for eigenvalues is obtained.

Key Words and Phrases: Schrodinger operator, Airy equation, Airy functions, eigenvalues.

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1. Introduction and main result

The spectral properties of the Airy operator $L_D y = -y'' + xy$, $y(0) = 0$ or $L_N y = -y'' + xy$, $y'(0) = 0$ were studied in the works of quite a few authors (see [1, 2, 3, 4, 5, 6, 7, 8, 9] also the literature there). The interest is also the corresponding Schrodinger operator on the whole axis.

In the space $L_2(-\infty, \infty)$ we consider the operator$L$, generated by the differential expression

$$l(y) = -y'' + |x|\, y$$

with the domain

$$D(L) = \left\{ y \in L_2(-\infty, \infty) : y \in W^2_{2,loc},\ l(y) \in L_2(-\infty, \infty) \right\}.$$

Note that the operator L is densely defined, because its domain contains infinitely differentiable functions compactly supported on $(-\infty, \infty)$, the set of these functions is well known to be dense in $L_2(-\infty, \infty)$, since its domain of definition contains infinitely differentiable functions with compact support on the interval, the set of which is dense in. Moreover, L is a self-adjoint operator. Obviously, the spectrum of the operator is discrete and consists of eigenvalues $\lambda_n, n = 1, 2, ...$, where $\lambda_n \to \infty$ for $n \to \infty$.

We will be interested the asymptotic behavior of eigenvalues λ_n.

First, consider the equation

$$-y'' + xy = \lambda y,\ -\infty < x < \infty, \quad \lambda \in C. \tag{1}$$

It is well known [10] that this equation has two linearly independent solutions in the form $Ai\,(x-\lambda)\,,\ Bi\,(x-\lambda)$, where $Ai\,(z)\,,\ Bi\,(z)$ are the Airy functions of the first and second kind, respectively.

We note some properties of these functions. As is known (see [8, 10]), both functions are entire functions of order $3/2$ and type $2/3$. The function $Ai\,(z)$ admits the following asymptotic representations as $|z|\to\infty$

$$\begin{gathered}\mathrm{A}i\,(z)\sim\pi^{-\frac{1}{2}}z^{-\frac{1}{4}}e^{-\zeta}\left[1+O\left(\zeta^{-1}\right)\right],\ |\arg z|<\pi,\\ Ai'\,(z)\sim-\pi^{-\frac{1}{2}}z^{\frac{1}{4}}e^{-\zeta}\left[1+O\left(\zeta^{-1}\right)\right],\ |\arg z|<\pi,\end{gathered}$$

$$\mathrm{A}i\,(-z)\sim\pi^{-\frac{1}{2}}z^{-\frac{1}{4}}\sin\left(\zeta+\frac{\pi}{4}\right)\left[1+O\left(\zeta^{-1}\right)\right],\ |\arg z|<\frac{2\pi}{3},$$

$$Ai'\,(-z)\sim-\pi^{-\frac{1}{2}}z^{\frac{1}{4}}\cos\left(\zeta+\frac{\pi}{4}\right)\left[1+O\left(\zeta^{-1}\right)\right],\ |\arg z|<\frac{2\pi}{3},$$

where $\zeta=\frac{2}{3}z^{\frac{3}{2}}$. In the sector $|z|<\frac{\pi}{3}$ the function $Bi\,(z)$ has an asymptotic representation

$$Bi\,(z)\sim\pi^{-\frac{1}{2}}z^{-\frac{1}{4}}e^{\zeta}\left[1+O\left(\zeta^{-1}\right)\right].$$

Thus, the functions $Bi\,(z)$ grow exponentially as $|z|\to\infty$ along any ray in this sector. For the Wronskian of functions $Ai\,(z)\,,Bi\,(z)$ the equality

$$\{Ai\,(z)\,,Bi\,(z)\}=Ai\,(z)\,Bi'\,(z)-Ai'\,(z)\,Bi\,(z)=\pi^{-1}\tag{2}$$

is valid.

We now consider the equation

$$-y''+|x|\,y=\lambda y,\ -\infty<x<\infty,\quad\lambda\in C.\tag{3}$$

According to the general theory (see [11]), equation (3) has two linearly independent solutions $\psi_{\pm}\,(x,\lambda)$, which for each $\lambda, Im\lambda>0$ satisfy the conditions $\psi_{\pm}\,(x,\lambda)\in L_2\,(0,\pm\infty)$. Since equation (3) does not change when x replaced by $-x$, the function $\psi_{\pm}\,(-x,\lambda)$ is also its solution. Therefore, we can assume that $\psi_{-}\,(x,\lambda)=\psi_{+}\,(-x,\lambda)$.

On the other hand, since $Ai\,(x-\lambda)\in L_2\,(0,\infty)$, the functions $\psi_{+}\,(x,\lambda)\,,\ Ai\,(x-\lambda)$ coincide up to a factor. Based on these considerations, for $x\geq 0$ we set $\psi_{+}\,(x,\lambda)=Ai\,(x-\lambda)$. Further, when $x\leq 0$ looking at the solution $\psi_{+}\,(x,\lambda)$ in the form

$$\psi_{+}\,(x,\lambda)=\alpha Ai\,(-x-\lambda)+\beta Bi\,(-x-\lambda)\,,$$

since the functions $Ai\,(-x-\lambda)\,,\,Bi\,(-x-\lambda)$ form the fundamental system of solutions of equation (1) for $x\leq 0$. Taking into account that the solution $\psi_{+}\,(x,\lambda)$ and its derivative $\psi'_{+}\,(x,\lambda)$ are continuous at a point $x=0$, to determine the coefficients α,β we obtain the following system of equations

$$\begin{cases}\ \ Ai\,(-\lambda)\,\alpha+Bi\,(-\lambda)\,\beta=Ai\,(-\lambda)\\ Ai'\,(-\lambda)\,\alpha+Bi'\,(-\lambda)\,\beta=-Ai'\,(-\lambda)\end{cases}$$

Solving the last system with respect to the coefficients α, β and taking into account equality (2), we obtain

$$\alpha = -\pi \left(Ai\left(-\lambda\right) Bi\left(-\lambda\right)\right)',$$
$$\beta = -2\pi Ai\left(-\lambda\right) Ai'\left(-\lambda\right).$$

So, we have proved the following theorem.

Theorem 1. *Equation* (3) *has special solutions* $\psi_{\pm}(x,\lambda)$*, which can be represented in the form*

$$\psi_{+}(x,\lambda) = \begin{cases} Ai\left(x-\lambda\right), x \geq 0, \\ -\pi \left(Ai\left(-\lambda\right) Bi\left(-\lambda\right)\right)' Ai\left(-x-\lambda\right) - 2\pi Ai\left(-\lambda\right) Ai'\left(-\lambda\right) Bi\left(-x-\lambda\right), x < 0 \end{cases}$$

$$\psi_{-}(x,\lambda) = \begin{cases} -\pi \left(Ai\left(-\lambda\right) Bi\left(-\lambda\right)\right)' Ai\left(x-\lambda\right) - 2\pi Ai\left(-\lambda\right) Ai'\left(-\lambda\right) Bi\left(x-\lambda\right), x \geq 0, \\ Ai\left(-x-\lambda\right), x < 0 \end{cases}$$

We return now to the study of the spectrum of the operator L. From the fact that $\psi_{\pm}(x,\lambda) \in L_2(0,\pm\infty)$, if $\lambda = \lambda_n$ is an eigenvalue, then the solutions $\psi_{+}(x,\lambda_n)$ and $\psi_{-}(x,\lambda_n)$ are linearly dependent. In fact, since

$$\psi_{+}(x,\lambda_n) = \begin{cases} Ai\left(x-\lambda_n\right), x \geq 0, \\ (-1)^{n-1} Ai\left(-x-\lambda_n\right), x < 0, \end{cases}$$

$$\psi_{-}(x,\lambda_n) = \begin{cases} (-1)^{n-1} Ai\left(x-\lambda_n\right), x \geq 0, \\ Ai\left(-x-\lambda_n\right), x < 0, \end{cases}$$

then following equality holds

$$\psi_{+}(x,\lambda_n) = (-1)^{n-1} \psi_{-}(x,\lambda_n).$$

From these arguments it follows that the eigenvalues of the operator coincide with the zeros of the function

$$\Delta(\lambda) = \left\{\psi_{+}(x,\lambda), \psi_{-}(x,\lambda)\right\}.$$

Taking advantage of the fact that the Wronskian of the two solutions does not depend on x, we obtain

$$\Delta(\lambda) = \left.\left\{\psi_{+}(x,\lambda), \psi_{-}(x,\lambda)\right\}\right|_{x=0} = -2Ai\left(-\lambda\right) Ai'\left(-\lambda\right). \tag{4}$$

From the last formula and the known properties of the zeros of functions $Ai(\lambda)$, $Ai'(\lambda)$ (see [10]) it follows that the eigenvalues $\lambda_n, n = 1, 2, ...$ of the operator L are located only on the positive semi-axis and holds the following asymptotic equality

$$\lambda_n = \left(\frac{3\pi\left(2n-1\right)}{8}\right)^{\frac{2}{3}} \left(1 + O\left(n^{-2}\right)\right), n \to \infty.$$

Let us prove that the eigenvalues of the operator L are simple. We introduce normalization numbers $\alpha_n, n = 1, 2, ...$, setting

$$\alpha_n = \sqrt{\int_{-\infty}^{\infty} |\psi_\pm(x,\lambda_n)|^2 dx}. \tag{5}$$

Let us agree with dots to denote differentiation with respect to λ, and strokes with respect to x:

$$u' = \frac{\partial}{\partial x}u,\ \dot{u} = \frac{\partial}{\partial \lambda}u.$$

Since $\psi_\pm(x,\lambda)$ decreases exponentially for $x \to \pm\infty$, from the standard (see, e.g., [12]) identity

$$f^2 = \left\{\dot{f}, f\right\}'$$

and (5) it follows that

$$\begin{aligned}
&(\alpha_n)^2 = \textstyle\int_{-\infty}^{\infty} \psi_+^2(x,\lambda_n)\,dx = \int_0^{\infty} \psi_+^2(x,\lambda_n)\,dx + \int_{-\infty}^{0} \psi_-^2(x,\lambda_n)\,dx = \\
&= \left\{\dot{\psi}_+(x,\lambda_n), \psi_+(x,\lambda_n)\right\}\Big|_0^{\infty} + \left\{\dot{\psi}_-(x,\lambda_n), \psi_-(x,\lambda_n)\right\}\Big|_{-\infty}^{0} = \\
&= -\left\{\dot{\psi}_+(x,\lambda_n), \psi_+(x,\lambda_n)\right\}\Big|_{x=0} + \left\{\dot{\psi}_-(x,\lambda_n), \psi_-(x,\lambda_n)\right\}\Big|_{x=0} = \\
&= -(-1)^{n-1}\left\{\dot{\psi}_+(x,\lambda_n), \psi_-(x,\lambda_n)\right\}\Big|_{x=0} - \\
&(-1)^{n-1}\left\{\psi_+(x,\lambda_n), \dot{\psi}_-(x,\lambda_n)\right\}\Big|_{x=0} = -(-1)^{n-1}\dot{\Delta}(\lambda_n).
\end{aligned}$$

Therefore, $\dot{\Delta}(\lambda_n) \neq 0$, i.e. the eigenvalues of the operator L are simple.

Thus, the following theorem holds.

Theorem 2. *The spectrum of the operator L consists of a sequence of simple real eigenvalues $\lambda_n, n \geq 1$, located on the positive semi-axis and*

$$\lambda_n = \left(\frac{3\pi(2n-1)}{8}\right)^{\frac{2}{3}} \left(1 + O\left(n^{-2}\right)\right), n \to \infty.$$

Remark 1. *In the space $L_2(0,\infty)$ we consider the operators $L_D y = -y'' + |x|\,y$, $y(0) = 0$ and $L_N y = -y'' + |x|\,y$, $y'(0) = 0$. Formula (4) shows that the spectrum of the operator consists of the union of the spectra of the operators L_D and L_N.*

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