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On an Inverse Boundary-value Problem for the Equation of Motion of a Homogeneous Beam

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Abstract. In this work, a classical solution of an inverse boundary-value problem for the equation of motion of a homogeneous beam with periodic boundary conditions is studied. Firstly, the original problem is reduced to an equivalent (in a defined sense) problem, for which the existence and uniqueness theorem of the solution is proved. Further, using the unique solvability of the equivalent problem, the classical solvability of the original problem is showed.

Key Words and Phrases: Inverse boundary-value problem, classical solution, Fourier's method, homogeneous beam.
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1. Introduction

Recently, there are many cases in which the needs in a practice lead to the problems of determining the coefficients of a differential equation (ordinary or in partial derivatives) from some known functional of its solution. Such problems are called inverse problems of mathematical physics. The applied importance of inverse problems is so great (they arise in various fields of human activity, such as seismology, mineral exploration, biology, medicine, desalination of seawater, movement of liquid in a porous medium, etc.) which puts them a series of the most actual problems of modern mathematics. The presence in the inverse problems of additional unknown functions requires that in the complement to the boundary conditions that are natural for a particular class of differential equations, impose some additional conditions - overdetermination conditions. The basics of the theory and practice of investigating inverse problems of mathematical physics were established and developed in the fundamental works of the outstanding scientists A.N.Tikhonov [1], M.M.Lavrent'ev [2], V.K.Ivanov [3], and their followers.

Inverse problems associated with equations of various types, have been studied by many papers and monographs, in particular, [4]-[13]. But the problem statement and the proof techniques used in this paper are different from those presented in these works. The technique used in this paper is based on the passing from the original inverse problem to the new equivalent one, the study of the solvability of the equivalent problem, and then in the reverse transition to the original problem.

Moreover, the vibrations and wave movements of an elastic beam on an elastic base were investigated by Yu.A. Mitropolsky [14], J.M.Thompson [15], B.S.Bardin [16], V.Z. Vlasov [17], D.V.Kostin [18], T.P.Goy [19], Ya.T.Mehraliyev [20], and et al. The simplest nonlinear model of the motion of a homogeneous beam is described by the equation

$$\frac{\partial^2 w}{\partial t^2}+\frac{\partial^4 w}{\partial x^4}+k\frac{\partial^2 w}{\partial x^2}+\alpha w+w^3=0,$$

where ω is beam deflection (after the displacements of the points of the midline of the elastic beam located along the x-axis). Note that a similar equation also arises in the theory of crystals, in which ω is parameter of order [21].

2. Statement of the problem

This paper is concerned with the following inverse problem of finding a pair $\{u(x,t),p(t)\}$ in the domain $D_T=\{(x,t):0\le x\le 1, 0\le t\le T\}$ for the following system

$$u_{tt}(x,t)+u_{xxxx}(x,t)+\beta u_{xx}(x,t)+\alpha u(x,t)+u^3(x,t)=p(t)g(x,t)+f(x,t),\ (x,t)\in D_T, \quad (1)$$

$$u(x,0)+\delta u(x,T)=\varphi(x),\ u_t(x,0)+\delta u_t(x,T)=\psi(x),\quad 0\le x\le 1, \quad (2)$$

$$u(0,t)=u(1,t),\quad u_x(0,t)=u_x(1,t),$$
$$u_{xx}(0,t)=u_{xx}(1,t),\quad u_{xxx}(0,t)=u_{xxx}(1,t),\quad 0\le t\le T, \quad (3)$$

$$u(x_0,t)=h(t),\quad 0\le t\le T, \quad (4)$$

where $x_0\in(0,1)$ is fixed number, $\alpha>0,\ \beta>0$, and δ are given numbers, and $\beta<4\alpha,\ g(x,t),\ f(x,t),\ \varphi(x),\ \psi(x),\ h(t)$ are known functions.

We introduce the set of functions

$$\tilde{C}^{2,4}(D_T)=\{u(x,t):u(x,t)\in C^2(D_T),u_{xxxx}(x,t)\in C(D_T)\}.$$

Definition 1. *The pair $\{u(x,t),p(t)\}$ defined on D_T is said to be a classical solution of the problem (1)-(4), if the functions $u(x,t)\in\tilde{C}^{2,4}(D_T)$ and $p(t)\in C[0,T]$ satisfies Eq. (1), condition (2) on $[0,1]$, and the statements (3)-(4) on the interval $[0,T]$.*

It's easy to prove that

Lemma 1. *Suppose that $f(x,t),\ g(x,t)\in C(D_T),\ g(0,t)\ne 0,\ 0\le t\le T,\ \varphi(x),\ \psi(x)\in C[0,1],\ h(t)\in C^2[0,T],\ \delta\ne\pm 1$, and the condition*

$$\varphi(x_0)=h(0)+\delta h(T),\ \psi(x_0)=h'(0)+\delta h'(T)$$

holds. Then the problem of finding a classical solution of (1)-(4) is equivalent to the problem of determining the functions $u(x,t)\in\tilde{C}^{2,4}(D_T)$ and $p(t)\in C[0,T]$ from the (1)-(3), and satisfying the condition

$$h''(t)+u_{xxxx}(x_0,t)+\beta u_{xx}(x_0,t)+\alpha h(t)+u^3(x_0,t)$$
$$=p(t)g(x_0,t)+f(x_0,t),\ 0\le t\le T. \quad (5)$$

3. Classical solvability of inverse boundary-value problem

It is known that [22] the system

$$1, \cos\lambda_1 x, \sin\lambda_1 x, ..., \cos\lambda_k x, \sin\lambda_k x, ... \tag{6}$$

are a bases in $L_2(0,1)$, for $\lambda_k = 2k\pi$ $(k = 0, 1, ...)$.

Since the system (6) forms a basis in $L_2(0,1)$, then it is obvious that the first component of the solution $\{u(x,t), p(t)\}$ has the form:

$$u(x,t) = \sum_{k=0}^{\infty} u_{1k}(t)\cos\lambda_k x + \sum_{k=1}^{\infty} u_{2k}(t)\sin\lambda_k x,\ \ \lambda_k = 2k\pi,\ \ k = 0, 1, ..., \tag{7}$$

where

$$u_{10}(t) = \int_0^1 u(x,t)dx,\ \ u_{1k}(t) = 2\int_0^1 u(x,t)\cos\lambda_k x dx,\ \ k = 0, 1, ...,$$

$$u_{2k}(t) = 2\int_0^1 u(x,t)\sin\lambda_k x dx,\ \ k = 0, 1, ...$$

Then applying the formal scheme of the Fourier method, from (1) and (2) we have

$$u''_{10}(t) + \alpha u_{10}(t) = F_{10}(t; u, p),\ \ 0 \le t \le T, \tag{8}$$

$$u''_{ik}(t) + (\lambda_k^4 - \beta\lambda_k^2 + \alpha)u_{ik}(t) = F_{ik}(t; u, p),\ \ 0 \le t \le T;\ \ i = 1, 2;\ \ k = 1, 2, ..., \tag{9}$$

$$u_{10}(0) + \delta u_{10}(T) = \varphi_{10},\ \ u'_{10}(0) + \delta u'_{10}(T) = \psi_{10}, \tag{10}$$

$$u_{ik}(0) + \delta u_{ik}(T) = \varphi_{ik},\ \ u'_{ik}(0) + \delta u'_{ik}(T) = \psi_{ik},\ \ i = 1, 2;\ \ k = 1, 2, ..., \tag{11}$$

where

$$F_{1k}(t; u, p) = p(t)g_{1k}(t) + f_{1k}(t) - G_{1k}(t, u),\ \ k = 0, 1, 2...,$$

$$g_k(t) = m_k \int_0^1 g(x,t)\cos\lambda_k x dx,$$

$$F_{1k}(t; u, p) = f_{1k}(t) + p(t)u_{1k}(t),\ \ k = 0, 1, 2...,$$

$$f_{10}(t) = \int_0^1 f(x,t)dx,\ \ f_{1k}(t) = 2\int_0^1 f(x,t)\cos\lambda_k x dx,\ \ k = 1, 2, ...,$$

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$$G_{10}(t,u)=\int_0^1 u^3(x,t)dx,\ G_{1k}(t,u)=2\int_0^1 u^3(x,t)\cos\lambda_k x dx,\ k=1,2,...,$$

$$\varphi_{10}=\int_0^1 \varphi(x)dx,\ \psi_{10}=\int_0^1 \psi(x)dx,$$

$$\varphi_{1k}=2\int_0^1 \varphi(x)\cos\lambda_k x dx,\ \psi_{1k}=2\int_0^1 \psi(x)\cos\lambda_k x dx,\ k=1,2,...,$$

$$F_{2k}(t;u,p)=p(t)g_{2k}(t)+f_{2k}(t)-G_{2k}(t,u),\ k=0,1,2,...,$$

$$f_{2k}(t)=2\int_0^1 f(x,t)\sin\lambda_k x dx,\ g_{2k}(t)=2\int_0^1 f(x,t)\sin\lambda_k x dx,\ k=1,2,...,$$

$$G_{2k}(t,u)=2\int_0^1 u^3(x,t)\sin\lambda_k x dx,\ k=1,2,...,$$

$$\varphi_{2k}=2\int_0^1 \varphi(x)\sin\lambda_k x dx,\ \psi_{2k}=2\int_0^1 \psi(x)\sin\lambda_k x dx,\ k=1,2,....$$

Solving problem (8) - (11), we find

$$u_{10}(t)=\frac{1}{\sqrt{\alpha}\rho_0(T)}\{\sqrt{\alpha}(\cos\sqrt{\alpha}t+\delta\cos\sqrt{\alpha}(T-t))\varphi_0$$

$$+(\sin\sqrt{\alpha}t-\delta\sin\sqrt{\alpha}(T-t))\psi_0-\delta\int_0^T F_0(\tau;u,p)(\sin\sqrt{\alpha}(T+t-\tau)$$

$$+\delta\sin\sqrt{\alpha}(t-\tau))d\tau\}+\frac{1}{\sqrt{\alpha}}\int_0^t F_0(\tau;u,p)\sin\sqrt{\alpha}(t-\tau)d\tau, \tag{12}$$

$$u_{ik}(t)=\frac{1}{\beta_k\rho_k(T)}\{\beta_k(\cos\beta_k t+\delta\cos\beta_k(T-t))\varphi_{ik}+(\sin\beta_k t$$

$$\left.-\delta\sin\beta_k(T-t))\psi_{ik}-\delta\int_0^T F_{ik}(\tau;u,p)(\sin\beta_k(T+t-\tau)+\delta\sin\beta_k(t-\tau))d\tau\right\}$$

$$+\frac{1}{\beta_k}\int_0^t F_{ik}(\tau;u,p)\sin\beta_k(t-\tau)d\tau,\ i=1,2;\ k=1,2,\ldots, \tag{13}$$

where

$$\rho_0(T)=1+2\delta\cos\sqrt{\alpha}T+\delta^2,\ \beta_k=\sqrt{\lambda_k^4-\beta\lambda_k^2+\alpha},$$

$$\rho_k(T)=1+2\delta\cos\beta_k T+\delta^2,\ k=1,2,\ldots.$$

Differentiating twice (13) gives

$$u'_{ik}(t)=\frac{1}{\rho_k(T)}(\beta_k(-\sin\beta_k t+\delta\sin\beta_k(T-t))\varphi_{ik}+(\cos\beta_k t$$

$$\left.+\delta\cos\beta_k(T-t))\psi_{ik}-\delta\int\limits_0^T F_{ik}(\tau;u,p)(\cos\beta_k(T+t-\tau)+\delta\cos\beta_k(t-\tau))d\tau\right\}$$

$$+\int\limits_0^t F_{ik}(\tau;u,p)\cos\beta_k(t-\tau)d\tau,\ i=1,2;\quad k=1,2,\ \ldots, \tag{14}$$

$$u''_{ik}(t)=F_{ik}(t;u,p)-\frac{\beta_k}{\rho_k(T)}\{\beta_k(\cos\beta_k t+\delta\sin\beta_k(T-t))\varphi_{ik}$$

$$+(\sin\beta_k t-\delta\sin\beta_k(T-t))\psi_{ik}$$

$$\left.-\delta\int\limits_0^T F_{ik}(\tau;u)(\sin\beta_k(T+t-\tau)+\delta\sin\beta_k(t-\tau))d\tau\right\}$$

$$-\beta_k\int\limits_0^t F_{ik}(\tau;u)\sin\beta_k(t-\tau)d\tau,\ i=1,2;\ k=1,2,\ldots. \tag{15}$$

In order to determine the first component of the solution of the problem (1)-(3), (5) we substitute of $u_{1k}(t)$ $(k=0,1,2,...)$ and $u_{2k}(t)$ $(k=1,2,...)$ into (7), we obtain

$$u(x,t)=\frac{1}{\sqrt{\alpha}\rho_0(T)}\{\sqrt{\alpha}(\cos\sqrt{\alpha}t+\delta\cos\sqrt{\alpha}(T-t))\varphi_{10}$$

$$+(\sin\sqrt{\alpha}t-\delta\sin\sqrt{\alpha}(T-t))\psi_{10}$$

$$\left.-\delta\int\limits_0^T F_{10}(\tau;u,p)(\sin\sqrt{\alpha}(T+t-\tau)+\delta\sin\sqrt{\alpha}(t-\tau))d\tau\right\}$$

$$+\frac{1}{\sqrt{\alpha}}\int\limits_0^t F_{10}(\tau;u,p)\sin\sqrt{\alpha}(t-\tau)d\tau$$

$$+\sum_{k=1}^{\infty}\left\{\frac{1}{\beta_k\rho_k(T)}[\beta_k(\cos\beta_k t+\delta\cos\beta_k(T-t))\varphi_{1k}+(\sin\beta_k t\right.$$

$$+\delta \sin \beta_k (T-t))\psi_{1k} - \delta \int_0^T F_{1k}(\tau; u, p)(\sin \beta_k (T+t-\tau) + \delta \sin \beta_k (t-\tau))d\tau \Bigg]$$

$$+\frac{1}{\beta_k} \int_0^t F_{1k}(\tau; u, p) \sin \beta_k (t-\tau) d\tau \Bigg\} \cos \lambda_k x$$

$$+\sum_{k=1}^{\infty} \left\{ \frac{1}{\beta_k \rho_k(T)} [\beta_k (\cos \beta_k t + \delta \cos \beta_k (T-t))\varphi_{2k} + (\sin \beta_k t \right.$$

$$+\delta \sin \beta_k (T-t))\psi_{2k} - \delta \int_0^T F_{2k}(\tau; u, p)(\sin \beta_k (T+t-\tau) + \delta \sin \beta_k (t-\tau))d\tau \Bigg]$$

$$\left. +\frac{1}{\beta_k} \int_0^t F_{2k}(\tau; u, p) \sin \beta_k (t-\tau) d\tau \right\} \sin \lambda_k x. \tag{16}$$

Now, from (5), taking into account (6), we have

$$p(t) = [g(0,t)]^{-1} \{h''(t) + \alpha h(t) - f(x_0, t) + u^3(x_0, t)$$

$$+\sum_{k=1}^{\infty} (\lambda_k^4 - \beta \lambda_k^2) u_{1k}(t) \cos \lambda_k x_0 + \sum_{k=1}^{\infty} (\lambda_k^4 - \beta \lambda_k^2) u_{2k}(t) \sin \lambda_k x_0. \tag{17}$$

In this way to obtain the equation for the second component of the solution to the problem (1) - (3), (5) we substitute expression (13) into (17) and get

$$p(t) = [g(0,t)]^{-1} \left\{ h''(t) + \alpha h(t) - f(x_0, t) + u^3(x_0, t) + \sum_{k=1}^{\infty} (\lambda_k^4 - \beta \lambda_k^2) \right.$$

$$\times \left[\frac{1}{\beta_k \rho_k(T)} [\beta_k (\cos \beta_k t + \delta \cos \beta_k (T-t))\varphi_{1k} + (\sin \beta_k t - \delta \sin \beta_k (T-t))\psi_{1k} \right.$$

$$-\delta \int_0^T F_{1k}(\tau; u, p)(\sin \beta_k (T+t-\tau) + \delta \sin \beta_k (t-\tau))d\tau \Bigg]$$

$$\left. +\frac{1}{\beta_k} \int_0^t F_{1k}(\tau; u, p) \sin \beta_k (t-\tau) d\tau \right] \cos \lambda_k x_0 + \sum_{k=1}^{\infty} (\lambda_k^4 - \beta \lambda_k^2)$$

$$\times\left[\frac{1}{\beta_k\rho_k(T)}[\beta_k(\cos\beta_k t+\delta\cos\beta_k(T-t))\varphi_{2k}+(\sin\beta_k t-\delta\sin\beta_k(T-t))\psi_{2k}\right.$$

$$\left.-\delta\int\limits_0^T F_{2k}(\tau;u,p)(\sin\beta_k(T+t-\tau)+\delta\sin\beta_k(t-\tau))d\tau\right]$$

$$\left.+\frac{1}{\beta_k}\int\limits_0^t F_{2k}(\tau;u,p)\sin\beta_k(t-\tau)d\tau\right]\sin\lambda_k x_0. \qquad (18)$$

Thus, finding the solution of problem (1) - (3), (5) is reduced to the finding solution of system (16), (18) with respect to unknown functions $u(x,t)$ and $p(t)$.

The following lemma plays an important role in studying the uniqueness of the solution to problem (1) - (3), (5):

Lemma 2. *If $\{u(x,t),p(t)\}$ is a solution of (1)-(3), (5), then the functions*

$$u_{10}(t)=\int\limits_0^1 u(x,t)dx,\;\; u_{1k}(t)=2\int\limits_0^1 u(x,t)\cos\lambda_k x dx,\;\; k=1,2,...,$$

$$u_{2k}(t)=2\int\limits_0^1 u(x,t)\sin\lambda_k x dx,\;\; k=1,2,...,$$

satisfy the system (12) and (13) on the interval $[0,T]$.

Remark 1. *It follows from Lemma 2 that in order to prove the uniqueness of a solution to the problem (1) - (3), (5) it is sufficient to prove the uniqueness of a solution to system (13), (15).*

Now, to study problem (1) - (3), (5), we consider the following spaces.

Denote by $B^5_{2,T}$ an aggregate of all the functions of the form

$$u(x,t)=\sum_{k=0}^{\infty}u_{1k}(t)\cos\lambda_k x+\sum_{k=1}^{\infty}u_{2k}(t)\sin\lambda_k x,\;\; \lambda_k=2\pi k,$$

considered in D_T, where each of the functions $u_{1k}(t)$ $(k=0,1,2,...)$ and $u_{2k}(t)$ $(k=1,2,...)$ is continuous on $[0,T]$, and

$$J_T(u)\equiv\|u_{10}(t)\|_{C[0,T]}+\left(\sum_{k=1}^{\infty}(\lambda_k^5\|u_{1k}(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}+\left(\sum_{k=1}^{\infty}(\lambda_k^5\|u_{2k}(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}<+\infty.$$

The norm in this set is defined as follows

$$\|u(x,t)\|_{B^5_{2,T}} = J(u).$$

Now, we denote by E^5_T the space of vector-functions $z(x,t) = \{u(x,t), p(t)\}$, which $u(x,t) \in B^5_{2,T},\ p(t) \in C[0,T]$.

The norm in the set E^5_T will be

$$\|z(x,t)\|_{E^5_T} = \|u(x,t)\|_{B^5_{2,T}} + \|p(t)\|_{C[0,T]}\,.$$

It is known that $B^5_{2,T}$ and E^5_T are the Banach spaces [23].

We now consider the operator

$$\Phi(u,p) = \{\Phi_1(u,p), \Phi_2(u,p)\},$$

in the space E^5_T, where

$$\Phi_1(u,p) = \tilde{u}(x,t) \equiv \sum_{k=0}^{\infty} \tilde{u}_{1k}(t)\cos\lambda_k x + \sum_{k=1}^{\infty} \tilde{u}_{2k}(t)\sin\lambda_k x, \quad \Phi_2(u,p) = \tilde{p}(t),$$

where the functions $\tilde{u}_{10}(t),\ \tilde{u}_{ik}(t)\ (i = 1,2;\ k = 1,2,...)$, and $\tilde{p}(t)$ are equal to the right-hand sides of (12), (13), and (15), respectively.

Then we obtain

$$\|\tilde{u}_{10}(t)\|_{C[0,T]} = \frac{\rho(T)}{\sqrt{\alpha}}\left\{\sqrt{\alpha}(1+|\delta|)\,|\varphi_{10}|\right.$$

$$\left. +(1+|\delta|)\,|\psi_{10}| + (1+|\delta|\,(1+|\delta|)\sqrt{T}\left(\int_0^T |F_{10}(\tau,u,p)|^2\right)^{\frac{1}{2}}\right., \tag{19}$$

$$\left(\sum_{k=1}^{\infty}(\lambda_k^5\,\|\tilde{u}_{ik}(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}} \le \sqrt{3}\rho(T)(1+|\delta|)\left(\sum_{k=1}^{\infty}(\lambda_k^5\,|\varphi_{ik}|)^2\right)^{\frac{1}{2}}$$

$$+\sqrt{3}\rho(T)(1+|\delta|)\varepsilon\left(\sum_{k=1}^{\infty}(\lambda_k^3|\psi_{ik}|)^2\right)^{\frac{1}{2}} + \sqrt{3T}(1+|\delta|\rho(T)(1+|\delta|)$$

$$\times\varepsilon\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k^3|F_{ik}(\tau;u,p)|)^2 d\tau\right)^{\frac{1}{2}}, \ i = 1,2, \tag{20}$$

$$\|\tilde{p}(t)\|_{C[0,T]} \le \left\|[g(0,t)]^{-1}\right\|_{C[0,T]}\left\{\left\|h''(t) + \alpha h(t) + u^3(x_0,t) + f(x_0,t)\right\|_{C[0,T]}\right.$$

$$+\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}(1+\beta)\left[\rho(T)(1+|\delta|)\sum_{i=1}^{2}\left(\sum_{k=1}^{\infty}(\lambda_k^5\,|\varphi_{ik}|)^2\right)^{\frac{1}{2}}\right.$$

$$+\rho(T)(1+|\delta|)\varepsilon\sum_{i=1}^{2}\left(\sum_{k=1}^{\infty}(\lambda_k^3\,|\psi_{ik}|)^2\right)^{\frac{1}{2}}$$

$$+\sqrt{T}(1+|\delta|\,\rho(T)(1+|\delta|)\varepsilon\sum_{i=1}^{2}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k^3\,|F_{ik}(\tau;u,p)|)^2d\tau\right)^{\frac{1}{2}}\Bigg]\Bigg\},\tag{21}$$

where

$$\rho(T)\equiv\sup_k\rho_k^{-1}(T)\le\frac{1}{(1+\delta^2-2\,|\delta|)},\quad \sup_k\left(\frac{\lambda_k^2}{\sqrt{\lambda_k^4-\beta\lambda_k^2+\alpha}}\right)=\frac{1}{\varepsilon}.$$

Suppose that the data of problem (1) - (3), (5) satisfy the conditions

(A_1) $\varphi(x)\in C^4[0,1],\ \varphi^{(5)}(x)\in L_2(0,1),\ \varphi(0)=\varphi(1),\ \varphi'(0)=\varphi'(1),$ $\varphi''(0)=\varphi''(1),\varphi'''(0)=\varphi'''(1),\ \varphi^{(4)}(0)=\varphi^{(4)}(1);$

(A_2) $\varphi(x)\in C^4[0,1],\ \psi'''(x)\in L_2(0,1),\ \psi(0)=\psi(1),\ \psi'(0)=\psi'(1),\ \psi''(0)=\psi''(1);$

(A_3) $f(x,t),\ f_x(x,t),\ f_{xx}(x,t)\in C(D_T),\ f_{xxx}(x,t)\in L_2(D_T)$, and $f(0,t)=f(1,t),$ $f_x(0,t)=f_x(1,t),\ f_{xx}(0,t)=f_{xx}(1,t),\ 0\le t\le T;$

(A_4) $g(x,t),\ g_x(x,t),\ g_{xx}(x,t)\in C(D_T),\ g_{xxx}(x,t)\in L_2(D_T)$, and $g(0,t)=g(1,t),$ $g_x(0,t)=g_x(1,t),\ g_{xx}(0,t)=g_{xx}(1,t)=0,\ g(0,t)\ne 0,\ 0\le t\le T;$

(A_5) $\alpha>0,\ \beta>0,\ \delta\ne\pm1,\ \beta<4\alpha,\ h(t)\in C^2[0,T],\ 0\le t\le T.$

Then from relations (16) - (18), correspondingly we have

$$\|\tilde{u}_0(t)\|_{C[0,T]}=\frac{\rho(T)}{\sqrt{\alpha}}\{\sqrt{\alpha}(1+|\delta|)\,\|\varphi\|_{L_2(0,1)}+(1+|\delta|)\,\|\psi\|_{L_2(0,1)}$$

$$+(1+|\delta|\,(1+|\delta|)\sqrt{T}\left\|p(t)g(x,t)+f(x,t)+u^3\right\|_{L_2(D_T)}\},\tag{22}$$

$$\left(\sum_{k=1}^{\infty}(\lambda_k^5\,\|\tilde{u}_{ik}(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}\le\sqrt{3}\rho(T)(1+|\delta|)\left\|\varphi^{(5)}(x)\right\|_{L_2(0,1)}$$

$$+\sqrt{3}\rho(T)(1+|\delta|)\varepsilon\left\|\psi'''(x)\right\|_{L_2(0,1)}+\sqrt{3T}(1+|\delta|\rho(T)(1+|\delta|)\varepsilon$$

$$\times\left\|p(t)g_{xxx}(x,t)+f_{xxx}(x,t)+6u_x^3+18u\cdot u_x\cdot u_{xx}+3u^2\cdot u_{xxx}\right\|_{L_2(D_T)},\tag{23}$$

$$\|\tilde{p}(t)\|_{C[0,T]}\le\left\|[g(x_0,t)]^{-1}\right\|_{C[0,T]}\{\left\|h''(t)+\alpha h(t)+u^3(x_0,t)+f(x_0,t)\right\|_{C[0,T]}$$

$$+2(1+\beta)\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}[\rho(T)(1+|\delta|)\left\|\varphi^{(5)}(x)\right\|_{L_2(0,1)}$$

$$+\rho(T)(1+|\delta|)\varepsilon\left\|\psi'''(x)\right\|_{L_2(0,1)}+\sqrt{T}(1+|\delta|\rho(T)(1+|\delta|)\varepsilon$$

$$\times\left\|p(t)g_{xxx}(x,t)+f_{xxx}(x,t)+6u_x^3+18u\cdot u_x\cdot u_{xx}+3u^2\cdot u_{xxx}\right\|_{L_2(D_T)}]\}. \qquad (24)$$

We denote by

$$A_1(T)=\frac{\rho(T)}{\sqrt{\alpha}}\{\sqrt{\alpha}(1+|\delta|)\|\varphi\|_{L_2(0,1)}+(1+|\delta|)\|\psi\|_{L_2(0,1)}$$

$$+(1+|\delta|(1+|\delta|)\sqrt{T}\|f(x,t)\|_{L_2(D_T)}\}$$

$$+2\sqrt{3}\rho(T)(1+|\delta|)\varepsilon\left\|\psi'''(x)\right\|_{L_2(0,1)}+2\sqrt{3}\rho(T)(1+|\delta|)\left\|\varphi^{(5)}(x)\right\|_{L_2(0,1)}$$

$$+6\sqrt{T}(1+|\delta|\rho(T)(1+|\delta|)\varepsilon\|f_{xxx}(x,t)\|_{L_2(D_T)},$$

$$B_1(T)=6\sqrt{T}(1+|\delta|\rho(T)(1+|\delta|)\varepsilon(\|g_{xxx}(x,t)\|_{L_2(D_T)}+1)$$

$$+\frac{\rho(T)(1+|\delta|(1+|\delta|))\sqrt{T}}{\sqrt{\alpha}}(\|g(x,t)\|_{L_2(D_T)}+1),$$

$$A_2(T)=\left\|[g(x_0,t)]^{-1}\right\|_{C[0,T]}\left\{\left\|h''(t)+\alpha h(t)+f(x_0,t)\right\|_{C[0,T]}\right.$$

$$+2\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}(1+\beta)[\rho(T)(1+|\delta|)\left\|\varphi^{(5)}(x)\right\|_{L_2(0,1)}$$

$$+\rho(T)(1+|\delta|)\varepsilon\left\|\psi'''(x)\right\|_{L_2(0,1)}$$

$$+\sqrt{T}(1+|\delta|\,\rho(T)(1+|\delta|)\varepsilon\|f_{xxx}(x,t)\|_{L_2(D_T)}],$$

$$B_2(T)=\left\|[g(x_0,t)]^{-1}\right\|_{C[0,T]}\left[2\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}(1+\beta)\right.$$

$$\times\sqrt{T}(1+|\delta|\rho(T)(1+|\delta|)\varepsilon(\|g_{xxx}(x,t)\|_{L_2(D_T)}+1)],$$

and rewrite (22) - (24) as

$$\|\tilde{u}(x,t)\|_{B_{2,T}^5}\le A_1(T)+B_1(T)(\|p(t)\|_{C[0,T]}$$

$$+\left\|u^3\right\|_{L_2(D_T)}+\left\|6u_x^3+18u\cdot u_x\cdot u_{xx}+3u^2\cdot u_{xxx}\right\|_{L_2(D_T)}), \qquad (25)$$

$$\|\tilde{p}(t)\|_{C[0,T]}\le A_2(T)+B_2(T)(\left\|u^3(0,t)\right\|_{C[0,T]}+\|p(t)\|_{C[0,T]}$$

$$+\left\|6u_x^3+18u\cdot u_x\cdot u_{xx}+3u^2\cdot u_{xxx}\right\|_{L_2(D_T)}). \qquad (26)$$

From the inequalities (25), (26), we conclude

$$\|\tilde{u}(x,t)\|_{B^5_{2,T}} + \|\tilde{p}(t)\|_{C[0,T]} \leq A(T) + B(T)(\left\|u^3(x_0,t)\right\|_{C[0,T]} + \|p(t)\|_{C[0,T]}$$

$$+ \left\|u^3\right\|_{L_2(D_T)} + \left\|6u_x^3 + 18u \cdot u_x \cdot u_{xx} + 3u^2 \cdot u_{xxx}\right\|_{L_2(D_T)}), \tag{27}$$

where

$$A(T) = A_1(T) + A_2(T),\ B(T) = B_1(T) + B_2(T).$$

Thus, the following assertion is valid

Theorem 1. *If conditions* $(A_1) - (A_5)$ *and*

$$64B(T)(A(T)+2)^3 < 1. \tag{28}$$

holds, then problem (1)-(3), (5) has a unique solution in the ball $K = K_R(||z||_{E^5_T} \leq R = A(T) + 2)$ *of the space* E^5_T.

Proof. In the space E^5_T, we consider the equation

$$z = \Phi z, \tag{29}$$

where $z = \{u, p\}$, the components $\Phi_i(u,p),\ i = 1,2,$ of operator $\Phi(u,p)$, defined by the right sides of equations (16) and (18), respectively.

Now, consider the operator $\Phi(u,p)$ in the ball $K = K_R$ of the space E^5_T. Similarly to (27), we obtain that for any $z = \{u,p\},\ z_1 = \{u_1,p_1\},\ z_2 = \{u_2,p_2\} \in K_R$ the following inequalities hold:

$$\|\Phi z\|_{E^5_T} \leq A(T) + 64B(T)R^3, \tag{30}$$

$$\|\Phi z_1 - \Phi z_2\|_{E^5_T} \leq 64B(T)R^2 \left(\|p_1(t) - p_2(t)\|_{C[0,T]} + \|u_1(x,t) - u_2(x,t)\|_{B^5_{2,T}} \right). \tag{31}$$

Then by (28), from (30) and (31) it follows that the operator Φ acts in the ball $K = K_R$, and satisfy the conditions of the contraction mapping principle. Therefore the operator Φ has a unique fixed point $\{u,p\}$, in the ball $K = K_R$, which is a solution of equation (29), i.e. in the ball $K = K_R$ is the unique solution of the systems (16), (18).

Then the function $u(x,t)$, as an element of space $B^5_{2,T}$, is continuous and has continuous derivatives $u_x(x,t), u_{xx}(x,t), u_{xxx}(x,t)$, and $u_{xxxx}(x,t)$ in D_T.

From (9) it is easy to see that

$$\left(\sum_{k=1}^{\infty} (\lambda_k \left\|u''_{ik}(t)\right\|_{C[0,T]})^2 \right)^{\frac{1}{2}} \leq 2(1+\beta+\alpha)(\rho(T)(1+|\delta|) \left\|\varphi^{(5)}(x)\right\|_{L_2(0,1)}$$

$$+\rho(T)(1+|\delta|)\varepsilon \left\|\psi'''(x)\right\|_{L_2(0,1)} + T(1+|\delta|\rho(T)(1+|\delta|)\varepsilon$$

$$\times \left\|p(t)g_{xxx}(x,t) + f_{xxx}(x,t) + 6u_x^3 + 18u \cdot u_x \cdot u_{xx} + 3u^2 \cdot u_{xxx}\right\|_{L_2(D_T)})$$

$$+2\left\|\left\|p(t)g_x(x,t)+f_x(x,t)+3u^2\cdot u_x\right\|_{C[0,T]}\right\|_{L_2(0,1)},\ i=1,2.$$

Hence, we conclude that the function $u_{tt}(x,t)$ is continuous in the domain D_T.

Further, it is easy to verify that equation (1), and conditions (2), (3), and (5) are satisfied in the usual sense. Consequently, $\{u(x,t),p(t)\}$ is a solution of (1)-(3), (5), and by Lemma 2 it is unique in the ball $K=K_R$. The proof is complete.

From Lemma 1 and Theorem 1, implies the unique solvability of the original problem (1) - (4).

In summary, we conclude the following result.

Theorem 2. *Suppose that all assumptions of Theorem 1, and*

$$\varphi(x_0)=h(0)+\delta h(T),\ \psi(x_0)=h'(0)+\delta h'(T).$$

hold. Then problem (1)-(4) has a unique classical solution in the ball $K=K_R(||z||_{E_T^5}\leq R=A(T)+2)$.

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