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A Variational View on Dupuit's formula

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Abstract. In this paper, Dupuit's formula on discharge from the well is studied in dependence of bottom hole zone and layer geometries. The term "conductivity" has been used to propose a new result in this regard. The obtained result useful for deriving of new Dupuit's formulas suitable to a concrete bottom-hole zone and layer constructions. It happens thanking a variational nature of conductivity of layer. Also same approach is considered for porous medium obeying Forthamel's low.

Key Words and Phrases: filtration, porous media, viscous flow, velocity of fluid, liquid, non-Newtonian fluids.

2010 Mathematics Subject Classifications: 76A02, 76S99, 76M30, 76S05

1. Introduction

The Dupuit formula relating to a debit and a depression in the oil wells is well-known (see e.g. [1, p. 61] or [4, p.40]) . Let $\Delta p = P_k - P_c$ be a debit,- it is difference of pressures on the bottom hole zone and in the end of layer. Then the discharge Q from the well over cylindrical well-bore of radius r_c, height h is found as

$$Q = \frac{2\pi kh\Delta p}{\mu \ln \frac{R_k}{r_c}}, \tag{1}$$

where R_k is the limit radius of layer, k is its permeability, μ is viscosity of fluid (oil).

There are a lot of versions of formula (1) relating to a single hole and multi-hole cases, where different form bottom-hole zones is considered (see e.g. [2]). From those it follows that the coefficient of proportionality of discharge Q on depression Δp significantly depends on geometry of layer both at infinity γ and in the inter-layer surfaces Γ. In the paper, to characterize the impact of those geometries, we have employed the mathematical term "*conductivity*". Using this term we derive a Dupuit formula, which characterizes the coefficient of proportionality in the dependence of discharge via the depression, provided arbitrary bottom-hole zone and layer to be considered. Though this formula contains the abstract mathematical term conductivity, in general, it may be exactly calculated finding a solution of variational problem (6) below. Solution of variational problem allows to find the conductivity-$\mathbb{P}(G)$ in order to be inserted in to (7). For example, in case of formula

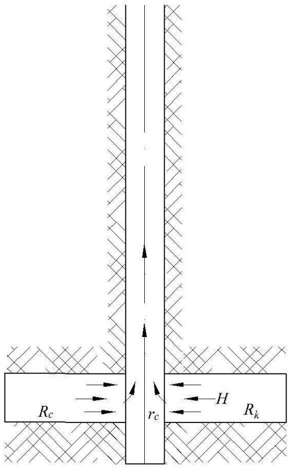

Figure 1: A circular cylinder of radius r_c height h.

(1) bottom-hole zone is a circular cylinder of height H radius r_c and layer limits in the infinity is a sphere of radius R_k (see, Figure 1)

It is a well known fact in potential theory of mathematics (see e.g. [5]) that the Wiener's capacity of a body coincides with capacity of its boundary surface. This property can also be attributed to the conductivity too. Also it is known that the capacity of an 2-dimensional surface is positive. From the Dupuit's type formula founded in the paper it is seen that the discharge increases as the contact surface ∂W of layer Ω with bottom hole zone W increase. This proves that the considered approach is true in the sense that, the discharge Q remains constant if the volume of bottom-hole zone W decreases but the capacity of its surface remains constant. In other words, it follows from formula (7) below that by taking the volume of contact zone W as for as small, but the contact surface ∂W sufficiently "big" we will increase the productivity of well. This result proves the increase of productivity of rocky and stony layers in the hydraulic fracture method exploitations. Though the interior volume in the hydraulic fractures (that stands a bottom-hole zone of well W) is almost zero over interior of the fracture, the total conductivity of the contact surface ∂W may be sufficiently larger (see, Figure 2)

This explains a reason of increase of productivity of rock and stone wells in exploitation by hydraulic fracture technologies (see e.g. [6, 7]).

In this paper, we have considered also a case of porous medium layer. In this case too a proper conductivity is introduced in order to characterize the productivity of wells. In dependence of the geometry of bottom hole zone and layer a Dupuit formula for a porous medium layer obeying Forchamel's low of filtration has been produced.

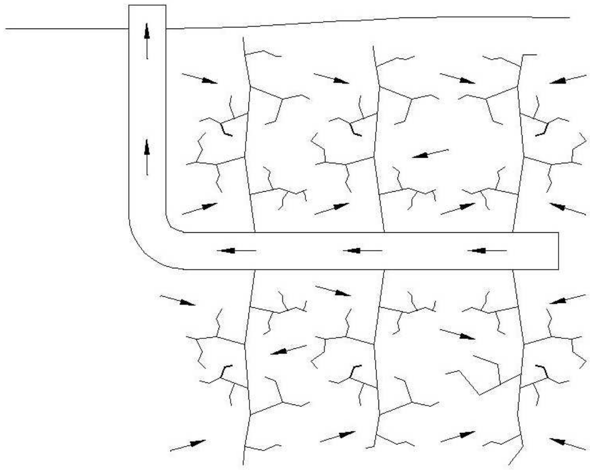

Figure 2: The hydraulic fractured well pipe.

2. A conductivity characterization of Dupuit's formula for Darcy flittering medium

Assume that the considered layer is restricted from upper and bottom by inter-layer surfaces Γ. The fluid filtrating from the medium (layer) through $\Omega\setminus$ and obeying the Darcy low arrives to the bottom-hole zone W (see, Figure 3).

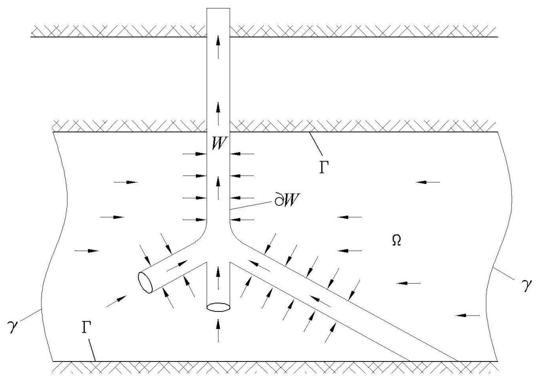

Figure 3: The multy-pipe bottom-hole well.

The filtration equation corresponding to this process in the steady stage is

$$\operatorname{div}\left(\frac{k}{\mu}\nabla\mathcal{P}\right)=0,\quad (x,y,z)\in\Omega\setminus W, \tag{2}$$

where $\nabla=\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)$, μ is viscosity of liquid, k -its permeability; for simplicity, having considered homogeneous and incompressible fluid, we can assume these quantities as constant. Let the bottom-hole pressure be P_c on W, the pressure at the end of medium on γ be P_k.

Denoting the discharge of well as Q we have $Q = \iint\limits_{\partial W} \rho v_n ds,$ where ds is an element of small area of surface $\partial W, Q$ -amount of fluid outing from well at unit time (the productivity of well), ρ is fluid density, $v_n = v \cdot \overline{n}$ is the liquid velocity passing through the bottom-hole zone, $\overline{n}$-unit normal to ∂W ordered out the layer. By the Darcy low [3], $v = -\frac{k}{\mu}$ grad p. For $\rho,\ k,\ \mu$ to be constants we have

$$Q = -\frac{\rho k}{\mu} \iint\limits_{\partial W} \frac{\partial p}{\partial n} ds, \tag{3}$$

while the boundary conditions are

$$p|_{\partial W} = P_c, \quad p|_{\gamma} = P_k, \quad \left.\frac{\partial p}{\partial n}\right\rfloor_{\Gamma} = 0;$$

Γ-the interlayer surface, γ is a limit surface of layer on infinity.

Introducing the auxiliary function $\mathcal{P} = \frac{P_k - p}{P_k - P_c}$, we have

$$\frac{\partial^2 \mathcal{P}}{\partial x^2} + \frac{\partial^2 \mathcal{P}}{\partial y^2} + \frac{\partial^2 \mathcal{P}}{\partial z^2} = 0. \tag{4}$$

and the conditions

$$\mathcal{P}|_{\partial W} = 1, \quad \mathcal{P}|_{\infty} = 0, \quad \left.\frac{\partial \mathcal{P}}{\partial n}\right\rfloor_{\Gamma} = 0.$$

Now, multiply equation (4) by $\mathcal{P}$ and integrate over the domain $G = \Omega \backslash W$. Then since $\mathcal{P}$ equals one on ∂W, it follows from Green's formula that

$$\iint\limits_{\partial W} \frac{\partial \mathcal{P}}{\partial n} ds = \iiint\limits_{\Omega \backslash \mathrm{W}} |\nabla \mathcal{P}|^2 dxdydz$$

Using (3) and the notation for $\mathcal{P}$ the left hand side equals $\frac{\mu Q}{\rho k (P_k - P_c)}$. Therefore,

$$\frac{\mu Q}{\rho k (P_k - P_c)} = \iiint\limits_{\Omega \backslash W} |\nabla \mathcal{P}|^2 \, dxdydz \tag{5}$$

It is proved in the potential theory in mathematics [5] that, the right hand side is conductivity $\mathbb{P}(G)$ of domain $G = \Omega \setminus W$. Where also was proved that solutions of (4) are minimizers of the functional

$$\mathbb{P}(G) = \inf \quad \iiint\limits_{G} |\nabla \psi|^2 \, dxdydz \ , \tag{6}$$

over the class of functions ψ that are greater than one in W, and vanishes at the end of medium (not the interlayer surface!). Observe, the inter-layer surfaces Γ are free from the conditions for a minimizer ψ. From (5) we get $\mathbb{P}(\Omega\backslash W) = \frac{\mu Q}{\rho k(P_k - P_c)}$ or

$$Q = \frac{(P_k - P_c)\,\rho k}{\mu} \cdot \mathbb{P}(G) \tag{7}$$

The obtained formula (7) is one of the main results of the paper. In applications it can be found (or estimated) as a solution of variation problem (6) by approximate or accurate calculations for the minimizing functions ψ.

Example 1. Let the wellbore be a cylinder of radius r_c, height H. It is not difficult to show that $\mathbb{P}(\Omega\backslash W) \simeq \frac{H}{\ln \frac{R_k}{r_c}}$. Taking into account the last from (7) it follows

$$Q \simeq \frac{(P_k - P_c)\,\rho k H}{\mu \ln \frac{R_k}{r_c}}. \tag{8}$$

This formula is known as Dupuit's formula [4]. To prove it let us calculate the conductivity $\mathbb{P}(\Omega\backslash W)$ in formula (7). For that, we search for a minimum of variation problem (6) in the class of functions $F = f_z(x, y)\ cos\frac{\pi z}{l}$, where $f_z(x, y)$ is a function of variables x, y greater than one on lateral surface of cylinder and is zero on the infinity. Inserting the function f_z in (6), we get (8).

Exampe 2. Let well-head be a sphere of radius r_c with center at zero and the medium is a ball of radius R_k also with center in the origin. This means $\Omega = Q(0, R_k), W = Q(0, r_c)$ and $G = Q(0, R_k) \setminus Q(r_c)$. To calculate $\mathbb{P}(G)$ for this case in order to get the analog of formulas (1) or (8). Take the function

$$\psi(r) = \left(1 - \frac{r_c}{R_k}\right)^{-1} \left(\frac{r_c}{r} - \frac{r_c}{R_k}\right), \quad r_c < r < R_k, \quad r = \sqrt{x^2 + y^2 + z^2}$$

and calculate integral (6) in the right hand side

$$\mathbb{P}(G) = \iiint\limits_{Q(0,R_k)\backslash Q(0,r_c)} |\nabla\psi|^2 dxdydz = \iint\limits_{r=r_c} \frac{\partial\psi}{\partial r} ds = \frac{4\pi r_c R_k}{R_k - r_c}$$

Inserting this into (7) we get the following Dupuit's type formula

$$Q = \frac{(P_k - P_c)\,\rho k}{\mu} \cdot \frac{4\pi r_c R_k}{R_k - r_c}. \tag{9}$$

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