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On Estimation of Surface Trigonometric Integrals

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Abstract. In this article new upper bounds for the multiple trigonometric integrals are found when the phase function's gradient defines a non-degenerating mapping.

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1. Introduction

An integral of a view

$$\int_{\Omega} G(\bar{x}) e^{2\pi i F(\bar{x})} d\bar{x} \tag{1}$$

is called a multiple trigonometric integral; here Ω denotes some domain of n dimensional space $\mathbb{R}^n$, and on the functions $G(x)$ and $F(x)$ one imposes definite conditions on boundedness or smoothness. Many investigations (see [1, 2, 3, 4, 7, 8, 9, 10, 11, 18, 19]) were devoted to estimations of trigonometric integrals. The first result in this direction belongs to Van der Corput and E.Landau (see [11]). The result established in the work [4] where the authors have received a non-improvable estimation for trigonometric integrals has important applications. The multidimensional case also was investigated in the literature. Unlike one-dimensional case, estimating of multiple trigonometric integrals of a view (1) in which Ω is some Jordan domain with a smooth boundary and the functions $G(x)$,$F(x)$ are from a certain class of smoothness is much more difficult.

The scheme of finding of estimates for integrals of a view (1) is similar to the scheme of one-dimensional case. After some transformations (see [11]) the integral reduces to the view

$$\int_a^b V(u) e^{2\pi i u} du,$$

where $V(u)$ represents the surface integral depending on parameter u.

Let Ω be a bounded closed domain of n-dimensional space $\mathbb{R}^n$, $n \geq 2$. Let's assume that in Ω an $n-1$-dimensional surface be given by means of a polynomial equation

$$f(\bar{x}) = 0 \tag{2}$$

with the gradient $\nabla f = (\partial f/dx_1, ..., \partial f/dx_n)$ which has everywhere in Ω a non-vanishing norm. In this article we consider surface trigonometric integrals taken over hypersurface Π given by the polynomial equation (2):

$$\int_{\Pi} g(\bar{x}) e^{2\pi i F(\bar{x})} ds; \tag{3}$$

here $g(\bar{x})$ is some algebraic function. Such integrals arise after of transformations by using Stokes type formulae. Trivial estimation of integral (3) can be obtained as follows

$$\int_{\Pi} g(\bar{x}) e^{2\pi i F(\bar{x})} ds \leq \int_{\Pi} |g(\bar{x})| ds.$$

Non-trivial estimation for the integrals of such type can be useful in applications to the questions connected with the distribution of integral points in multidimensional domains.

2. Auxiliary statements

Let Ω be a bounded closed domain of n-dimensional space $\mathbb{R}^n$, $n > 1$. Let's assume that in Ω some r-dimensional surface be given by means of a system of polynomial equations

$$f_j(\bar{x}) = 0, j = 1, ..., n - r, 0 \leq r \leq n, \tag{4}$$

with a Jacoby matrix

$$J = J(\bar{x}) = \| \frac{\partial f_j}{\partial xi} \|, i = 1, ..., n, j = 1, ..., n - r$$

which has everywhere in Ω a maximal rank.

Let $A_0 = A_0(\bar{x})$ be some functional matrix written down in a form

$$A_0 = A_0 = \| f_{ij}(\bar{x}) \|, \; 1 \leq i \leq r, \; 1 \leq j \leq m, rm \geq n$$

with smooth entries. Arranging the entries of columns of this matrix in a line as below

$$f_{11}(\bar{x}), ..., f_{r1}(\bar{x}), f_{12}(\bar{x}), ..., f_{r2}(\bar{x}), ..., f_{1m}(\bar{x}), ..., f_{rm}(\bar{x}),$$

let's take the transposed Jacoby matrix of this system of functions designating it as A_1:

$$A_1 = A_1(\bar{x}) = \left\| \begin{matrix} \frac{\partial f_{11}}{\partial x_1} & \cdots & \frac{\partial f_{r1}}{\partial x_1} & \cdots & \frac{\partial f_{1m}}{\partial x_1} & \cdots & \frac{\partial f_{rm}}{\partial x_1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f_{11}}{\partial x_n} & \cdots & \frac{\partial f_{r1}}{\partial x_n} & \cdots & \frac{\partial f_{1m}}{\partial x_n} & \cdots & \frac{\partial f_{rm}}{\partial x_n} \end{matrix} \right\|.$$

Then, entries of columns of this matrix, consequently as above, we arrange in a line, and take the transposed Jacoby matrix $A_2 = A_2(\bar{x}) = A_1'(\bar{x})$ of the received system of functions. Let's continue this procedure while we have not received a matrix $A_k = A_{k-1}'(\bar{x})$ for a given $k \geq 1$. The last matrix defined by such procedure consists of all possible partial derivatives of the same order k of entries of the matrix $A_0 = A_0(\bar{x})$ and has the size $n \times n^{k-1}rm$. Let's assume that $A_j(\bar{x})$ has in Ω a maximal rank equal to n. Let's designate by $G_j(\bar{x})$ the product of the last (smallest) r singular numbers of the matrix $A_j(\bar{x})$, $j = 0, ..., k$. We put

$$E = E(H) = \{\bar{x} \in \Omega | G_0(\bar{x}) \leq H\}, \; H > 0.$$

If $\varphi_{ik}(\bar{x})$ are entries of the matrix $A_j(\bar{x})$ we will accept the following designations

$$L_j(\bar{x}) = \left(\sum_{i,k} |\varphi_{ik}(\bar{x})|^2 \right),$$

$$L = \max_j \max_{\bar{x}\in\Omega} L_j(\bar{x}), \;\; G_j = \min_{\bar{x}\in\Omega} G_j(\bar{x}), j = 0, ..., k.$$

The cases $r = n-1$ and $r = n-2$ we will consider separately. Assume that the domain Ω can be dissected into such parts that on each of them the equation (2) allows one sheeted and one valued solvability, and in every of them one of minors of the matrix $A_j(\bar{x})$ (also one of partial derivatives of the function) has the maximal absolute values among all minors. So, doesn't destroying a generality, we assume that in Ω some of minors, say the minor placed on the first $n-1$ columns of the Jacoby matrix, has positive maximal absolute values. Then, by the theorem on implicit functions ([5, 12, 15, 17]), we may solve the equation (2) with respect to the first $n-1$ variables. Denote by $\bar{\xi} = (\xi_2, ..., \xi_n)$ a vector of independent variables. Then, x_1 is possible to represent as a function $x_1 = x_1(\bar{\xi})$ of these independent variables. Denote by $A_0(\bar{\xi})$ the matrix constructed from the matrix $A_0(\bar{x})$ by replacing of the variable x_1 by the function $x_1 = x_1(\bar{\xi})$. In other words we consider the functional matrix $A_0(\bar{\xi})$ as a matrix depending on $\bar{\xi}$. Denote by $G_{(1)}$ the minimal value of Gram determinant for gradients of entries of the matrix $A_0(\bar{\xi})$ (differentiation is taken with regard to $\bar{\xi}$), i.e.

$$G_{(1)} = \min_{\bar{\xi}} \det \left(A_{1\bar{\xi}} \cdot A^t_{1\bar{\xi}} \right).$$

Thus, $A_{1\bar{\xi}}$ means the matrix of a size $(n-1) \times rm$ received from A_0 by differentiation in regard to $\bar{\xi}$, $A_{1\bar{\xi}} = A'_0(\bar{\xi})$. So, the matrix $A_1(\bar{x})$ being considered as a matrix of $\bar{\xi}$, differs from$A_{1\bar{\xi}}$. Similarly, we can, beginning from the matrix A_{j-1}, form a matrix $A_{j\bar{\xi}}$ assuming that $G_{(j)} > 0$ for all considered $j > 0$. For a positive number $a > 0$ we write $h(a) = a + a^{-1}$. We have $a \leq h(a)$, $h(a) = h(a^{-1})$, and $h(ab) \leq h(a)h(b)$, for $a, b > 0$.

Lemma 1. *Let Π_H be a part of a surface (4) included in $E(H)$, $k > 1$ and $G_{(k)} > 0$. Then under the conditions above we have:*

$$\mu(\Pi_H) \leq KH^{1/k} \cdot G_{(k)}^{-1/k} \cdot \mathrm{Q}_k^n;$$

$$\mathrm{Q}_k = \log \tilde{H}; \; \tilde{H} = \max\left\{ h(H), h(C_{(1)}), ..., h(C_{(k-1)}), h(G_{(k)}), h(L) \right\},$$

and K is a constant, and numbers $C_{(2)}, ..., C_{(k-1)}$ are defined by equalities

$$C_{(1)} = H^{1/2} C_{(2)}^{1/2}, C_{(2)} = H^{1/3} C_{(3)}^{1/3}, ..., G_{(k)} = H^{1/k} C_{(k-1)}^{1/k}.$$

The proof of the lemma 1 is given in [11, 13]. Following lemma is a generalization of this lemma ([11, 13]).

Lemma 2. *Under the conditions of the lemma 1 there exist an absolute constant* K_1 *such that:*

$$\mu\left(\Pi_H\right) \le K_1 H^{1/k} \cdot G_k^{-1/k} \cdot \tilde{Q}_{\mathrm{k}}^n.$$

Let $F(\bar{x})$ be some polynomial. Let's consider the trigonometric integral (3), in the domain Ω with a boundary consisted of finite number of algebraic surfaces. Gradient of this function is a matrix A_0:

$$A_0 = \nabla F = \left\| \frac{\partial F}{\partial x_1}, ..., \frac{\partial F}{\partial x_n} \right\|.$$

Let everywhere in Ω

$$\|\nabla F\| = \sqrt{\left(\frac{\partial F}{\partial x_1}\right)^2 + \cdots + \left(\frac{\partial F}{\partial x_n}\right)^2} \ne 0.$$

We assume that the boundary of the domain Ω is a union of surfaces defined by finite number of algebraic equations of a view $H(\bar{x}) = 0$. Not breaking a generality, we can take this number equal to 1. Assume, also, that the Jacoby matrix of the system of functions F, H has rank 2.

It is clear that the matrix $A_1(\bar{x})$ looks like

$$A_1 = A_1(\bar{x}) = \left\| \begin{matrix} \frac{\partial^2 F}{\partial x_1^2} \cdots \frac{\partial^2 F}{\partial x_1 \partial x_r} \\ \cdots \quad \cdots \\ \frac{\partial^2 F}{\partial x_r \partial x_1} \cdots \frac{\partial^2 F}{\partial x_r^2} \end{matrix} \right\|, \tag{5}$$

and the matrix $A_{k-1}(\bar{x})$ is combined of all partial derivatives of order $k \ge 2$ of the function $F(\bar{x})$. Let now $\tilde{G}_{k-1}$ be a minimal value of the product of $n-2$ least singular numbers of the matrix A_{k-1}. Similarly, we can, beginning from the matrix A_{j-1}, form a matrix $A_{j\bar{\xi}}$ assuming that $\tilde{G}_{(j)} > 0$ for all considered $j > 0$. Now we formulate analogs of the lemmas 1 and 2 for the case $r = n-2$ designating the numbers $G_{(j)}$ and G_j as $\tilde{G}_{(j)}$ and $\tilde{G}_j$, respectively.

Lemma 3. *Let* Π_H *be a part of a surface* (4) *included in* $E(H)$ *and* $\tilde{G}_1 > 0$. *Then for the area* $\mu(\Pi_H)$ *we have the bound*

$$\mu(\Pi_H) \le C_0 H \tilde{G}_{(1)}^{-1} \tilde{\wp}^r,$$

where

$$\tilde{\wp} = r^2 \log\left[h\left(\tilde{G}_{(1)}\right) h\left(H\right) h\left(L\right)\right],$$

and C_0 *is an absolute constant.*

Lemma 4. *Let $k \geq 1$ and $\tilde{G}_{(k)} > 0$. Then under the conditions of the lemma 1 we have:*

$$\mu\left(\Pi_H\right) << H^{1/k} \tilde{G}_{(k)}^{-1/k} \wp_k^r;$$

$$\tilde{\wp}_k = 3r^2 \log \tilde{H};\ \tilde{H} = \max\left\{h\left(H\right), h\left(\tilde{C}_{(1)}\right), ..., h\left(\tilde{C}_{(k-1)}\right), h\left(\tilde{G}_{(k)}\right), h\left(L\right)\right\},$$

and numbers $\tilde{C}_{(1)}, ..., \tilde{C}_{(k-1)}, \tilde{G}_{(k)}$ are defined by equalities

$$\tilde{C}_{(1)} = H^{1/2} \tilde{C}_{(2)}^{1/2}, ..., \tilde{C}_{(k-1)} = H^{1/k} \tilde{G}_{(k)}^{1/k}$$

Lemma 5. *Let $k \geq 1$ and $\tilde{G}_k > 0$. Then, under the conditions of the lemma 2, one has:*

$$\mu\left(\Pi_H\right) << H^{1/k} \tilde{G}_k^{-1/k} \wp_k^r.$$

Lemma 6. *There exist such a dissection of the domain Ω into the union of no more than finite number of subdomains so that the surface integral $\varphi(u) = \int_{F(\bar{x})=u} \frac{g(\bar{x})ds}{\|\nabla F\|}$, respectively, breaks into the sum of the surface integrals being monotonous functions of a variable u, moreover, the number of addends of the last sum depends on the degree of a polynomial F only.*

Proof. Proof of this lemma we will spend using reasonings of the proof of analogical lemma from the work [11]. Having given to the variable u some increment, we can write

$$\varphi(u + \Delta u) - \varphi(u) = \int_{F(\bar{x})=u+\Delta u} \frac{g(\bar{x})ds}{\|\nabla F\|} - \int_{F(\bar{x})=u} \frac{g(\bar{x})ds}{\|\nabla F\|}.$$

As the domain Ω is closed, the gradient of functions $F(\bar{x})$ and $g(\bar{x})$ and their partial derivatives of the second order are bounded. Consider the Taylor decomposition of the function $F(\bar{x})$ in a neighborhood of the point $\bar{x}$, lying on the surface $F(\bar{x}) = u$, in the gradient direction:

$$F(\bar{x} + \lambda \nabla F) - F(\bar{x}) = \lambda \nabla F \cdot \nabla F + o(\lambda).$$

Let's pick up λ so that the point $F(\bar{x} + \lambda \nabla F)$ was placing on the surface $F = u + \Delta u$. Then, we get

$$\Delta u = \lambda \nabla F \cdot \nabla F + o(\lambda).$$

When Δu is sufficiently small, the second term on the right part is small also. So,

$$\lambda = \frac{\Delta u}{\nabla F \cdot \nabla F} + o(\Delta u) = \frac{\Delta u}{\|\nabla F\|^2} + o(\Delta u).$$

After of shifting of the argument in the gradient direction the function $\frac{g(\bar{x})}{\|F\|}$ takes on an increment δ which can be written as follows:

$$\delta = \nabla\left(\frac{g(\bar{x})}{\|F\|}\right) \cdot \lambda \nabla F(1 + o(\lambda)) =$$

$$= \lambda \sum_{i=1}^{n} \frac{\partial F}{\partial x_i} \left(\frac{\partial g/\partial x_i}{\|F\|} - \frac{g}{\|F\|^3} \left(\sum_{j=1}^{n} \frac{\partial F}{\partial x_j} \frac{\partial^2 F}{\partial x_j \partial x_i} \right) \right) + o(\lambda)) =$$

$$= \lambda \frac{\nabla F \cdot \nabla g}{\|F\|} - \lambda g \sum_{i=1}^{n} \frac{\partial F}{\partial x_i} \left(\frac{g}{\|F\|^3} \left(\sum_{j=1}^{n} \frac{\partial F}{\partial x_j} \frac{\partial^2 F}{\partial x_j \partial x_i} \right) \right) + o(\lambda)).$$

Using definition of the matrix A_1 given above and denoting $\bar{\nabla} = \nabla F / \|\nabla F\|$, we can rewrite the last equality as below:

$$\delta = \frac{\lambda}{\|F\|} \left(\nabla F \cdot \nabla g - g(A_1 \bar{\nabla}, \bar{\nabla}) \right) = \frac{\Delta u}{\|F\|^3} \left(\nabla g \cdot \nabla F - g(A_1 \bar{\nabla}, \bar{\nabla}) \right).$$

Under the conditions imposed on a gradient, as shown above, the domain Ω may be dissected into finite number of subdomains which pairwisely intersecting by parts of the boundary only, and where the equation $F(\bar{x}) = u$ allows one sheeted solvability with respect to one and the same variable. Let's consider one of them where the mentioned equation is solved with respect, say, to x_1:

$$x_1 = \psi(x_2, ..., x_n);\ (x_2, ..., x_n) \in \omega,$$

and ω is an domain of changing for independent variables. Having fixed any point $\bar{\xi}_0 \in \omega$, we will define the mapping $\bar{\psi}$ in $\omega - \bar{\xi}_0 = \{\Delta y \in \mathbb{R}^{n-1} | \bar{\xi}_0 + \Delta y \in \omega\}$ which puts to each point Δy in correspondence the point $(\psi(\xi_0 + \Delta y), \xi_0 + \Delta y)$ on the surface $F = u$, and will consider tangential linear mapping

$$\Phi : \Delta y \mapsto \psi(\bar{\xi}_0) + \psi(\bar{\xi}_0) \cdot \Delta y; \Delta y \in \omega - \bar{\xi}_0. \tag{6}$$

The image of this mapping is a tangential linear variety (hyper plane) to the surface $F = u$ in the point $(\psi(\bar{\xi}_0), \bar{\xi}_0)$. Let's notice that the point $(\Phi(\bar{\xi}), \bar{\xi})$ of the tangential hyper plane will situated from the corresponding point $(\psi(\bar{\xi}), \bar{\xi})$ on the surface $F = u$ at a distance $o\left(\left|\Phi(\Delta y) - \bar{\psi}(\Delta y)\right|\right)$ which is of order $o(\Delta u)$. At each point $\bar{x}$ of the surface $F = u$ the gradient ∇F is orthogonal to the tangential hyper plane. Really,

$$\nabla F \cdot \Phi'(\vec{\xi}) \Delta x = \left(\frac{\partial F}{\partial x_1}, ..., \frac{\partial F}{\partial x_n} \right) \cdot$$

$$\cdot \left(\left(\frac{\partial F}{\partial x_1} \right)^{-1} \left(-\frac{\partial F}{\partial x_2} \Delta x_2 -, , , -\frac{\partial F}{\partial x_n} \Delta x_n \right), \Delta x_2, ..., \Delta x_n \right) = 0.$$

When λ is defined as above, the point $\bar{x} + \lambda \nabla F$ where $\bar{x} \in \Pi(u)$, belongs to the surface $\Pi(u + \Delta u)$; here by $\Pi(u)$ we designate the surface defined by the equation $F = u$ in a wider open domain $\Omega' \supset \Omega$. For any open domain Ω' the surface $\Pi(u + \Delta u) \bigcap \Omega$ entirely lies in Ω' for all enough small values of $|\Delta u|$. The mapping $\Psi : \Pi(u) \to \Omega'$ defined as $\Psi(\bar{x}) = \bar{x} + \lambda \nabla F$ is one-one mapping when $|\Delta u|$ is sufficiently small. Really,

$$\Psi(\bar{x}) = \bar{x} + \left(\frac{\Delta u}{\|\nabla F\|^2} + o(|\Delta u|) \right) \nabla F = \bar{x} + \Delta u \frac{\Delta F}{\|\nabla F\|^2} + o(|\Delta u|),$$

and at sufficiently small $|\Delta u|$ the Jacoby matrix of this mapping can be represented as a sum of identity matrix and a Jacoby matrix of the mapping

$$\bar{x} \mapsto \Psi(\bar{x}) - \bar{x}.$$

Note that when u and Δu are fixed then we have $\Psi(\bar{x}) - \bar{x} = \lambda(\bar{x})\nabla F(\bar{x})$, and we can define partial derivatives of the function $\lambda(\bar{x})$ from the identity

$$F(\bar{x} + \lambda(\bar{x})\nabla F(\bar{x})) - F(\bar{x}) = \Delta u.$$

If we take partial derivatives both sides of this identity with respect to the variables of $\bar{x}$ then we get the system of linear equations from which we can define required partial derivatives. Since the domain is closed and the matrix $A_1(\bar{x})$ (see (4)) is not degenerating, then as it follows from Cramer's Rule all of obtained partial derivatives will be bounded. So, at sufficiently small values of Δu, determinant of the Jacoby matrix of the mapping $\bar{x} \mapsto \Psi(\bar{x})$ tends to 1 as $\Delta u \to 0$, i.e. this determinant will be distinct from zero everywhere in considered domain. So, Ψ is a bijective mapping for sufficiently small $|\Delta u|$.

We put: $D(u) = \{\bar{x} \in \Omega | F(\bar{x}) = u\}$. Then, the surface $D(u + \Delta u)$ tends to $D(u)$ as $\Delta u \to 0$ (pointwisely and uniformly). $\Psi(D(u))$ is a closed subset of $D(u + \Delta u)$. Further, a prototype $D(u + \Delta u)$ of the same mapping we will designate as $D'(u + \Delta u)$. Then, we have:

$$\varphi(u + \Delta u) - \varphi(u) = \int_{D'(u+\Delta u)\bigcap D(u)} \left(\frac{g(\bar{x} + \lambda\nabla F)}{\|\nabla F(\bar{x} + \lambda\nabla F)\|} - \frac{g(\bar{x})}{\|\nabla F(\bar{x})\|} \right) ds +$$

$$+ \int_{D(u+\Delta u)\setminus\Psi(D(u))} \frac{g(\bar{x})ds}{\|\nabla F(\bar{x})\|} - \int_{D(u)\setminus D'(u+\Delta u)} \frac{g(\bar{x})ds}{\|\nabla F(\bar{x})\|}. \tag{7}$$

Substituting the value found above for an increment, we find for the first surface integral the following expression:

$$-\Delta u(1 + o(1)) \int_{F(\bar{x})=u} \frac{\left(\nabla g \cdot \nabla F - g(A_1\bar{\nabla}, \bar{\nabla})\right)}{\|F\|^3} ds.$$

Consider now two remained surface integrals on the right hand side of the equality (6). They will be transformed by one and the same way. The first integral is taken over the surface $D(u + \Delta u)\setminus\Psi(D(u))$ which is included between the boundaries $D(u + \Delta u)$ and $\Psi(D(u))$. It is clear that this piece narrowing, will be pulled off along $n - 2$-dimensional surface of an intersection $D(u+\Delta u)\bigcap\partial\Omega$, which tends to the limiting position $D(u)\bigcap\partial\Omega$ (it may be empty), as $\Delta u \to 0$.

Let's denote ω' an $n-1$-dimensional domain being a projection of the $D(u+\Delta u)\setminus\Psi(D(u))$ (we will use designation ψ' instead of ψ for the solution of the equation $F(\bar{x}) = u + \Delta u$). Dissect now the projection of the boundary $D(u + \Delta u)\bigcap\partial\Omega$ into the small parts $E_i, i = 1, ..., N$ with the maximal diameter not exceeding Δu. Now taking any point $(\psi'(\bar{\xi_i}), \bar{\xi_i})$ on E_i draw the ray lying on the tangential hyper plane, being orthogonal to the boundary $D(u + \Delta u)\bigcap\partial\Omega$ and intersecting the last at this point. The set of all such rays set up a surface. We restrict this surface by a such way that the projection of the got piece of the surface was coincide with ω'. This surface, consisted of pieces set up by all restricted rays with top points at E_i. The piece corresponding E_i we denote as $F_i = F_i(u, \Delta u)$. They set up something like a tiled covering for the surface $D(u+\Delta u)\backslash\Psi(D(u))$, area of which differs from the area of the surface $D(u+\Delta u)\backslash\Psi(D(u))$ by a value $o(\Delta u)$. Let $\bar{\xi}_i \in E_i$ be any point, ρ_i be a vector lying on the constructed tangential space to the surface $F = u + \Delta u$ at the point $(\psi'(\bar{\xi}_i), \bar{\xi}_i)$, orthogonal to $D(u+\Delta u)\bigcap\partial\Omega$, and with the endpoint at $\bar{\eta}_i$ of the boundary of corresponding piece $F_i = F_i(u, \Delta u)$. For small Δu we have: $|F_i| = |E_i|\, h_i$ (here $|E_i|$ expresses $n-2$-dimensional volume of E_i), and $h_i = |\rho_i|\,(1 + o(1))$, i.e. h_i plays a role of height of F_i which approximately we take as a cylindroid with the base $\Delta_i = \{(\psi'(\bar{\xi}_i), \bar{\xi}_i) | \bar{\xi}_i \in E_i)\}$ (with an error of order $o(\Delta u)$ for $n-2$-dimensional volume). Then, we have:

$$\int_{D(u+\Delta u)\backslash\Psi(D(\bar{u}))} \frac{g(\bar{x})ds}{\|\nabla F(\bar{x})\|} = \sum_{j=1}^{N} \int_{(\Delta_i)} \frac{g(\bar{x})ds}{\|\nabla F(\bar{x})\|}(1 + o(1)).$$

Intersection of tangential hyper planes, respectively, to $\partial\Omega$ and $D(u+\Delta u)$ at the point $(\psi'(\bar{\xi}_i), \bar{\xi}_i)$ is a tangential $n-2$- dimensional subspace to $D(u+\Delta u)\bigcap\partial\Omega$ at the same point. Let's consider three points: a point $P_i = (\psi'(\bar{\xi}_i), \bar{\xi}_i)$, a point $\bar{\eta}_i$ and a point $\Psi^{-1}(\eta_i)$. Let α_i be an angle between an external normal vector $\bar{n}$ to the boundary of Ω and a gradient ∇F. An angle between the segment $[\bar{\eta}_i, P_i]$ and the gradient ∇F, at small ∇u, differs from the angle α_i by a value $o(\Delta u)$ (or their sum is close to π). From a rectangular triangle we receive (the told above segment $[\Psi^{-1}(\eta_i), P_i]$ is here an hypotenuse):

$$\bar{h}_i = |\lambda| \cdot \|\nabla F\|\, ctg\alpha_i(1 + o(1)) = \frac{\Delta u}{\|\nabla F\|} ctg\alpha_i(1 + o(1)).$$

As $\cos\alpha_i = \bar{n}\cdot\nabla F, ctg\alpha_i = \bar{n}\cdot\nabla F/\sqrt{1-(\bar{n}\cdot\nabla F)^2}$, then we have:

$$\int_{D(u+\Delta u)\backslash\Psi(D(u))} \frac{g(\bar{x})ds}{\|\nabla F(\bar{x})\|} = \sum_{j=1}^{N} \int_{(\Delta_i)} \frac{g(\bar{x})ds}{\|\nabla F(\bar{x})\|}(1 + o(1)) =$$

$$= \sum_{j=1}^{N} \Delta u \int_{(\Delta_i)} \frac{g(\bar{x})ctg\alpha_i d\sigma}{\|\nabla F(\bar{x})\|^2}(1 + o(1)) = \Delta u(1 + o(1)) \int_{Z} \frac{g(\bar{x})\bar{\nabla}\cdot\bar{n}}{\sqrt{1-(\bar{\nabla}\cdot\bar{n})^2}} \frac{d\sigma}{\|\nabla F\|^2},$$

where $d\sigma$ designates $n-2$-dimensional element of the volume, and Z denotes an intersection of surfaces $F = u$ and $\partial\Omega$ (it can consist of several pieces). The similar formula is true for the third surface integral in (6). Therefore, from the formula (6) one can derive:

$$\varphi'(u) = \lim_{\Delta u\to 0} \frac{\varphi(u+\Delta u) - \varphi(u)}{\Delta u} = -\int_{F(\bar{x})=u} \frac{\left(\nabla g\cdot\nabla F - g(A_1\bar{\nabla}, \bar{\nabla})\right)}{\|F\|^3} ds +$$

$$+ \int_{Z} \frac{g(\bar{x})\bar{\nabla}\cdot\bar{n}}{\sqrt{1-(\bar{\nabla}\cdot\bar{n})^2}} \frac{d\sigma}{\|\nabla F\|^2}, \qquad (8)$$

and the sign before the integral is counted by the scalar product $\bar{\nabla} \cdot \bar{n}$.

To apply the Stokes formula ([5, p. 645], [16, p. 261]) to the second integral at the right side of (8), we note that the boundary Z is defined by the system of equations of a view $F = u$, $H = c$. Gram determinant of the functions standing at the left sides of the equations is non-zero. By this reason surface integral is possible to represent as below:

$$\int_Z \frac{g(\bar{x})\bar{\nabla} \cdot \bar{n}}{\sqrt{1-(\bar{\nabla} \cdot \bar{n})^2}} \frac{d\sigma}{\|\nabla F\|^2} = \int_{\partial D(u)} \frac{g(\bar{x})\bar{\nabla} \cdot \bar{n}}{\sqrt{1-(\bar{\nabla} \cdot \bar{n})^2}} \frac{\sqrt{G_0}}{|J_0|} \frac{d\xi_3 \cdots d\xi_n}{\|\nabla F\|^2},$$

and the variables $\xi_3, ..., \xi_n$ denote independent variables after of suitable solution of the considered system, say, with respect to the first two variables. So, we get integral of a differential form:

$$\eta = W d\xi_3 \wedge \cdots \wedge d\xi_n;\ W = \frac{g(\bar{x})\bar{\nabla} \cdot \bar{n}}{\sqrt{1-(\bar{\nabla} \cdot \bar{n})^2}} \frac{\sqrt{G_0}}{|J_0|} \frac{1}{\|\nabla F\|^2}$$

and G_0 is a Gram determinant of considered functions F, H, J_0 is a determinant

$$J_0 = \begin{vmatrix} \frac{\partial F}{\partial x_1} & \frac{\partial F}{\partial x_2} \\ \frac{\partial H}{\partial x_1} & \frac{\partial H}{\partial x_2} \end{vmatrix}.$$

Now we have

$$d\eta = \left(\frac{\partial W}{\partial x_1} dx_1 + \frac{\partial W}{\partial x_2} dx_2 + \cdots + \frac{\partial W}{\partial x_n} dx_n \right) \wedge d\xi_3 \wedge \cdots \wedge d\xi_n.$$

Further at the surface $F = u$, after of solving this equation, the variable x_1 stands a function of independent variables $\xi_2, ..., \xi_n$ (we suppose that this is possible, not breaking a generality). Then,

$$d\eta = \left(\frac{\partial W}{\partial x_1} dx_1 + \frac{\partial W}{\partial x_2} dx_2 + \cdots + \frac{\partial W}{\partial x_n} dx_n \right) \wedge d\xi_3 \wedge \cdots \wedge d\xi_n =$$

$$d\eta = \left(\frac{\partial W}{\partial x_1} \left(\frac{\partial x_1}{\partial \xi_2} d\xi_2 + \cdots + \frac{\partial x_1}{\partial \xi_n} d\xi_n \right) + \frac{\partial W}{\partial x_2} dx_2 + \cdots + \frac{\partial W}{\partial x_n} dx_n \right)$$
$$\wedge d\xi_3 \wedge \cdots \wedge d\xi_n =$$
$$= \left(\frac{\partial W}{\partial x_1} \frac{\partial x_1}{\partial \xi_2} + \frac{\partial W}{\partial x_2} \right) d\xi_2 \wedge d\xi_3 \wedge \cdots \wedge d\xi_n + \cdots +$$
$$+ \left(\frac{\partial x_1}{\partial \xi_n} \frac{\partial W}{\partial x_1} + \frac{\partial W}{\partial x_n} \right) d\xi_n \wedge d\xi_3 \wedge \cdots \wedge d\xi_n =$$
$$= \left(\frac{\partial W}{\partial x_1} \frac{\partial x_1}{\partial \xi_2} + \frac{\partial W}{\partial x_2} \right) d\xi_2 \wedge d\xi_3 \wedge \cdots \wedge d\xi_n.$$

Now in consent with the Stokes formula (see [12, p. 261]):

$$\int_{\partial D(u)} \eta = \int_{D(u)} d\eta.$$

It is obviously, that right hand side of this relation is possible to represent as a surface integral taken over the surface $F = u$ after of multiplying and dividing by a positive element of area. Then, from (8) we derive:

$$\varphi'(u) = \int_{F(\bar{x})=0} G_1(\bar{x}) ds, \tag{9}$$

where

$$G_1(\bar{x}) = \frac{\partial F/\partial x_1}{\|\nabla F\|}\left(\frac{\partial W}{\partial x_1}\frac{\partial x_1}{\partial \xi_2} + \frac{\partial W}{\partial x_2}\right) - \frac{\left(\nabla g \cdot \nabla F - g(A_1\bar{\nabla}, \bar{\nabla})\right)}{\|F\|^3}.$$

It is clear that the function G_1 is an algebraic function in Ω. Now, let's dissect the domain Ω into a finite number of such subdomains Ω_i in every of which the function G_1 keeps own sign invariable. Then, the integral (8) splits into the sum of several surface integrals:

$$\varphi'(u) = \sum \varphi_i'(u),\ \varphi_i'(u) = \int_{\Omega_i, F(\bar{x})=u} G(\bar{x}) ds \tag{10}$$

(notice that when we consider the sum of the integrals $\int_{S\subset Z}$ taken on the different sides of the piece S of a surface, the normal vector $\bar{n}$ changes the sign, and consequently, such a sum is equal to zero); the number of domains on the right part of (9) depends on Ω and a degree of the polynomial F. Let's designate, in the consent with (9)

$$\varphi(u) = \sum \varphi_i(u), \varphi_i(u) = \int_{\Omega_i, F(\bar{x})=u} \frac{g(\bar{x}) ds}{\|\nabla F\|}.$$

Thus, the equality $\phi'(u) = \sum_i \phi_i'(u) = \sum_i \int_{\Omega_i, F=u} G(\bar{x}) \, ds$ is true. Since the function under the surface integral does not change its sign, the function is a monotone function. The lemma 6 is proved.

Lemma 7. *Let Ω be a bounded closed domain of n-dimensional space $\mathbb{R}^n$, $n > 1$. Let's assume that in Ω some r-dimensional surface be given by means of a system of equations*

$$f_j(\bar{x}) = 0, j = 1, ..., n-r, 0 \le r \le n,$$

with a Jacoby matrix

$$J = J(\bar{x}) = \|\frac{\partial f_j}{\partial x_i}\|, i = 1, ..., n, j = 1, ..., n-r$$

which has, everywhere in Ω, a maximal rank and smooth entries. Let, further a mapping $\bar{\xi} \mapsto \bar{x}$ maps some domain $\Omega' \subset \mathbb{R}$ into Ω with non-degenerating in Ω' Jacoby matrix

$$Q = Q(\bar{\xi}) = \|\frac{\partial f_j}{\partial xi}\|$$

with continuous entries. Then for any continuous in the Ω function $f(\bar{x})$ the formula

$$\int_M f(\bar{x}) \frac{ds}{\sqrt{G}} = \int_{M'} \mid \det Q \mid f(\bar{x}(\bar{\xi})) \frac{d\sigma}{\sqrt{G'}}, G' = \det(JQ \cdot Q^t J^t)$$

holds; here M' denotes a pre-image of the piece of the surface on given surface, $d\sigma$ designates the surface element in coordinates $\bar{\xi}$.

Proof of this lemma is given in [11, p.92].

3. Basic results

Consider now the integral (3):

$$\int_{\Pi} g(\bar{x}) e^{2\pi i F(\bar{x})} ds.$$

Our goal is proving following theorems concerning estimations of surface trigonometric integrals. Let's denote

$$H = \max_{\bar{x}\in\Omega} \|\nabla F\|, g_0 = \max_{\bar{x}\in\Omega} |g(\bar{x})|.$$

Designate by G_{k-2} and $\tilde{G}_{k-2}$ a minimal value of the product of, respectively, $n-1$ and $n-2$ least singular numbers of the matrix A_{k-2}.

Theorem 1. *If $k > 2$ then there exist a positive constant $c_0 = c_0(r, k, \deg F)$ such that*

$$\left|\int_{\Pi} g(\bar{x}) e^{2\pi i F(\bar{x})} ds\right| \leq c_0 g_0 \max\left(G_1^{-1}, H^{(n-1)/(k-1)} G_{k-2}^{-1/(k-1)} \cdot \mathrm{Q}_{k-2}^{n-1}\right);$$

$$\mathrm{Q}_{k-2} = \log \tilde{H};\ \tilde{H} = \max\left\{h(H), h(G_{(1)}), ..., h(G_{(k-2)}), h(L)\right\}.$$

Theorem 2. *Suppose that the Jacoby matrix Λ_0 of the system of functions $f(\bar{x})$, $F(\bar{x})$ has a rank 2. If $k > 2$ and $n \geq 3$ then there exist a positive constant $c_1 = c_1(r, k, \deg F)$ such that*

$$\left|\int_{\Pi} g(\bar{x}) e^{2\pi i F(\bar{x})} ds\right| \leq c_1 g_0 \max\left(\tilde{G}_1^{-1}, H^{(n-3)/(k-1)} \tilde{G}_{k-2}^{-1/(k-1)} \wp_{k-2}^{n-2}\right);$$

$$\tilde{\wp}_{k-2} = \log \tilde{H};\ \tilde{H} = \max\left\{h\left(H\right), h\left(\tilde{G}_{(1)}\right), ..., h\left(\tilde{G}_{(k-2)}\right), h\left(L\right)\right\},$$

Note. When $k = 2$ estimations of these theorems remains valid if to take the first expression in the sign of maximum.

Proofs of the theorems. Using the formula of the lemma 1 of the work [13] we can represent the integral

$$\int_{\Pi} g(\bar{x}) e^{2\pi i F(\bar{x})} ds$$

as a limit

$$\lim_{h\to 0} \frac{1}{2h} \int_{|f(\bar{x})|\leq h, \bar{x}\in\Omega} g(\bar{x}) \|\nabla f\| e^{2\pi i F(\bar{x})} d\bar{x}. \tag{11}$$

For every $h > 0$ the condition $|f(\bar{x})| \leq h$ defines some closed subdomain in Ω. We suppose, in agree with the lemma 6 above, that in the considered domain the surface integral

$$\int_{F(\bar{x})=u} \frac{\|\nabla f\| g(\bar{x}) ds}{\|\nabla F\|}$$

is a monotone function of u. We can apply the reasonings of the work [13] to transform the integral under the limit (11) as follows

$$\int_{|f(\bar{x})|\le h, \bar{x}\in\Omega} g(\bar{x})\,\|\nabla f\|\, e^{2\pi i F(\bar{x})} d\bar{x} = \int_m^M e^{2\pi i u} \left(\int_{|f(\bar{x})|\le h, F(\bar{x})=u} \frac{\|\nabla f\|\, g(\bar{x}) ds}{\|\nabla F\|} \right) du.$$

So, we have:

$$\int_\Pi g(\bar{x}) e^{2\pi i F(\bar{x})} ds = \lim_{h\to 0} \frac{1}{2h} \times$$

$$\times \int_m^M \left(\int_{F(\bar{x})=u, |f(\bar{x})|\le h} \frac{\|\nabla f\|\, g(\bar{x}) ds}{\|\nabla F\|} \right) (\cos 2\pi u + i \sin 2\pi u)\, du.$$

Applying of the lemma 3, [13] allows us to pass to the limit under the sign of integration. Then we get:

$$\int_\Pi g(\bar{x}) e^{2\pi i F(\bar{x})} ds = \int_m^M (\cos 2\pi u + i \sin 2\pi u) \times$$

$$\times \lim_{h\to 0} \frac{1}{2h} \left(\int_{F(\bar{x})=u, |f(\bar{x})|\le h} \frac{\|\nabla f\|\, g(\bar{x}) ds}{\|\nabla F\|} \right) du.$$

Using the known method of estimation of this integral (see [2]), one may get a following bound

$$\left| \int_\Pi g(\bar{x}) e^{2\pi i F(\bar{x})} ds \right| \le 2 \max_u \left| \lim_{h\to 0} \frac{1}{2h} \left(\int_{F(\bar{x})=u, |f(\bar{x})|\le h} \frac{\|\nabla f\|\, g(\bar{x}) ds}{\|\nabla F\|} \right) \right| \le$$

$$\le 2 g_0 \max_u \left(\int_{\Pi, F(\bar{x})=u} \frac{ds}{\|\nabla F\|} \right). \tag{12}$$

Assume that $K \le H = \max_{\bar{x}\in\Omega} \|\nabla F\|$. As the norm of the gradient is a square root of the polynomial $\|\nabla F\|^2$, then the subset of the domain Ω where $\|\nabla F\| = 0$, as a closed subset, is a Jourdan set with zero measure. Then writing $\Omega' = \{\bar{x} \in \Omega |\, \|\nabla F\| > 0\}$ we find

$$\left| \int_{\Pi \bigcap \Omega} g(\bar{x}) e^{2\pi i F(\bar{x})} ds \right| = \left| \int_{\Pi \bigcap \Omega'} g(\bar{x}) e^{2\pi i F(\bar{x})} ds \right| =$$

$$= \sum_{j=1}^{\infty} \lim_{h\to 0} \frac{1}{2h} \left| \int_{|f(\bar{x})|\le h, \bar{x}\in\Omega^{(j)}} g(\bar{x})\,\|\nabla f\|\, e^{2\pi i F(\bar{x})} d\bar{x} \right|; \tag{13}$$

here the subdomains $\Omega^{(j)}$ defined as below

$$\Omega^{(j)} = \{\bar{x} \in \Omega | 2^{-j} K \le \|\nabla F\| \le 2^{1-j} K\}.$$

To estimate the integral over $\Omega^{(j)}$ firstly let's make change of variables $\Phi : \bar{x} \mapsto \nabla F(\bar{x})$:

$$u_1 = \frac{\partial F}{\partial x_1}, ..., u_r = \frac{\partial F}{\partial x_r}.$$

Then we have:

$$\left| \int_{\Pi \bigcap \Omega^{(j)}} g(\bar{x}) e^{2\pi i F(\bar{x})} ds \right| =$$

$$= \lim_{h \to 0} \frac{1}{2h} \left| \int_{\substack{\left|f(\Phi^{-1}(\bar{u}))\right| \leq h \\ 2^{-j} K \leq \|\bar{u}\| \leq 2^{1-j} K}} g(\Phi^{-1}(\bar{u})) \|\nabla f\| e^{2\pi i F(\Phi^{-1}(\bar{u}))} (\det A_1)^{-1} d\bar{u} \right| \leq$$

$$\leq \lim_{h \to 0} \frac{1}{2h} \left| \int_{\substack{f(\Phi^{-1}(\bar{u})) = 0 \\ 2^{-j} K \leq \|\bar{u}\| \leq 2^{1-j} K}} g(\Phi^{-1}(\bar{u})) \|\nabla f\| (\det A_1)^{-1} d\bar{u} \right| =$$

$$= \int_{f(\Phi^{-1}(\bar{u}))=0, 2^{-j}K \leq \|\bar{u}\| \leq 2^{1-j}K} \frac{\|\nabla f\| g(\Phi^{-1}(\bar{u}))(\det A_1)^{-1} ds}{\left\|A_1^{-1}(\nabla f)\right\|} \leq$$

$$\leq g_0 R \int_{f(\Phi^{-1}(\bar{u}))=0, 2^{-j}K \leq \|\bar{u}\| \leq 2^{1-j}K} ds; \qquad (14)$$

here

$$R = \max_{\bar{x} \in \Omega} \frac{\|\nabla f\| (\det A_1)^{-1}}{\left\|A_1^{-1}(\nabla f)\right\|}.$$

It is easy to note that

$$\left\|A_1^{-1}(\nabla f)\right\| \geq \lambda_1^{-1} \|\nabla f\|,$$

where λ_1 is a maximal singular number of the matrix A_1. Then we realize that

$$R \leq G_1^{-1},$$

and G_1 is a minimal value of the product of all singular numbers of the matrix A_1, with exception of λ_1.

Consider now the surface integral at last chain of (14). The algebraic equation

$$f(x_1, x_2, ..., x_n) = 0$$

has a set of solutions consisted of finite number of connected hypersurfaces (see [12]) of a view $x_1 = \varphi(x_2, ..., x_n)$. This connected sets will be mapped one-valudely to connected $n-1$ dimensional manifolds of a view $\bar{u} = \Phi(\bar{x}) = (\varphi_1(\bar{x}), ..., \varphi_n(\bar{x}))$ with

$$\varphi_i(\bar{x}) = \frac{\partial F}{\partial x_i}(\varphi(x_2, ..., x_n), x_2, ..., x_n)$$

Then these manifolds are defined by the equation

$$f(\Phi^{-1}(\bar{u})) = 0. \tag{15}$$

From the compactness it follows that the set of solutions of this equation decomposes into n subsets every of which is a finite union of open simple connected components. In every component partial derivatives of the left hand side of the equation (15) takes maximal absolute values with respect to one of the variables $u_1, u_2, ..., u_n$. Since the mapping Φ is one to one mapping then all of open components is possible to include into one subset. Then, surface integral splits into the union of n integrals of following view:

$$\int_{2^{-j}K \leq \|\bar{u}\| \leq 2^{1-j}K} du_1...du_{n-1} \leq c_0 \left(2^{1-j}K\right)^{n-1}.$$

So, summing this estimation for all $j = 1, 2, ...,$ we get the estimation

$$\left| \int_{\Pi \bigcap \Omega} g(\bar{x}) e^{2\pi i F(\bar{x})} ds \right| \leq 4 c_0 g_0 K^{n-1} G_1^{-1}. \tag{16}$$

Taking some parameter $T > 0$ we estimate the part of the integral over the subset $\Pi \bigcap \Omega_1$, where $G_1 \geq T$, as below

$$\left| \int_{\Pi \bigcap \Omega_1} g(\bar{x}) e^{2\pi i F(\bar{x})} ds \right| \leq 4 c_0 g_0 K^{n-1} T^{-1}.$$

The integral over remaining part of the surface where $G_1 \leq T$ we estimate applying the lemma 2 as follows:

$$\left| \int_{\Pi \bigcap \Omega_1} g(\bar{x}) e^{2\pi i F(\bar{x})} ds \right| << T^{1/k-2} \cdot G_{k-2}^{-1/(k-2)} \cdot \tilde{Q}_{k-2}^{n-1}.$$

Define now the parameter T from the equality

$$K^{n-1} T^{-1} = T^{1/(k-2)} G_{k-2}^{-1/(k-2)} \Rightarrow T = K^{\frac{(k-2)(n-1)}{k-1}} G_{k-2}^{1/(k-1)}.$$

Then we find:

$$\int_{\Pi \bigcap \Omega} g(\bar{x}) e^{2\pi i F(\bar{x})} ds << K^{\frac{n-1}{k-1}} G_{k-2}^{-1/(k-1)} \cdot \tilde{Q}_{k-2}^{n-1}.$$

Theorem 1 is proven.

Consider now the estimation of the integral under the limit (11) by another method. We have

$$\left| \int_{\Pi \bigcap \Omega^{(j)}} g(\bar{x}) e^{2\pi i F(\bar{x})} ds \right| \leq 2 g_0 \max_u \left(\int_{\Pi \bigcap \Omega^{(j)}, F(\bar{x}) = u} \frac{ds}{\|\nabla F\|} \right) \leq$$

$$\leq 2 g_0 K^{-1} \max_u \left(\int_{\Pi \bigcap \Omega^{(j)}, F(\bar{x}) = u} ds \right). \tag{17}$$

Now we apply the lemma 7, and make change of variables $u_1 = \frac{\partial F}{\partial x_1}, ..., u_r = \frac{\partial F}{\partial x_r}$. Then this surface will be transformed into the surface defined by the system of equations

$$f(\Phi^{-1}(\bar{u})) = 0,\ F(\Phi^{-1}(\bar{u})) = 0. \tag{18}$$

By the conditions of the theorem the Jacoby matrix Λ_0 of the system of functions $f(\bar{x})$, $F(\bar{x})$ has a rank 2. Applying the lemma 7, we get

$$\int_{\Pi \bigcap \Omega^{(j)}, F(\bar{x})=u} ds \leq \int_{2^{-j}K \leq \|\nabla F\| \leq 2^{1-j}K} 1 \times$$

$$\times \frac{\sqrt{\det(\Lambda_0 \cdot \Lambda_0^t)}}{|\det A_1| \sqrt{\left|\det(\Lambda_0 A_1^{-1} \cdot (A_1^t)^{-1} \Lambda_0^t)\right|}} d\sigma; \tag{19}$$

here $d\sigma$ is an surface element at the transformed surface (18), and the sign ^{t} over the matrix means a transposition. Consider square root of the determinant at the denominator of the expression under integral. There is an integral representation (see [13, p. 131) for it:

$$\frac{1}{\sqrt{\left|\det(\Lambda_0 A_1^{-1} \cdot (A_1^t)^{-1} \Lambda_0^t)\right|}} =$$

$$= \pi \iint_{\left\| (A_1^t)^{-1} \Lambda_0^t \begin{pmatrix} x \\ y \end{pmatrix} \right\| \leq 1} dxdy = \frac{\pi}{\sqrt{\det(\Lambda_0 \cdot \Lambda_0^t)}} \int_{\left\| (A_1^t)^{-1} \bar{u} \right\| \leq 1} ds;$$

here the last integral is a surface integral taken over the two-dimensional subspace of $\mathbb{R}^n$ which is a linear span of the gradient vectors of the functions $f(\bar{x})$, $F(\bar{x})$. If we substitute this surface integral by maximal its value taken over all two dimensional subspaces, we get, in accordance with the theorem 6, §11, ch. 7 (in the suitable form) of the book [6, p.148] (see also [14, 20]), exactly the product of inverted minimal singular numbers of the matrix A_1^{-1}, i. e. maximal singular numbers of the matrix A_1. So, the integral at the right hand side of the equality (19) can be represented as follows:

$$\int_{2^{-j}K \leq \sqrt{u_3^2 + \cdots + u_n^2} \leq 2^{1-j}K} \frac{d\sigma}{\Sigma_{n-2}(A_1)},$$

where $\Sigma_{n-2}(A_1)$ means the product of least $n-2$ singular numbers of the matrix A_1. Hence, we have the bound

$$\int_{2^{-j}K \leq \sqrt{u_3^2 + \cdots + u_n^2} \leq 2^{1-j}K} \frac{d\sigma}{\Sigma_{n-2}(A_1)} \leq$$

$$\leq C_n^2 \frac{\Gamma(1 + (n-2)/2)}{\pi^{(n-2)/2}} (2^{1-j}K)^{n-2} \tilde{G}_1^{-1} << K^{n-2} \tilde{G}_1^{-1}.$$

here $\tilde{G}_1 = \min_{\bar{x}\in\Omega} \Sigma_{n-2}(A_1)$ denotes the minimal value of product of last $n-2$ (smallest) singular numbers of the matrix A_1. Therefore, we have

$$\left|\int_{\|\nabla F\|\le 2^{1-j}K} g(\bar{x})e^{2\pi i F(\bar{x})}ds\right| \le 2^{n-1}\frac{\Gamma(1+(n-2)/2)}{\pi^{(n-2)/2}} g_0 K^{-1}(2^{1-j}K)^{n-2}\tilde{G}_1^{-1}.$$

Summarizing over all $j = 1, 2, ...,$ we obtain:

$$\left|\int_{\Pi\bigcap\Omega} g(\bar{x})e^{2\pi i F(\bar{x})}ds\right| \le C g_0 K^{n-3}\tilde{G}_1^{-1}; \tag{20}$$

$$C = 2^{2n}\frac{\Gamma(1+(n-2)/2)}{\pi^{(n-2)/2}}.$$

This estimation is got using constraints over the gradient and the matrix A_1. Applying the lemma 4 we can prove the estimation in the terms of high order derivatives. This lemma can be applied by following way. Denote by Ω_1 subdomain in Ω for all points of which the condition $\tilde{G}_1 \le T$ is satisfied. We have, in consent with the lemma 4, the bound

$$\mu\left(\Pi_H \bigcap \Omega_1\right) << T^{1/(k-2)}\tilde{G}_{k-2}^{-1/(k-2)}\wp_{k-2}^{n-2};$$

$$\tilde{\wp}_{k-2} = 3(n-2)^2\log\tilde{H};\ \tilde{H} = \max\left\{h\left(H\right), h\left(G_1\right), ..., h\left(G_{k-2}\right), h\left(L\right)\right\}.$$

The value of the parameter T can be defined by the condition

$$K^{n-3}T^{-1} = T^{1/(k-2)}\tilde{G}_{k-2}^{-1/(k-2)}.$$

We have:

$$T = K^{\frac{(k-2)(n-3)}{k-1}}\tilde{G}_{k-2}^{1/(k-1)}.$$

So, we find when $n \ge 2$:

$$\left|\int_{\Pi\bigcap\Omega} g(\bar{x})e^{2\pi i F(\bar{x})}ds\right| << K^{\frac{n-3}{k-1}}\tilde{G}_{k-2}^{-1/(k-1)}\wp_{k-2}^{n-2}. \tag{21}$$

Theorem 2 is now proven.

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