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On Riesz-Thorin Type Theorems in the Besov-Morrey Spaces

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Abstract. In this paper is studied some differential properties of functions belonging to the intersection of Besov-Morrey spaces $B^{l^\mu}_{p_\mu,\theta_\mu,\varphi,\beta}(G_\varphi)$ $(\mu = 1, 2, \dots, N)$

Key Words and Phrases: Besov-Morrey spaces, integral representation, generalized Hölder condition.
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1. Introduction

In this paper we study differential and differential-difference properties of functions from intersection Besov-Morrey spaces

$$B^{l^\mu}_{p_\mu,\theta_\mu,\varphi,\beta}(G_\varphi) \tag{1}$$

was introduced in paper [13]. Note that the paper [13] was proved embedding theorems, but in this paper we prove interpolation type theorem in Besov-Morrey space $B^l_{p,\theta,\varphi,\beta}(G_\varphi)$. Such type theorems were first proved in [2] and later in [1, 4, 5, 7, 8, 9, 10, 11, 13, 14, 15].

Let $G \subset R^n$, $\varphi(t) = (\varphi_1(t), \dots, \varphi_n(t))$, $\varphi_j(t) > 0$,$\varphi'_j(t) > 0$; $j = 1, 2, \dots, n$ $(t > 0)$ is continuously differentiable functions. Assume that $\lim_{t\to+0} \varphi_j(t) = 0$ and $\lim_{t\to+\infty} \varphi_j(t) = K_j$, $0 < K_j \le \infty$,$(j = 1, \dots, n)$. We denote the set of such vector-functions φ by A. We assume that $|\varphi([t]_1)|^{-\beta} = \prod_{j=1}^{n} (\varphi_j([t]_1))^{-\beta_j}$, $\beta_j \in [0, 1]$ $(j = 1, 2, \dots, n)$ and $[t]_1 = \min\{1, t\}$.

For any $x \in R^n$ we put

$$G_{\varphi(t)}(x) = G \cap I_{\varphi(t)}(x) =$$
$$= G \cap \left\{ y : |y_j - x_j| < \frac{1}{2}\varphi_j(t), (j = 1, 2, ..., n) \right\},$$

Let $l \in (0, \infty)^n$, $m_i \in N$, $k_i \in N_0$, $1 \le p < \infty$, $1 \le \theta \le \infty$. The space $B^l_{p,\theta,\varphi,\beta}(G_\varphi)$ is defined [13] as a linear normed space of functions f, on G, with the finite norm $(m_i > l_i - k_i > 0$ $(i = 1, ..., n))$:

$$\|f\|_{B^l_{p,\theta,\varphi,\beta}(G_\varphi)} = \|f\|_{p,\varphi,\beta;G} +$$

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$$+\sum_{i=0}^{n}\left\{\int_{0}^{t_0}\left[\frac{\left\|\Delta_i^{m_i}(\varphi_i(t),G_{\varphi(t)})D_i^{k_i}f\right\|_{p,\varphi,\beta}}{(\varphi_i(t))^{(l_i-k_i)}}\right]^{\theta}\frac{d\varphi_i(t)}{\varphi_i(t)}\right\}^{\frac{1}{\theta}}, \tag{2}$$

where $t_0>0$ is a fixed number and

$$\|f\|_{p,\varphi,\beta;G}=\|f\|_{L_{p,\varphi,\beta}(G)}=\sup_{\substack{x\in G,\\ t>0}}\left(|\varphi([t]_1)|^{-\beta}\|f\|_{p,G_{\varphi(t)}(x)}\right). \tag{3}$$

Let $\lambda_\mu\geq 0$ $(\mu=1,\ldots,N)$, $\sum\limits_{\mu=1}^{N}\lambda_\mu=1$, $\frac{1}{p}=\sum\limits_{\mu=1}^{N}\frac{\lambda_\mu}{p_\mu}$, $\frac{1}{\theta}=\sum\limits_{\mu=1}^{N}\frac{\lambda_\mu}{\theta_\mu}$, $\frac{1}{r}=\sum\limits_{\mu=1}^{N}\frac{\lambda_\mu}{r_\mu}$, $l=\sum\limits_{\mu=1}^{N}\lambda_\mu l^\mu$ and let $\Omega(\cdot,y)$, $M_i\left(\cdot,y,z\right)\in C_0^\infty\left(R^n\right)$, be such that

$$S\left(M_i\right)\subset I_{\varphi(T)}=\left\{y:|y_j|<\frac{1}{2}\varphi_j(T),\ j=1,2,...,n\right\},\ 0<T\leq 1.$$

We put

$$V=\bigcup_{0<t\leq T}\left\{y:\left(\frac{y}{\varphi(t)}\right)\in S\left(M_i\right)\right\}.$$

noting $V\subset I_{\varphi(t)}$ and $U\subset G$, we assume that $U+V\subset G$.

Lemma 1. *Let* $1\leq p_\mu\leq q_\mu\leq r_\mu\leq\infty$, $0<\eta,t\leq T\leq 1$, $\nu=(\nu_1,\ldots,\nu_n)$, $\nu_j\geq 0$ *be integer* $(j=1,2,\ldots,n)$;$\Delta_i^{m_i}\left(\varphi_i(t)\right)f\in L_{p_\mu,\varphi,\beta}(G)$ *and let*

$$B(x)=\prod_{j=1}^{n}\int_{R^n}\int_{R^n}f(x+y+z)\Omega^{(\nu)}\left(\frac{y}{\varphi(T)},\frac{\rho\left(\varphi(T),x\right)}{2\varphi(T)}\right)$$

$$\times\Omega\left(\frac{z}{\varphi\left(T\right)},\frac{\rho\left(\varphi(T),x\right)}{2\varphi(T)}\right)dydz, \tag{4}$$

$$B_\eta^i\left(x\right)=\int_0^\eta L_i\left(x,t\right)\prod_{j=1}^{n}\left(\varphi_j(t)\right)^{-\nu_j-2}\frac{\varphi_i'(t)}{\varphi_i(t)}dt \tag{5}$$

$$B_{\eta T}^i\left(x\right)=\int_\eta^T L_i\left(x,t\right)\prod_{j=1}^{n}\left(\varphi_j(t)\right)^{-\nu_j-2}\frac{\varphi_i'(t)}{\varphi_i(t)}dt \tag{6}$$

$$Q_T^i=\int_0^T\prod_{j=1}^{n}\left(\varphi_j(t)\right)^{-\nu_j-(1-\beta_jp)\left(\frac{1}{p}-\frac{1}{q}\right)}\frac{\varphi_i'(t)}{\left(\varphi_i(t)\right)^{1-l_i}}dt<\infty \tag{7}$$

$$L_i\left(x,t\right)=\int_{R^n}\int_{-\infty}^{+\infty}M_i\left(\frac{y}{\varphi(t)},\frac{\rho\left(\varphi(t),x\right)}{\varphi(t)}\right)\zeta_i\left(\frac{u}{\varphi_i(t)},\frac{\rho_i\left(\varphi_i(t),x\right)}{2\varphi_i(t)},\frac{1}{2}\rho_i'\left(\varphi_i(t),x\right)\right)\times$$

$$\times\Delta_i^{m_i}\left(\varphi_i\left(\delta\right)u\right)f\left(x+y+ue_i\right)dudy \tag{8}$$

Then for any $\bar{x}\in U$ *the following inequalities are true*

$$\sup_{\overline{x}\in U}\|B\|_{qU_{\psi(\xi)}(\overline{x})}\le C_1\prod_{\mu=1}^{N}\{\|f\|_{p_\mu,\varphi,\beta;G}\}^{\lambda_\mu}\times$$

$$\times\prod_{j=1}^{n}\left(\varphi_j(t)\right)^{-\nu_j-(1-\beta_j p)\left(\frac{1}{p}-\frac{1}{q}\right)}\prod_{j=1}^{n}\left(\psi_j[\xi]_1\right)^{\beta_j\frac{p}{q}}, \tag{9}$$

$$\sup_{\overline{x}\in U}\left\|B_\eta^i\right\|_{qU_{\psi(\xi)}(\overline{x})}\le C_2\prod_{\mu=1}^{N}\left\{\left\|\left(\varphi_i(t)\right)^{-l_i^\mu}\Delta_i^{m_i}\left(\varphi_i(t),G_{\varphi(t)}\right)f\right\|_{p_\mu,\varphi,\beta;G}\right\}^{\lambda_\mu}$$

$$\times Q_\eta^i\prod_{j=1}^{n}\left(\psi_j\left([\xi]_1\right)\right)^{\beta_j\frac{p}{q}}, \tag{10}$$

$$\sup_{\overline{x}\in U}\left\|B_{\eta T}^i\right\|_{qU_{\psi(\xi)}(\overline{x})}\le C_3\prod_{\mu=1}^{N}\left\{\left\|\left(\varphi_i(t)\right)^{-l_i^\mu}\Delta_i^{m_i}\left(\varphi_i(t),G_{\varphi(t)}\right)f\right\|_{p_\mu,\varphi,\beta;G}\right\}^{\lambda_\mu}$$

$$\times Q_{\eta T}^i\prod_{j=1}^{n}\left(\psi_j\left([\xi]_1\right)\right)^{\beta_j\frac{p}{q}}, \tag{11}$$

where $U_{\psi(\xi)}\left(\overline{x}\right)=\left\{x:|x_j-\overline{x}_j|<\frac{1}{2}\psi_j\left(\xi\right),j=1,2,...,n\right\}$ $\psi\in A$, C_1 *and* C_2 *-the constants independent of* φ, ξ, η *and* T.

Proof. Apply the generalized Minkowski inequality for $\bar{x}\in U$ we obtain

$$\left\|B_\eta^i\right\|_{q,U_{\psi(\xi)}(\overline{x})}\le\int_0^\eta\|L_i(\cdot,t)\|_{q,U_{\psi(\xi)}(\overline{x})}\prod_{\mu=1}^{N}(\varphi_j(t))^{-2-\nu_j}\frac{\varphi_i'(t)}{\varphi_i(t)}dt \tag{12}$$

estimate the norm $\|L_i(\cdot,t)\|_{p,U_{\psi(\xi)}(\overline{x})}$. Applying the Holder inequality with exponents

$$\alpha_\mu=\frac{q_\mu}{\lambda_\mu q},\mu=1,2,\ldots,N;\left(\sum_{\mu=1}^{N}\frac{1}{\alpha_\mu}=q,\ \sum_{\mu=1}^{N}\frac{\lambda_\mu}{q_\mu}=1\right)$$

for $|L_i(x,t)|$ we obtain

$$\|L_i(\cdot,t)\|_{q,U_{\psi(\xi)}(\bar{x})}\le C_1\prod_{\mu=1}^{N}\left\{\|L_i(\cdot,t)\|_{q_\mu,U_{\psi(\xi)}(\bar{x})}\right\}^{\lambda_\mu}. \tag{13}$$

By virtue of the Holder inequality, for $(q_\mu\le r_\mu)$ $(\mu=1,2,\ldots,N)$ we have

$$\|L_i\left(\cdot,t\right)\|_{p_\mu,U_{\psi(\xi)}(\bar{x})}\le\prod_{\mu=1}^{N}(\psi_j(\xi))^{\left(\frac{1}{p_\mu}-\frac{1}{r_\mu}\right)}\|L_i\left(\cdot,t\right)\|_{r_\mu,U_{\psi(\xi)}(\bar{x})}. \tag{14}$$

Now estimate the norm $\|L_i(\cdot,t)\|_{r_\mu,U_{\psi(\xi)}(\overline{x})}$.Let χ be a characteristic function of the set $S\left(M_i\right)$. Again applying the Holder inequality for representing the function in the function in the form (8) in the case $1\le p_\mu\le r_\mu\le\infty, s_\mu\le r_\mu,\ \frac{1}{s_\mu}=1-\frac{1}{p_\mu}+\frac{1}{r_\mu}$ $(\mu=1,2,\ldots,N)$, we get

$$\|L_i\left(\cdot,t\right)\|_{r_\mu,U_{\psi(\xi)}(\bar{x})}$$

$$\le C\sup_{x\in U_{\psi(\xi)}}\left(\int\limits_{R^n}\left|\int\limits_{-\infty}^{\infty}\Delta^{m^i}\left(\varphi_i(\delta)u\right)f\left(x+y+ue_i\right)du\right|^{p_\mu}\chi\left(\frac{y}{\varphi(t)}\right)dy\right)^{\frac{1}{p_\mu}-\frac{1}{r_\mu}}\times$$

$$\times\sup_{x\in V}\left(\int\limits_{U_{\psi(\xi)}}\left|\int\limits_{-\infty}^{\infty}\zeta_i\left(\frac{u}{\varphi_i(t)},\frac{\rho\left(\varphi_i(t),x\right)}{2\varphi_i(t)},\frac{1}{2}\rho'\left(\varphi_i(t),x\right)\right)\right.\right.$$

$$\left.\left.\times\Delta^{m^i}\left(\varphi_i(\delta)u\right)f\left(x+y+ue_i\right)du\right|^{p_\mu}X\left(\frac{y}{\varphi(t)}\right)dy\right)^{\frac{1}{p_\mu}-\frac{1}{r_\mu}}$$

$$\times\sup_{y\in V}\left(\int\limits_{U_{\psi(\xi)}(\bar{x})}\left|\int\limits_{-\infty}^{\infty}\zeta_i\left(\frac{u}{\varphi_i(t)},\frac{\rho\left(t,x\right)}{2\varphi_i(t)},\frac{1}{2}\rho'\left(\varphi_i(t),x\right)\right)\right.\right.$$

$$\left.\left.\times\Delta^{m^i}\left(\varphi_i(\delta)u\right)f\left(x+y+ue_i\right)du\right|^{p_\mu}\chi\left(\frac{y}{\varphi(t)}\right)dy\right)^{\frac{1}{r_\mu}}$$

$$\times\left(\int\limits_{R^n}\left|\widetilde{M}_i\left(\frac{y}{\varphi(t)}\right)\right|^{s_\mu}dy\right)^{\frac{1}{s_\mu}}. \tag{15}$$

It is assumed that $|M_i(x,y)|\le C|\widetilde{M}_i(x)|$, $\widetilde{M}_i\in C_0^\infty(R^n)$. For any $x\in U$ we have

$$\int\limits_{R^n}\left|\int\limits_{-\infty}^{\infty}\zeta_i\left(\frac{u}{\varphi_i(t)},\frac{\rho\left(\varphi_i(t),x\right)}{2\varphi_i(t)},\frac{1}{2}\rho'\left(\varphi_i(t),x\right)\right)\right.$$

$$\left.\times\Delta^{m^i}\left(\varphi_i(\delta)u\right)f\left(x+y+ue_i\right)du\right|^{p_\mu}X\left(\frac{y}{\varphi(t)}\right)dy$$

$$\le\int\limits_{(U+V)_{\varphi(t)}(\bar{x})}\left|\int\limits_{-\infty}^{\infty}\zeta_i\left(\frac{u}{\varphi_i(t)},\frac{\rho\left(\varphi_i(t),x\right)}{2\varphi_i(t)},\frac{1}{2}\rho'\left(\varphi_i(t),x\right)\right)\right.$$

$$\left.\times\Delta^{m^i}\left(\varphi_i(\delta)u\right)f\left(x+y+u^{e^i}\right)du\right|^{p_\mu}dy\le$$

$$\le\int\limits_{G_{\varphi(t)}(\bar{x})}\left|\int\limits_{-\infty}^{\infty}\zeta_i\left(\frac{u}{\varphi_i(t)},\frac{\rho\left(\varphi_i(t),x\right)}{2\varphi_i(t)},\frac{1}{2}\rho'\left(\varphi_i(t),x\right)\right)\right.$$

$$\times \Delta^{m^i}\left(\varphi_i(\delta)u\right) f\left(y+u^{e^i}\right) du\Big|^{p_\mu} dy \le$$

$$\le (\varphi_i(t))^{l_i^\mu p_\mu} \|(\varphi_i(t))^{-l_i^\mu} \Delta_i^{m_i}(\varphi_i(t), G_{\varphi(t)}) f\|_{p_\mu, G_{\varphi(t)}(x)}^{p_\mu} \le$$

$$\le (\varphi_i(t))^{l_i^\mu p_\mu} \|(\varphi_i(t))^{-l_i^\mu} \Delta_i^{m_i}(\varphi_i(t), G_{\varphi(t)}) f\|_{p_\mu,\varphi,\beta}^{p_\mu} \prod_{j=1}^{n} (\varphi_j(t))^{\beta_j p_\mu}. \tag{16}$$

for $y \in V \quad \left((U+V)_{\psi(\xi)} \subset G_{\varphi(t)}\right)$

$$\int\limits_{U_{\psi_\xi(\bar{x})}} \left| \int\limits_{-\infty}^{\infty} \zeta_i \left(\frac{u}{\varphi_i(t)}, \frac{\rho\left(\varphi_i(t), x\right)}{2\varphi_i(t)}, \frac{1}{2}\rho'\left(\varphi_i(t), x\right) \right) \right.$$

$$\left. \times \Delta^{m^i}\left(\varphi_i(\delta)u\right) f\left(x+y+ue_i\right) du \right|^{p_\mu} X\left(\frac{y}{\varphi(t)}\right) dy$$

$$\le \int\limits_{(U+V)_{\varphi(t)}(\bar{x}+y)} \left| \int\limits_{-\infty}^{\infty} \zeta_i \left(\frac{u}{\varphi_i(t)}, \frac{\rho\left(\varphi_i(t), x\right)}{2\varphi_i(t)}, \frac{1}{2}\rho'\left(\varphi_i(t), x\right) \right) \right.$$

$$\left. \times \Delta^{m^i}\left(\varphi_i(\delta)u\right) f\left(x+u^{e^i}\right) du \right|^{p_\mu} dy \le$$

$$\le (\varphi_i(t))^{l_i^\mu p_\mu} \|(\varphi_i(t))^{-l_i^\mu} \Delta_i^{m_i}(\varphi_i(t), (U+V)_{\psi(\xi)}) f\|_{p_\mu,(U+V)_{\psi(\xi)}}^{p_\mu} \le$$

$$\le (\varphi_i(t))^{l_i^\mu p_\mu} \|(\varphi_i(t))^{-l_i^\mu} \Delta_i^{m_i}(\varphi_i(t), G_{\varphi(t)}) f\|_{p_\mu,\varphi,\beta}^{p_\mu} \prod_{j=1}^{n} (\psi_j([\xi]_1))^{\beta_j p_\mu}. \tag{17}$$

$$\int\limits_{R^n} \left| \widetilde{M_i}\left(\frac{y}{\varphi(t)}\right) \right|^{s_\mu} dy = \left\| \widetilde{M_i} \right\|_{s_\mu}^{s_\mu} \prod_{j=1}^{n} \varphi_j(t). \tag{18}$$

From inequalities (15)-(18), we have

$$\|L_i(\cdot,t)\|_{r_\mu, U_{\psi(\xi)}(\bar{x})} \le C_1 \left\| \widetilde{M_i} \right\|_{s_\mu} \|(\varphi_i(t))^{-l_i^\mu} \Delta_i^{m_i}(\varphi_i(t), G_{\varphi(t)}) f\|_{p_\mu,\varphi,\beta} \times$$

$$\times (\varphi_i(t))^{l_i^\mu} \prod_{j=1}^{n} (\varphi_j(t))^{\frac{1}{s_\mu} + \beta_j p_\mu \left(\frac{1}{p_\mu} - \frac{1}{r_\mu}\right)} \prod_{j=1}^{n} (\psi_j([\xi]_1))^{\frac{\beta_j p_\mu}{r_\mu}} \tag{19}$$

and by inequality (14) we have

$$\|L_i(\cdot,t)\|_{r_\mu, U_{\psi(\xi)}(\bar{x})} \le C_2 \left\| \widetilde{M_i} \right\|_{s_\mu} \|(\varphi_i(t))^{-l_i^\mu} \Delta_i^{m_i}(\varphi_i(t), G_{\varphi(t)}) f\|_{p_\mu,\varphi,\beta} \times$$

$$\times (\varphi_i(t))^{l_i^\mu} \prod_{j=1}^{n} (\varphi_j(t))^{\frac{1}{s_\mu} + \beta_j p_\mu \left(\frac{1}{p_\mu} - \frac{1}{r_\mu}\right)} \prod_{j=1}^{n} (\psi_j([\xi]_1))^{\frac{\beta_j p_\mu}{r_\mu}} \prod_{j=1}^{n} (\psi_j([\xi]_1))^{\frac{1}{q_\mu} - \frac{1}{r_\mu}} \tag{20}$$

From inequalities (12),(13) for $r_\mu = p_\mu$ and for any $\bar{x} \in U$ reduce to the estimation

$$\sup_{\bar{x}\in U} \|B^i_\eta\|_{q,U_{\psi(\xi)}(\bar{x})} \leq$$

$$\leq C_3 \prod_{\mu=1}^{N} \left\{ \|(\varphi_i(t))^{-l_i^\mu} \Delta_i^{m_i}(\varphi_i(t), G_{\varphi(t)}) f\|_{p_\mu,\varphi,\beta} \right\}^{\lambda_\mu} \prod_{j=1}^{n} (\psi_j([\xi]_1))^{\frac{\beta_j p_\mu}{r_\mu}}$$

In a similar way can prove inequality (9) and (11).

Corollary 1. *For $1 \leq \tau_1 \leq \tau_2 \leq \infty$ the following inequalities:*

$$\sup_{\bar{x}\in U} \|B\|_{q,\psi,\beta_1;U} \leq C^1 \prod_{\mu=1}^{N} \left\{ \|f\|_{p_\mu,\varphi,\beta;G} \right\}^{\lambda_\mu} \tag{21}$$

$$\sup_{\bar{x}\in U} \|B^i_\eta\|_{q,\psi,\beta_1;U} \leq$$

$$\leq C^2 \prod_{\mu=1}^{N} \left\{ \|(\varphi_i(t))^{-l_i^\mu} \Delta_i^{m_i}(\varphi_i(t), G_{\varphi(t)}) f\|_{p_\mu,\varphi,\beta} \right\}^{\lambda_\mu} \prod_{j=1}^{n} (\psi_j([\xi]_1))^{\frac{\beta_j p_\mu}{r_\mu}} \tag{22}$$

$$\sup_{\bar{x}\in U} \|B^i_{\eta,T}\|_{q,\psi,\beta_1;U} \leq$$

$$\leq C^2 \prod_{\mu=1}^{N} \left\{ \|(\varphi_i(t))^{-l_i^\mu} \Delta_i^{m_i}(\varphi_i(t), G_{\varphi(t)}) f\|_{p_\mu,\varphi,\beta} \right\}^{\lambda_\mu} \prod_{j=1}^{n} (\psi_j([\xi]_1))^{\frac{\beta_j p_\mu}{r_\mu}} \tag{23}$$

2. Main results

Prove two theorems on the properties of the functions from the space $\bigcap_{\mu=1}^{N} B^{l^\mu}_{p_\mu,\theta_\mu,\varphi,\beta}(G_\varphi)$.

Theorem 1. *Let $G \subset R^n$ satisfy the condition of flexible φ-horn [11], $1 \leq p_\mu \leq q_\mu \leq \infty$, $1 \leq \theta_\mu \leq \infty (\mu = 1, 2, \ldots, N)$; $\nu = (\nu_1, \nu_2, .., \nu_n)$, $\nu_j \geq 0$ integer $j = 1, 2, ..., n$, $Q^i_T < \infty$ $(i = 1, 2, ..., n)$ and let $f \in \bigcap_{\mu=1}^{N} B^{l^\mu}_{p_\mu,\theta_\mu,\varphi,\beta}(G_\varphi)$. Then the following embeddings hold*

$$D^\nu : \bigcap_{\mu=1}^{N} B^{l^\mu}_{p_\mu,\theta_\mu,\varphi,\beta}(G_\varphi) \hookrightarrow L_{q,\psi,\beta^1}(G)$$

i.e. for $f \in \bigcap_{\mu=1}^{N} B^{l^\mu}_{p_\mu,\theta_\mu,\varphi,\beta}(G_\varphi)$ there exists a generalized derivative $D^\nu f$ in G and the following inequalities are true

$$\|D^\nu f\|_{q,G} \leq C^1 B(T) \prod_{\mu=1}^{N} \left\{ \|f\|_{\bigcap_{\mu=1}^{N} B^{l^\mu}_{p_\mu,\theta_\mu,\varphi,\beta}(G_\varphi)} \right\}^{\lambda_\mu}, \tag{24}$$

$$\|D^{\nu} f\|_{q,\psi,\beta^1;G} \leq C^2 \prod_{\mu=1}^{N} \left\{ \|f\|_{\bigcap_{\mu=1}^{N} B^{l^\mu}_{p_\mu,\theta_\mu,\varphi,\beta}(G_\varphi)} \right\}^{\lambda_\mu}, \ p_\mu \leq q_\mu < \infty, \tag{25}$$

in particular, if

$$Q^i_{T,0} = \int_0^T \prod_{j=1}^{n} (\varphi_j(t))^{-\nu_j-(1-\beta_j p)\frac{1}{p}} \frac{\varphi_i'(t)}{(\varphi_i(t))^{1-\sum\limits_{\mu=1}^{n} l_i^\mu \lambda_\mu}} dt < \infty, \left(i = \overline{1,n}\right), \tag{26}$$

then $D^\nu f\,(x)$ *is continuous on* G, *and*

$$\sup_{x\in G} |D^\nu f(x)| \leq C_1 B_1^0(t) \prod_{\mu=1}^{N} \left\{ \|f\|_{\bigcap_{\mu=1}^{N} B^{l^\mu}_{p_\mu,\theta_\mu,\varphi,\beta}(G_\varphi)} \right\}^{\lambda_\mu} \tag{27}$$

where $0 < T \leq \min\{1, T_0\}$ *is a fixed number,* C_1, C_2 *are the constants independent of* f, C_1 *are independent also on* T.

Proof. At first note that in the conditions of our theorem there exists a generalized derivative $D^\nu f$. Indeed, from the condition $Q^i_T < \infty$ $\{i = 1, 2, ..., n\}$, it follows that for $f \in \bigcap\limits_{\mu=1}^{N} B^{l^\mu}_{p_\mu,\theta_\mu,\varphi,\beta}(G_\varphi) \to B^{l^\mu}_{p_\mu,\theta_\mu,\varphi,\beta}(G_\varphi) \to B^{l^\mu}_{p_\mu,\theta_\mu}(G_\varphi)$ there exists a generalized derivative $D^\nu f \in L_p(G)$ and for it integral representation with the kernels is valid [13].

$$D^\nu f(x) = f^{(\nu)}_{\varphi(T)}(x) + (-1)^{|\nu|} \sum_{i=1}^{n} \int_0^T \int_{-\infty}^{+\infty} \int_{R^n} K_i^{(\nu)} \left(\frac{u}{\varphi(t)}, \frac{\rho(\varphi(t), x)}{\varphi(t)} \right) \times$$

$$\times \zeta_i \left(\frac{u}{\varphi_i(t)}, \frac{\rho(\varphi_i(t), x)}{2\varphi_i(t)}, \frac{1}{2}\rho'(\varphi_i(t), x) \right) \Delta^{m^i} (\varphi_i(\delta)u) \times$$

$$\times f(x + y + ue_i) \prod_{j=1}^{n} (\varphi_j(t))^{-\nu_j-2} \frac{\varphi_i'(t)}{\varphi_i(t)} dtdudy, \tag{28}$$

$$f^{(\nu)}_{\varphi(T)}(x) = \prod_{j=1}^{n} (\varphi_j(t))^{-\nu_j-2} \int_{R^n} \int_{R^n} \Omega^{(\nu)} \left(\frac{u}{\varphi(T)}, \frac{\rho(\varphi(T), x)}{2\varphi(T)} \right) \times$$

$$\times \Omega^{(\nu)} \left(\frac{z}{\varphi(T)}, \frac{\rho(\varphi(T), x)}{2\varphi(T)} \right) f(x + y + z) dydz. \tag{29}$$

Based around the Minkowsky inequality, from identities (28) and (29) we get

$$\|D^\nu f\|_{q,G} \leq \left\| f^{(\nu)}_{\varphi(T)} \right\|_{q,G} + \sum_{i=1}^{n} \left\| B^i_T \right\|_{q,G}. \tag{30}$$

By means of inequality (9) for $U = G$, $M_i = \Omega$ we get

$$\left\| f^{(\nu)}_{\varphi(t)} \right\|_{q,G} \leq \prod_{\mu=1}^{N} \left\{ \|f\|_{p_\mu,\theta_\mu,\varphi,\beta;G} \right\}^{\lambda_\mu} \prod_{j=1}^{n} (\varphi_j(T))^{-\nu_j-(1-\beta_j p)\left(\frac{1}{p}-\frac{1}{q}\right)} \prod_{j=1}^{n} (\psi_j([\xi]_1))^{\beta_j \frac{p}{q}}, \quad (31)$$

and by means inequality (10) for $U = G$, $M_i = K_i^{(\nu)}$ $\eta = T$ we get

$$\left\| B^i_T \right\|_{q,G} \leq C_2 |Q^i_T| \prod_{\mu=1}^{N} \left\{ \left\| (\varphi_i(t))^{-l_i} \Delta_i^{m_i} \left(\varphi_i(t), G_{\varphi(t)} \right) f \right\|_{p,\varphi,\beta} \right\}^{\lambda_\mu}. \quad (32)$$

Substituting (31), (32) for $1 \leq \theta_\mu \leq \infty$, $p_\mu \leq \theta_\mu$ $(\mu = 1, 2, \ldots, N)$, we get inequality (24). By means of inequalities (21) and (22) for $\eta = T$ we get inequality (25) .

Now let conditions $Q^i_{T,0} < \infty$ $(i = 1, 2, \ldots, n)$. Then from identities (28), (29) and by the inequality (24) for $q = \infty, p \leq \theta$ we get

$$\left\| D^\nu f(x) - f^{(\nu)}_{\varphi(T)}(x) \right\|_{\infty,G} \leq C_1 \sum_{i=1}^{n} Q^i_{T,0}$$

$$\times \prod_{\mu=1}^{N} \left\{ \left(\int_0^{t_0} \left[\frac{\left\| \Delta_i^{m_i}(\varphi_i(t), G_{\varphi(t)}) f \right\|_{p_\mu,\varphi,\beta}}{(\varphi_i(t))^{l_i^\mu}} \right]^{\theta_\mu} \frac{d\varphi_i(t)}{\varphi_i(t)} \right)^{\frac{1}{\theta_\mu}} \right\}^{\lambda_\mu}.$$

As $T \to 0$, the left side of this inequality tends to zero, since $f^{(\nu)}_{\varphi(T)}$ is continuous on G and the convergence on $L_\infty(G)$ coincides with the uniform convergence. Then the limit function $D^\nu f$ is continuous on G.

Theorem 1 is proved.

Let γ be an n-dimensional vector.

Theorem 2. *Let all the conditions of Theorem 2.1 be satisfied. Then for $Q^i_T < \infty$ $(i = 1, 2, ..., n)$ the generalized derivative $D^\nu f$ satisfies on G the generalized Hölder condition, i.e. the following inequality is valid:*

$$\|\Delta(\gamma, G) D^\nu f\|_{q,G} \leq C \prod_{\mu=1}^{N} \left\{ \|f\|_{B^{l^\mu}_{p_\mu,\theta_\mu,\varphi,\beta}(G_\varphi)} \right\}^{\lambda_\mu} \cdot |H(|\gamma|, \varphi; T)|. \quad (33)$$

In particular, if $Q^i_{T,0} < \infty$, $(i = 1, 2, \ldots, n)$, then

$$\sup_{x \in G} |\Delta(\gamma, G) D^\nu f(x)| \leq C \prod_{\mu=1}^{N} \left\{ \|f\|_{B^{l^\mu}_{p_\mu,\theta_\mu,\varphi,\beta}(G_\varphi)} \right\}^{\lambda_\mu} \cdot |H_0(|\gamma|, \varphi, T)|, \quad (34)$$

where C - is a constant independent of f, $|\gamma|, \varphi, T$ and H.

$$H(|\gamma|, \varphi, T) = \max_i \left\{ |\gamma|, Q^i_{|\gamma|}, Q^i_{|\gamma|,T} \right\}$$

$$\left(H_0(|\gamma|, \varphi, T) = \max_i \left\{ |\gamma|, Q^i_{|\gamma|,0}, Q^i_{|\gamma|,T,0} \right\} \right)$$

Proof. According to lemma 8.6 from [3] there exists a domain

$$G_\omega \subset G\left(\omega = \vartheta r\left(x\right),\ \vartheta > 0,\ r\left(x\right) = \rho\left(x, \partial G\right),\ x \in G\right)$$

and assume that $|\gamma| < \omega$,then for any $x \in G_\omega$ the segment connecting the points $x, x+\gamma$ is contained in G. Consequently, for all the points of this segment, identities (28) and (29) with the same kernels are valid. After same transformations, we get

$$\left|\Delta\left(\gamma, G\right) D^\nu f\left(x\right)\right| \leq \prod_{j=1}^{n} \left(\varphi_j(t)\right)^{-1-\nu_j} =$$

$$= C_1 B\left(x,\gamma\right) + C_2 \sum_{i=1}^{n} \left(B_1\left(x,\gamma\right) + B_2\left(x,\gamma\right)\right), \tag{35}$$

where $0 < T \leq \{1, T_0\}$ we also assume that $|\gamma| < t$,consequently $|\gamma| < \min\left(\omega, T\right)$. If $x \in G \setminus G_\omega$, then

$$\Delta\left(\gamma, G\right) D^\nu f\left(x\right) = 0.$$

From (35) we get

$$\left\|\Delta\left(\gamma, G\right) D^\nu f\right\|_{q,G} \leq \left\|B\left(\cdot,\gamma\right)\right\|_{q,G_\omega}$$

$$+\sum_{i=1}^{n}\left(\left\|B_1\left(\cdot,\gamma\right)\right\|_{q,G_\omega} + \left\|B_2\left(\cdot,\gamma\right)\right\|_{q,G_\omega}\right), \tag{36}$$

$$B\left(x,\gamma\right) \leq \prod_{j=1}^{n}\left(\varphi_j(t)\right)^{-\nu_j-2} \int_0^{|\gamma|} d\zeta \int_{R^n}\int_{R^n} \left|f\left(x+\zeta e_\gamma + y\right)\right|$$

$$\times \left|D_j \Omega^{(\nu)}\left(\frac{y}{\varphi\left(T\right)}, \frac{\rho\left(\varphi(t), x\right)}{2\varphi(t)}\right)\Omega^{(\nu)}\left(\frac{z}{\varphi\left(T\right)}, \frac{\rho\left(\varphi(t), x\right)}{2\varphi(t)}\right)\right| dydz.$$

Taking into account $\xi e_\gamma + G_\omega \subset G$, and from inequality (21) for $U = G$, we have

$$\left\|B\left(\cdot,\gamma\right)\right\|_{q,G_\omega} \leq C_1 \left|\gamma\right| \left\|f\right\|_{p,\varphi,\beta;G}. \tag{37}$$

By means of inequality (22), for $U = G$, $\eta = |\gamma|$, $M_i = K_i^{(\nu)}$ we get

$$\left\|B_1\left(\cdot,\gamma\right)\right\|_{q,G_\omega} \leq C_2 \left|Q^i_{|\gamma|}\right| \left\|\left(\varphi_i(t)\right)^{-l_i} \Delta_i^{m_i}\left(\varphi_i(t), G_{\varphi(t)}\right) f\right\|_{p,\varphi,\beta;G} \tag{38}$$

and by means of inequality (23) for $U = G$, $\eta = |\gamma|$, $M_i = K_i^{(\nu)}$ we get

$$\left\|B_2\left(\cdot,\gamma\right)\right\|_{q,G_\omega} \leq C_3 \left|Q^i_{|\gamma|,T}\right| \left\|\left(\varphi_i(t)\right)^{-l_i} \Delta_i^{m_i}\left(\varphi_i(t), G_{\varphi(t)}\right) f\right\|_{p,\varphi,\beta;G}. \tag{39}$$

From inequalities (36)-(39) for cases $p_\mu \leq \theta_\mu$ we get the required inequality (33).

Now suppose that, $|\gamma| \geq \min\left(\omega, T\right)$. Then we have

$$\left\|\Delta\left(\gamma, G\right) D^\nu f\right\|_{q,G} \leq 2\left\|D^\nu f\right\|_{q,G} \leq C\left(\omega T\right)\left\|D^\nu f\right\|_{q,G} \left|h\left(\left|\gamma\right|, \varphi; T\right)\right|.$$

Estimating for $\left\|D^\nu f\right\|_{q,G}$ by means of inequality (24), in this case we get estimation. This completes the proof of Theorem 2.

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