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Inverse Boundary Value Problem for Two-Dimensional Pseudo Parabolic Equation of Third Order with Additional Integral Condition

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Abstract. Inverse boundary value problem for two-dimensional pseudo parabolic equation of third order with additional integral condition is considered. We first reduce our problem to some equivalent (in some sense) one. Using the Fourier method, the equivalent problem, in turn, is reduced to the system of integral equations. Then, using contraction mapping method, we prove the existence and uniqueness for the solution of the system of integral equations, which is also a unique solution of the equivalent problem. Finally, using equivalence, we prove the existence and uniqueness for the classical solution of the original problem.
Key Words and Phrases: inverse boundary value problem, two-dimensional pseudo parabolic equation of third order, Fourier method, classical solution.
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1. Introduction

By the inverse problem for partial differential equations, we mean a problem that requires to find, along with a solution itself, the right-hand side and (or) some coefficient(s) of the equation. Inverse problems arise in many fields of human activities, such as seismology, mineral exploration, biology, medicine, quality control of industrial products, etc. which makes them one of the most important problems in today's mathematics. If an inverse problem requires to find not only the solution itself, but also the right-hand side of the equation, then such an inverse problem is linear. And if it requires to find both the solution and at least one of the coefficients, then such an inverse problem is nonlinear. Many mathematicians have studied various inverse problems for some types of partial differential equations, such as A.N.Tikhonov [1], M.M.Lavrentiev [2,3], V.K.Ivanov [4] and their students. More details about these problems can be found in the monograph by A.M.Denisov [5].

Inverse problems for one-dimensional pseudo parabolic equations of third order have been considered in [6–8].

In this work, using Fourier method and contraction mapping principle, we prove the existence and uniqueness of the solution of the nonlocal inverse boundary value problem for a third order two-dimensional pseudo parabolic equation.

2. Problem statement and its reduction to the equivalent problem

Let $D_T = Q_{xy} \times \{0 \le t \le T\}$, where $Q_{xy} = \{(x,y) : 0 \le x \le 1,\ 0 \le y \le 1\}$. Also, let $\alpha(t) > 0$, $\beta(t) > 0$, $f(x,y,t)$,$\varphi(x,y)$,$h(t)$ be the given functions defined for $x \in [0,1]$, $y \in [0,1]$, $t \in [0,T]$. Consider the following inverse boundary value problem: find a pair $\{u(x,t)\,,\ p(t)\}$ of functions $u(x,t)$, $p(t)$ which satisfy the equation

$$u_t(x,y,t) - \alpha(t)(u_{txx}(x,y,t) + u_{tyy}(x,y,t)) - \beta(t)(u_{xx}(x,y,t) + u_{yy}(x,y,t)) =$$

$$= p(t)u(x,y,t) + f(x,y,t)\,, \tag{1}$$

nonlocal initial condition

$$u(x,y,0) + \delta u(x,y,T) = \varphi(x,y) \quad (0 \le x \le 1,\ 0 \le x \le 1)\ , \tag{2}$$

boundary conditions

$$u_x(0,y,t) = u(1,y,t) = 0 \ \ (0 \le y \le 1,\, 0 \le t \le T)\ , \tag{3}$$

$$u(x,0,t) = u_y(x,1,t) = 0 \ (0 \le y \le 1, \ \ 0 \le t \le T)\ , \tag{4}$$

and the additional condition

$$\int_0^1 \int_0^1 u(x,y,t)dxdy = h(t) \ (0 \le t \le T), \tag{5}$$

where $\delta \ge 0$ is a given number.

Denote

$$\tilde{C}^{2,2,1}(D_T) = \ \{u(x,y,t) : u(x,y,t) \in C^{2,2,1}(D_T),\, u_{txx}(x,y,t), u_{tyy}(x,y,t) \in C(D_T)\}\,.$$

Definition 1. *By the classical solution of the inverse boundary value problem* (1)-(5)*, we mean a pair* $\{u(x,y,t), p(t)\,\}$ *of functions* $u(x,t)$*,* $p(t)$ *such that* $u(x,y,t) \in \tilde{C}^{2,2,1}(D_T)$*,* $p(t) \in C[0,T]$ *and the relations* (1)-(5) *are satisfied in the usual sense.*

The following theorem is true.

Theorem 1. *Let* $0 < \alpha(t)$*,* $0 < \beta(t) \in C[0,T]$*,* $\varphi(x,y) \in C(Q_{xy})$*,* $f(x,y,t) \in C(D_T)$*,* $h(t) \in C^1[0,T]$ $\ h(t) \ne 0 \ \ (0 \le t \le T)$*,* $\delta \ge 0$*, and the coherence condition*

$$\int_0^1 \int_0^1 \varphi(x,y)dxdy = h(0) + \delta h(T)$$

be satisfied. Then the problem of finding the classical solution of the problem (1)-(5) *is equivalent to the one of determining the functions* $u(x,y,t) \in \tilde{C}^{2,2,1}(D_T)$*,*$p(t) \in C[0,T]$ *from the relations* (1)-(4)*,*

$$h'(t) - \alpha(t)\left(\int_0^1 u_{tx}(1,y,t)dy - \int_0^1 u_{ty}(x,0,t)dx\right) -$$

$$-\beta(t)\left(\int_0^1 u_x(1,y,t)dy-\int_0^1 u_y(x,0,t)dx\right)=$$

$$=p(t)h(t)+\int_0^1\int_0^1 f(x,y,t)dxdy \qquad (0\le t\le T). \tag{6}$$

Proof. Let $\{u(x,y,t),p(t))\}$ be a classical solution of the problem (1)- (5). On integrating the equation (1) with respect to x and y from 0 to 1, we have:

$$\frac{d}{dt}\int_0^1\int_0^1 u(x,y,t)dxdy-$$

$$-\alpha(t)\left(\int_0^1 u_{tx}(1,y,t)-u_{tx}(0,y,t)dy+\int_0^1 u_{ty}(x,1,t)-u_{ty}(x,0,t)dx\right)-$$

$$-\beta(t)\left(\int_0^1 u_x(1,y,t)-u_x(0,y,t)dy+\int_0^1 u_y(x,1,t)-u_y(x,0,t)dx\right)=$$

$$=p(t)\int_0^1\int_0^1 u(x,y,t)dxdy+\int_0^1\int_0^1 f(x,y,t)dxdy\ (0\le t\le T).$$

From the last relation, by (3),(4)we obtain

$$\frac{d}{dt}\int_0^1\int_0^1 u(x,y,t)dxdy-\alpha(t)\left(\int_0^1 u_{tx}(1,y,t)dy-\int_0^1 u_{ty}(x,0,t)dx\right)-$$

$$-\beta(t)\left(\int_0^1 u_{tx}(1,y,t)dy-\int_0^1 u_{ty}(x,0,t)dx\right)=$$

$$=p(t)\int_0^1\int_0^1 u(x,y,t)dxdy+\int_0^1\int_0^1 f(x,y,t)dxdy\ (0\le t\le T). \tag{7}$$

Now, taking $h(t)\in C^1[0,T]$ and differentiating (5), we have

$$\int_0^1\int_0^1 u_t(x,y,t)dxdy=h'(t)\ (0\le t\le T) \tag{8}$$

By (5) and (8), it follows from (7) that the relation (6) is valid.

Now let's assume that $\{u(x,t),p(t)\}$ is a soluton of the problem (1)-(4), (5). Then from (6) and (7) we obtain

$$\frac{d}{dt}\left(\int_0^1\int_0^1 u(x,y,t)dxdy-h(t)\right)=p(t)\left(\int_0^1\int_0^1 u(x,y,t)dxdy-h(t)\right)\quad (0\le t\le T). \tag{9}$$

By (2) and the coherence condition $\int_0^1\int_0^1 \varphi(x,y)dxdy=h(0)+\delta h(T)$, we have

$$\int_0^1\int_0^1 u(x,y,0)dxdy-h(0)+\delta\left(\int_0^1\int_0^1 u(x,y,T)dxdy-h(T)\right)=$$

$$= \int_0^1 \int_0^1 u(x,y,0) + \delta u(x,y,T) dxdy - (h(0) + \delta h(T)) =$$

$$= \int_0^1 \int_0^1 \varphi(x,y) dxdy - (h(0) + \delta h(T)) = 0. \qquad (10)$$

The differential equation (9) has the following general solution:

$$\int_0^1 \int_0^1 u(x,y,t) dxdy - h(t) = Ce^{\int_0^t p(\tau)d\tau} \ , \qquad (11)$$

where C is an arbitrary constant. Let's require that the solutions (9) satisfy the conditions (10). Then it is easy to obtain

$$C\left(1 + \delta e^{\int_0^t p(\tau)d\tau}\right) = 0 \ .$$

By $\delta \geq 0$, from the last relation we obtain $C = 0$. Substituting $C = 0$ in (11), we conclude that

$$\int_0^1 \int_0^1 u(x,y,t) dxdy - h(t) = 0,$$

i.e. the condition (5) holds. The theorem is proved.

3. The proof of the existence and uniqueness of the classical solution of the inverse boundary value problem

We will search for the first component $u(x,y,t)$ of the solution $\{u(x,y,t), p(t)\}$ of the problem (1)-(4), (6) in the following form:

$$u(x,y,t) = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} u_{k,n}(t) \cos \lambda_k x \sin \gamma_n y \ , \qquad (12)$$

where

$$\lambda_k = \frac{\pi}{2}(2k-1) \ (k = 1,2,...), \quad \gamma_n = \frac{\pi}{2}(2n-1) \ (n = 1,2,...) \, ,$$

$$u_{k,n}(t) = 4 \int_0^1 \int_0^1 u(x,y,t) \cos \lambda_k x \sin \gamma_n y dxdy \ \ (k = 1,2,...; \ n = 1,2,...).$$

Using the method of separation of variables to define the sought coefficients $u_{k,n}(t)$ $(k = 1,2,...; n = 1,2,...)$ of the function $u(x,t)$, from (1), (2) we obtain

$$\left(1 + \mu_{k,n}^2 \alpha(t)\right) u'_{k,n}(t) + \mu_{k,n}^2 \beta(t) u_{k,n}(t) =$$

$$= F_{k,n}(t; u, p) \ (k = 1,2,..., n = 1,2,...; \ 0 \leq t \leq T) \, , \qquad (13)$$

$$u_{k,n}(0) + \delta u_{k,n}(T) = \varphi_{k,n} (k = 1,2,...; n = 1,2,...) \, , \qquad (14)$$

where

$$\mu_{k,n}^2 = \lambda_k^2 + \gamma_n^2 \ (k = 1, 2, ...; n = 1, 2, ...),$$

$$F_{k,n}(t; u, p) = f_{k,n}(t) + p(t)u_{k,n}(t) \ (k = 1, 2, ...; n = 1, 2, ...),$$

$$f_{k,n}(t) = 4\int_0^1\int_0^1 f(x, y, t)\cos\lambda_k x \sin\gamma_n y dxdy \ \ (k = 1, 2, ...; \ n = 1, 2, ...),$$

$$\varphi_{k,n} = 4\int_0^1\int_0^1 \varphi(x, y)\cos\lambda_k x \sin\gamma_n y dxdy \ \ (k = 1, 2, ...; \ n = 1, 2, ...).$$

Solving the problem (13), (14), we find

$$u_{k,n}(t) = \frac{\varphi_{k,n} e^{-\int_0^t \frac{\mu_{k,n}^2\beta(s)ds}{1+\mu_{k,n}^2\alpha(s)}}}{1 + \delta e^{-\int_0^T \frac{\mu_{k,n}^2\beta(s)ds}{1+\mu_{k,n}^2\alpha(s)}}} + \int_0^t \frac{F_{k,n}(\tau; u, p)}{1 + \mu_{k,n}^2\alpha(\tau)} e^{-\int_\tau^t \frac{\mu_{k,n}^2\beta(s)ds}{1+\mu_{k,n}^2\alpha(s)}} d\tau -$$

$$-\frac{\delta e^{-\int_0^T \frac{\mu_{k,n}^2\beta(s)ds}{1+\mu_{k,n}^2\alpha(s)}}}{1 + \delta e^{-\int_0^T \frac{\mu_{k,n}^2\beta(s)ds}{1+\mu_{k,n}^2\alpha(s)}}} \int_0^T \frac{F_{k,n}(\tau; u, p)}{1 + \mu_{k,n}^2\alpha(\tau)} e^{-\int_\tau^t \frac{\mu_{k,n}^2\beta(s)ds}{1+\mu_{k,n}^2\alpha(s)}} d\tau \ \ (k = 1, 2, ...; n = 1, 2, ...). \quad (15)$$

Substituting the expressions $u_{k,n}(t)$ $(k = 1, 2, \ldots; n = 1, 2, ...)$ in (12), we have

$$u(x, y, t) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left\{ \frac{\varphi_{k,n} e^{-\int_0^t \frac{\mu_{k,n}^2\beta(s)ds}{1+\mu_{k,n}^2\alpha(s)}}}{1 + \delta e^{-\int_0^T \frac{\mu_{k,n}^2\beta(s)ds}{1+\mu_{k,n}^2\alpha(s)}}} + \int_0^t \frac{F_{k,n}(\tau; u, p)}{1 + \mu_{k,n}^2\alpha(\tau)} e^{-\int_\tau^t \frac{\mu_{k,n}^2\beta(s)ds}{1+\mu_{k,n}^2\alpha(s)}} d\tau - \right.$$

$$\left. -\frac{\delta e^{-\int_0^T \frac{\mu_{k,n}^2\beta(s)ds}{1+\mu_{k,n}^2\alpha(s)}}}{1 + \delta e^{-\int_0^T \frac{\mu_{k,n}^2\beta(s)ds}{1+\mu_{k,n}^2\alpha(s)}}} \int_0^T \frac{F_{k,n}(\tau; u, p)}{1 + \mu_{k,n}^2\alpha(\tau)} e^{-\int_\tau^t \frac{\mu_{k,n}^2\beta(s)ds}{1+\mu_{k,n}^2\alpha(s)}} d\tau \right\} \cos\lambda_k x \sin\gamma_n y. \quad (16)$$

Now, from (6), by (12), we obtain

$$h'(t) + \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}(-1)^{k+1}\left(\frac{\gamma_n}{\lambda_k} - \frac{\lambda_k}{\gamma_n}\right)\left(\alpha(t)u'_{k,n}(t) + \beta(t)u_{k,n}(t)\right) =$$

$$= p(t)h(t) + \int_0^1\int_0^1 f(x, y, t)dxdy \qquad (0 \le t \le T). \quad (17)$$

Further, from (13) we have

$$\mu_{k,n}^2\left(\alpha(t)u'_{k,n}(t) + \beta(t)u_{k,n}(t)\right) = F_{k,n}(t; u, p) \ \ - u'_{k,n}(t) = \frac{\mu_{k,n}^2\beta(t)}{1 + \mu_{k,n}^2\alpha(t)} u_{k,n}(t) +$$

$$+\frac{\mu_{k,n}^2\alpha(t)}{1+\mu_{k,n}^2\alpha(t)}F_{k,n}(t;u,p)(k=1,2,...,n=1,2,...;\ 0\le t\ \le T)\,,$$

or

$$\alpha(t)u'_{k,n}(t)+\beta(t)u_{k,n}(t)\ =\frac{\beta(t)}{1+\mu_{k,n}^2\alpha(t)}u_{k,n}(t)+$$

$$+\frac{\alpha(t)}{1+\mu_{k,n}^2\alpha(t)}F_{k,n}(t;u,p)(k=1,2,...,n=1,2,...;\ 0\le t\ \le T). \qquad (18)$$

From (17), taking into account (18), we obtain

$$p(t)=[h(t)]^{-1}\left\{h'(t)-\int_0^1\int_0^1 f(x,y,t)dxdy+\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}(-1)^{k+1}\left(\frac{\gamma_n}{\lambda_k}-\frac{\lambda_k}{\gamma_n}\right)\times\right.$$

$$\left.\times\left(\frac{\beta(t)}{1+\mu_{k,n}^2\alpha(t)}u_{k,n}(t)+\ \frac{\alpha(t)}{1+\mu_{k,n}^2\alpha(t)}F_{k,n}(t;u,p)\right)\right\} \qquad (19)$$

To obtain the equation for the second component $p(t)$ of the solution $\{u(x,t),p(t)\}$ of the problem (1)-(4), (5), we substitute the expression (15) in (19) to get

$$p(t)=[h(t)]^{-1}\left\{h'(t)-\int_0^1\int_0^1 f(x,y,t)dxdy+\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}(-1)^{k+1}\left(\frac{\gamma_n}{\lambda_k}-\frac{\lambda_k}{\gamma_n}\right)\times\right.$$

$$\times\left(\frac{\beta(t)}{1+\mu_{k,n}^2\alpha(t)}\left[\frac{\varphi_{k,n}e^{-\int_0^t\frac{\mu_{k,n}^2\beta(s)ds}{1+\mu_{k,n}^2\alpha(s)}}}{1+\delta e^{-\int_0^T\frac{\mu_{k,n}^2\beta(s)ds}{1+\mu_{k,n}^2\alpha(s)}}}+\int_0^t\frac{F_{k,n}(\tau;u,p)}{1+\mu_{k,n}^2\alpha(\tau)}e^{-\int_\tau^t\frac{\mu_{k,n}^2\beta(s)ds}{1+\mu_{k,n}^2\alpha(s)}}d\tau-\right.\right.$$

$$\left.-\frac{\delta e^{-\int_0^T\frac{\mu_{k,n}^2\beta(s)ds}{1+\mu_{k,n}^2\alpha(s)}}}{1+\delta e^{-\int_0^T\frac{\mu_{k,n}^2\beta(s)ds}{1+\mu_{k,n}^2\alpha(s)}}}\int_0^T\frac{F_{k,n}(\tau;u,p)}{1+\mu_{k,n}^2\alpha(\tau)}e^{-\int_\tau^t\frac{\mu_{k,n}^2\beta(s)ds}{1+\mu_{k,n}^2\alpha(s)}}d\tau\right]+$$

$$\left.\left.+\frac{\alpha(t)}{1+\mu_{k,n}^2\alpha(t)}F_{k,n}(t;u,p)\right)\right\}. \qquad (20)$$

Thus, the solution of the problem (1)-(4), (6) is reduced to the solution of the system (16), (20) with respect to the unknown functions $u(x,y,t)$ and $p(t)$.

To treat the uniqueness of the solution of (1)-(4), (6), we will significantly use the following lemma.

Lemma 1. *If $\{u(x,y,t),p(t)\}$ is any solution of the problem (1)-(4), (6), then the functions*

$$u_{k,n}(t)=4\int_0^1\int_0^1 u(x,y,t)\cos\lambda_k x\sin\gamma_n y dxdy\ \ (k=1,2,...;\ n=1,2,...)$$

satisfy the system (15) on $[0,T]$.

Proof. Let $\{u(x,y,t),p(t)\}$ be any solution of the problem (1)-(4), (6). Then, multiplying both sides of the equation (1) by the function $4\cos\lambda_k x\sin\gamma_n y$ $(k=1,2,...;\ n=1,2,...)$, integrating the obtained equality with respect to x and y from 0 to 1 and using the relations

$$4\int_0^1\int_0^1 u_t(x,y,t)\cos\lambda_k x\sin\gamma_n ydxdy =$$

$$=\frac{d}{dt}\left(4\int_0^1\int_0^1 u(x,y,t)\cos\lambda_k x\sin\gamma_n ydxdy\right)=u'_{k,n}(t)(k=1,2,...;\ n=1,2,...),$$

$$4\int_0^1\int_0^1 u_{xx}(x,y,t)\cos\lambda_k x\sin\gamma_n ydxdy =$$

$$=-\lambda_k^2\left(4\int_0^1\int_0^1 u(x,y,t)\cos\lambda_k x\sin\gamma_n ydxdy\right)=-\lambda_k^2 u_{k,n}(t)\,(k=1,2,...;\ n=1,2,...),$$

$$4\int_0^1\int_0^1 u_{yy}(x,y,t)\cos\lambda_k x\sin\gamma_n ydxdy =$$

$$-\gamma_n^2\left(4\int_0^1\int_0^1 u(x,y,t)\cos\lambda_k x\sin\gamma_n ydxdy\right)=-\gamma_k^2 u_{k,n}(t)\,(k=1,2,...;\ n=1,2,...),$$

$$4\int_0^1\int_0^1 u_{txx}(x,y,t)\cos\lambda_k x\sin\gamma_n ydxdy = -\lambda_k^2 u'_{k,n}(t)\,(k=1,2,...;\ n=1,2,...),$$

$$4\int_0^1\int_0^1 u_{tyy}(x,y,t)\cos\lambda_k x\sin\gamma_n ydxdy = -\gamma_n^2 u'_{k,n}(t)\,(k=1,2,...;\ n=1,2,...),$$

we get the validity of the equation (13).

Similarly, from (2) it follows that the condition (14) holds.

Thus, $u_{k,n}(t)\,(k=1,2,...;\ n=1,2,...)$ are the solutions of the problem (13), (14). Hence it directly follows that the functions $u_{k,n}(t)\,(k=1,2,...;\ n=1,2,...)$ satisfy the system (15) on $[0,T]$. The lemma is proved.

It is clear that if $u_{k,n}(t)=4\int_0^1\int_0^1 u(x,y,t)\cos\lambda_k x\sin\gamma_n ydxdy$ $(k=1,2,...;\ n=1,2,...)$ are the solutions of the system (15), then the pair $\{u(x,y,t),p(t)\}$ of the functions $u(x,y,t)=\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}u_{k,n}(t)\cos\lambda_k x\sin\gamma_n y$ and $p(t)$ is a solution of the system (16), (20).

Lemma 1 has the following corollary.

Corollary 1. *Let the system* (16), (20) *have a unique solution. Then the problem* (1)-(4), (6) *cannot have more than one solution, i.e. if the problem* (1)-(4),(6) *has a solution, then it is unique.*

1. Denote by $B_{2,T}^3$ [9] the totality of all functions $u(x,y,t)$ of the form

$$u(x,y,t)=\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}u_{k,n}(t)\cos\lambda_k x\sin\gamma_n y$$

in D_T, where each of the functions $u_{k,n}(t)\,(k=1,2,...;\ n=1,.2,..)$ is continuously differentiable on $[0,T]$ and

$$\left\{\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\mu_{k,n}^{3}\left\|u_{k,n}(t)\right\|_{C[0,T]}\right)^{2}\right\}^{\frac{1}{2}}<+\infty.$$

Define the norm on this set as follows:

$$\|u(x,y,t)\|_{B_{2,T}^{3}}=\left\{\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\mu_{k,n}^{3}\left\|u_{k,n}(t)\right\|_{C[0,T]}\right)^{2}\right\}^{\frac{1}{2}}.$$

2. Denote by E_T^3 the space consisting of topological product

$$B_{2,T}^{3}\times C[0,T]\,.$$

The norm of the element $z=\{u,p\}$ is defined by the formula

$$\|z\|_{E_T^3}=\|u(x,y,t)\|_{B_{2,T}^{3}}+\|p(t)\|_{C[0,T]}\,.$$

It is known that $B_{2,T}^{3}$ and $E_T^{3;}$ are Banach spaces.

Now let's consider in the space E_T^3 the operator

$$\Phi(u,p)=\{\Phi_1(u,p),\Phi_2(u,p)\},$$

where

$$\Phi_1(u,p)=\tilde{u}(x,y,t)\equiv\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\tilde{u}_{k,n}(t)\cos\lambda_k x\sin\gamma_n y\ \ ,$$

$$\Phi_2(u,p)=\tilde{p}(t),,$$

and $\tilde{u}_{k,n}(t)$ $(k=1,2,...;\ n=1,2,...)$ and $\tilde{p}(t)$ are equal to the right-hand sides of (15) and (20), respectively.

It is not difficult to see that

$$1+\mu_{k,n}^{2}\alpha(t)>\mu_{k,n}^{2}\alpha(t),\quad \frac{\mu_{k,n}^{2}\beta(t)}{1+\mu_{k,n}^{2}\alpha(t)}<\frac{\beta(t)}{\alpha(t)},\quad \frac{\mu_{k,n}^{2}\alpha(t)}{1+\mu_{k,n}^{2}\alpha(t)}<1,$$

$$1+\delta e^{-\int_0^T\frac{\mu_{k,n}^{2}\beta(s)ds}{1+\mu_{k,n}^{2}\alpha(s)}}\geq 1\,,\ \mu_{k,n}^{3}\leq(\lambda_k^2+\gamma_k^2)(\lambda_k+\gamma_n)=\lambda_k^3+\lambda_k^2\gamma_n+\gamma_k^2\lambda_k+\gamma_k^3.$$

From these relations we obtain

$$\left\{\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\mu_{k,n}^{3}\left\|\tilde{u}_{k,n}(t)\right\|_{C[0,T]}\right)^{2}\right\}^{\frac{1}{2}}\leq 3\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\lambda_k^3\ \ |\varphi_{k,n}|\right)^{2}\right)^{\frac{1}{2}}+$$

$$+3\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\lambda_k^2\gamma_n\ |\varphi_{k,n}|\right)^2\right)^{\frac{1}{2}}+3\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\lambda_k\gamma_n^2\ |\varphi_{k,n}|\right)^2\right)^{\frac{1}{2}}+$$

$$+3\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\gamma_n^3\ |\varphi_{k,n}|\right)^2\right)^{\frac{1}{2}}+3(1+\delta)\left\|\frac{1}{\alpha(t)}\right\|_{C[0,T]}\times$$

$$\times\left[\sqrt{T}\left(\left(\int_0^T\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\lambda_k\left|f_{k,n}(\tau)\right|\right)^2d\tau\right)^{\frac{1}{2}}+\left(\int_0^T\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\gamma_n\left|f_{k,n}(\tau)\right|\right)^2d\tau\right)^{\frac{1}{2}}\right)+\right.$$

$$\left.+T\left\|p(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\sum_{k=1}^{\infty}\left(\mu_{k,n}^3\ \left\|u_{k,n}(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}\right],\qquad(21)$$

$$\left\|\tilde{p}(t)\right\|_{C[0,T]}\leq\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left\{\left\|h'(t)-\int_0^1\int_0^1f(x,y,t)dxdy\right\|_{C[0,T]}+\right.$$

$$+\left(\sum_{k=1}^{\infty}\sum_{k=1}^{\infty}\mu_k^{-2}\right)^{\frac{1}{2}}\left[\left\|\frac{\beta(t)}{\alpha(t)}\right\|_{C[0,T]}\left(\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\lambda_k\left|\varphi_{k,n}\right|\right)^2\right)^{\frac{1}{2}}+\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\gamma_n\left|\varphi_{k,n}\right|\right)^2\right)^{\frac{1}{2}}+\right.\right.$$

$$+\left\|\frac{1}{\alpha(t)}\right\|_{C[0,T]}\left(\sqrt{T}\left(\int_0^T\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left|f_{k,n}(\tau)\right|^2d\tau\right)^{\frac{1}{2}}+\right.$$

$$\left.\left.+T\left\|p(t)\right\|_{C[0,T]}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\mu_k^3\ \left\|u_k(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}\right)\right]+\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\lambda_k\left|f_{k,n}(t)\right|\right)^2d\tau\right)^{\frac{1}{2}}+$$

$$\left.+\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\gamma_n\left|f_{k,n}(t)\right|\right)^2d\tau\right)^{\frac{1}{2}}+\left\|p(t)\right\|_{C[0,T]}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\mu_k^3\ \left\|u_k(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}\right\}.\qquad(22)$$

Assume that the data of the problem (1)-(4), (6) satisfy the following conditions:

$$1.\varphi(x,y),\varphi_x(x,y),\varphi_{xx}(x,y),\varphi_y(x,y),\varphi_{xy}(x,y),\varphi_{yy}(x,y)\in C(\bar{Q}_{xy}),$$

$$\varphi_{xxy}(x,y),\varphi_{xyy}(x,y),\varphi_{xxx}(x,y),\varphi_{yyy}(x,y)\in L_2(Q_{xy}),$$

$$\varphi_x(0,y)=\varphi(1,y)=\varphi_{xx}(1,y)=0\,(0\leq y\leq 1),$$

$$\varphi(x,0)=\varphi_y(x,1)=\varphi_{yy}(x,0)=0\,(0\leq x\ \leq 1).$$

$$2.f(x,y,t)\in C(D_T),\ f_x(x,y,t),f_y(x,y,t)\in L_2(D_T),$$

$$f\ (1,y,t)=f\ (x,0,t)=0\,(0\leq x,y\leq 1,\ \ 0\leq t\leq T).$$

$$3.\delta \geq 0, \quad 0 < \alpha(t) \in C[0,T], 0 < \beta(t) \in C[0,T], \quad h(t) \in C^1[0,T] \ ,$$
$$h(t) \neq 0 \ \ (0 \leq t \leq T).$$

Then from (21)- (22) we obtain

$$\|u(x,y,t)\|_{B^3_{2,T}} = \left\{ \sum_{k=1}^{\infty} \left(\lambda_k^3 \|u_{k,0}(t)\|_{C[0,T]} \right)^2 \right\}^{\frac{1}{2}} + \left\{ \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \left(\mu_k^3 \|u_{k,n}(t)\|_{C[0,T]} \right)^2 \ \right\}^{\frac{1}{2}} \leq$$
$$\leq A_1(T) + B_1(T) \|p(t)\|_{C[0,T]} \|u(x,y,t)\|_{B^3_{2,T}} \ , \tag{23}$$
$$\|\tilde{p}(t)\|_{C[0,T]} \leq A_2(T) + B_2(T) \|p(t)\|_{C[0,T]} \|u(x,y,t)\|_{B^3_{2,T}} \ , \tag{24}$$

where

$$A_1(T) = 5 \|\varphi_{xxx}(x,y)\|_{L_2(Q_{xy})} + 3 \|\varphi_{xyy}(x,y)\|_{L_2(Q_{xy})} + 3 \|\varphi_{xxy}(x,y)\|_{L_2(Q_{xy})} +$$
$$+3 \|\varphi_{xxx}(x,y)\|_{L_2(Q_{xy})} + (1+\delta) \left\| \frac{1}{\alpha(t)} \right\|_{C[0,T]} \sqrt{T} \left(5 \|f_x(x,y,t)\|_{L_2(D_T)} + 3 \|f_y(x,y,t)\|_{L_2(D_T)} \right),$$
$$B_1(T) = 5(1+\delta) \left\| \frac{1}{\alpha(t)} \right\|_{C[0,T]} T,$$
$$A_2(T) = \left\| [h(t)]^{-1} \right\|_{C[0,T]} \left\{ \left\| h'(t) - \int_0^1 \int_0^1 f(x,y,t) dx dy \right\|_{C[0,T]} + \right.$$
$$+ \left(\sum_{k=1}^{\infty} \sum_{k=1}^{\infty} \mu_k^{-2} \right)^{\frac{1}{2}} \left[\left\| \frac{\beta(t)}{\alpha(t)} \right\|_{C[0,T]} \left(\|\varphi_x(x,y)\|_{L_2(Q_{xy})} + \|\varphi_y(x,y)\|_{L_2(Q_{xy})} \right. \right.$$
$$\left. + \left\| \frac{1}{\alpha(t)} \right\|_{C[0,T]} \sqrt{T} \|f(x,y,t)\|_{L_2(D_T)} \right) + \left\| \|f_x(x,y,t)\|_{C[0,T]} \right\|_{L_2(Q_{xy})} +$$
$$\left. \left. + \left\| \|f_y(x,y,t)\|_{C[0,T]} \right\|_{L_2(0,1)} \right] \right\},$$
$$B_2(T) = \left\| [h(t)]^{-1} \right\|_{C[0,T]} \left(\sum_{k=1}^{\infty} \sum_{k=1}^{\infty} \mu_k^{-2} \right)^{\frac{1}{2}} \times$$
$$\times \left[\left\| \frac{1}{\alpha(t)} \right\|_{C[0,T]} \left\| \frac{\beta(t)}{\alpha(t)} \right\|_{C[0,T]} T + 1 \right].$$

From the inequalities (23)-(24) it follows

$$\|\tilde{u}(x,y,t)\|_{B^3_{2,T}} + \|\tilde{p}(t)\|_{C[0,T]} \leq$$

$$\leq A(T) + B(T)\left\|p(t)\right\|_{C[0,T]}\left\|u(x,t)\right\|_{B^3_{2,T}}\,, \tag{25}$$

where

$$A(T) = \sum_{i=1}^{2} A_i(T), \;\; B(T) = \sum_{i=1}^{2} B_i(T), \;.$$

So we can prove the following theorem.

Theorem 2. *Let the conditions 1-4 be satisfied and*

$$(A(T) + 2)^2 B(T) < 1. \tag{26}$$

Then the problem (1)-(4), (6) *has a unique solution in the ball* $K = K_R(\|z\|_{E^3_T} \leq R = A(T) + 2)$ *of the space* E^3_T.

Proof. Consider in the space E^3_T the equation

$$z = \$z, \tag{27}$$

where $z = \{u, p\}$, and the components $\Phi_i(u,p)(i = 1,2)$ of the operator $\$(u,p)$ are defined by the right-hand sides of the equations (16), (20), respectively. Consider the operator $\$(u,p)$ in the ball $K = K_R(\|z\|_{E^3_T} \leq R = A(T) + 2)$ of E^3_T.

Similar to (25), we obtain the following estimates for every $z, z_1, z_2 \in K_R$:

$$\|\Phi z\|_{E^3_T} \leq A(T) + B(T)\left\|p(t)\right\|_{C[0,T]}\left\|u(x,y,t)\right\|_{B^3_{2,T}}\,, \tag{28}$$

$$\|\Phi z_1 - \Phi z_2\|_{E^2_T} \leq B(T)R\left(\|p_1(t) - p_2(t)\|_{C[0,T]} + \|u_1(x,y,t) - u_2(x,y,t)\|_{B^3_{2,T}}\right). \tag{29}$$

Then from the estimates (28) and (29), by (26), it follows that the operator $ acts in the ball $K = K_R$ and is a contraction operator. Therefore, the operator $ has a unique fixed point $\{u, p\}$ in the ball $K = K_R$, which is a unique solution of the equation (27), i.e. a unique solution of the system (16), (20) in the ball $K = K_R$.

As an element of the space $B^3_{2,T}$, the function $u(x,y,t)$ is continuous and has continuous derivatives $u_x(x,y,t)$, $u_{xx}(x,y,t)$, $u_y(x,y,t)$, $u_{xy}(x,y,t)$, $u_{yy}(x,y,t)$ in D_T.

Now it is not difficult to see from (13) that

$$\left\{\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\mu_{k,n}\left\|u'_{k,n}(t)\right\|_{C[0,T]}\right)^2\right\}^{\frac{1}{2}} \leq \sqrt{2}\left\|\frac{1}{\alpha(t)}\right\|_{C[0,T]}\Big[\|u(x,y,t)\|_{B^3_{2,T}} +$$

$$+\left\|\|f(x,y,t) + p(t)u(x,y,t)\|_{C[0,T]}\right\|_{L_2(Q_{xy})}\Big].$$

Hence, it is clear that $u_t(x,y,t)$, $u_{txx}(x,y,t)$, $u_{tyy}(x,y,t)$ are continuous in D_T.

It is not difficult to verify that the equation (1) and the conditions (2)-(4), (6) are satisfied in the usual sense. Thus, the solution of the problem (1)–(4), (6) is a pair of functions $\{u(x,t), p(t)\}$. By the corollary of Lemma 1, this solution is unique in the ball $K = K_R$. The theorem is proved.

Using Theorems 1 and 2, we obtain the unique solvability of the problem (1)–(5).

Theorem 3. *Let all the conditions of Theorem 2 be satisfied and the coherence conditions*

$$\int_0^1 \int_0^1 \varphi(x,y)dxdy = h(0) + \delta h(T)$$

hold. Then the problem (1)-(5) *has a unique classical solution in the ball* $K = K_R(\|z\|_{E_T^3} \leq R = A(T)+2)$ *of the space* E_T^3.

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