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Asymptotic Behavior of the Distribution Function of the Riesz Transform

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Abstract. It is known that the Riesz transform of a Lebesgue integrable function is not Lebesgue integrable. In the present paper, we study the asymptotic behavior of the distribution function of the Riesz transform of a Lebesgue integrable function as $\lambda \to +\infty$ and as $\lambda \to 0+$.

Key Words and Phrases: Riesz transform, distribution function, asymptotic behavior, summability.

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1. Introduction

The j-th Riesz transform of a function $f \in L_p(R^d)$, $1 \leq p < +\infty$ is defined as the following singular integral:

$$R_j(f)(x) = \gamma_{(d)} \lim_{\varepsilon \to 0} \int_{\{y \in R^d : |x-y| > \varepsilon\}} \frac{x_j - y_j}{|x-y|^{d+1}} f(y) dy, \; j = \overline{1, d},$$

where $C_{(d)} = \frac{\Gamma((d+1)/2)}{\pi^{(d+1)/2}}$.

The Riesz transform is one of the important operators in harmonic analysis. It has been shown in [2, 6, 10, 11] that this transform plays an essential role in applications to the theory of elliptic partial differential equations.

From the theory of singular integrals (see [10]) it is known that the Riesz transform is a bounded operator in the space $L_p(R^d)$, $p > 1$, that is, if $f \in L_p(R^d)$, then $R_j(f) \in L_p(R^d)$ and the inequality

$$\|R_j f\|_{L_p} \leq C^{(p)} \|f\|_{L_p} \tag{1}$$

holds. In the case $f \in L_1(R^d)$ only the weak inequality holds:

$$m\{x \in R^d : \; |(R_j f)(x)| > \lambda\} \leq \frac{C_1}{\lambda} \|f\|_{L_1}, \tag{2}$$

where m stands for the Lebesgue measure, $C^{(p)}$, C_1 are constants independent of f.

In [3, 4, 5, 7, 8, 9, 10] the boundedness of the operator R_j in other function spaces (in the spaces of Sobolev, Besov, Campanato, Morrey, etc.) was studied.

Note that the Riesz transform of a function $f \in L_1(R^d)$ is not Lebesgue integrable. In work [1] using the notion of A-integrating functions, the analogue of Riss equality is proved for the class of functions $f \in L_1(R^d)$. In this paper we study the asymptotic behavior of the distribution function of the Riesz transform of a Lebesgue integrable function as $\lambda \to +\infty$ and as $\lambda \to 0+$ and find a necessary condition and a sufficient condition for the summability of the Riesz transform.

2. Asymptotic behavior of the distribution function of the Riesz transform as $\lambda \to +\infty$

In this section we studying the asymptotic behavior of the distribution function of the Riesz transform as $\lambda \to +\infty$.

Theorem 1. Let $f \in L_1(R^d)$. Then the equation

$$\lim_{\lambda \to +\infty} \lambda m\{x \in R^d : \ |(R_j f)(x)| > \lambda\} = 0 \tag{3}$$

holds.

Proof: Since $f \in L_1(R^d)$, then for every $\varepsilon > 0$ there exists $n \in N$ and $r > 0$ such that

$$\|f - [f]_r^n\|_{L_1} \leq \frac{\varepsilon}{4C_1}, \tag{4}$$

where $[f]_r^n(x) = [f]^n \chi(B(0;r))(x)$, $[f(x)]^n = f(x)$ for $|f(x)| \leq n$, $[f(x)]^n = 0$ for $|f(x)| > n$, $\chi(B(0;r))(x)$ - characteristic function of the ball $B(0;r) = \{x \in R^d : \ |x| < r\}$. It follows from (2) and (4) that for every $\lambda > 0$ the inequality

$$m\{x \in R^d : \ |R_j(f - [f]_r^n)(x)| > \frac{\lambda}{2}\} \leq \frac{2C_1}{\lambda}\|f - [f]_r^n\|_{L_1} \leq \frac{\varepsilon}{2\lambda} \tag{5}$$

holds. Since the function $[f]_r^n(x)$ is bounded, then we get that $[f]_r^n \in L_p(R^d)$ for each $p \geq 1$. It follows that $R_j[f]_r^n \in L_p(R^d)$ for each $p > 1$. Denote

$$F_1(x) = R_j([f]_r^n)(x) \cdot \chi(B(0;2r)), F_2(x) = R_j([f]_r^n)(x) \cdot \chi(R^d \backslash B(0;2r)).$$

Then

$$R_j([f]_r^n)(x) = F_1(x) + F_2(x),$$

The function $F_1(x)$ is concentrated on the closed ball $\overline{B(0;2r)}$, and the function $F_2(x)$ is concentrated on the set $R^d \backslash B(0;2r)$. For every $p > 1$ from the inclusion $R_j([f]_r^n) \in L_p(R^d)$ it follows that $F_1(x) \in L_p(R^d)$. Since the function $F_1(x)$ is concentrated on the bounded set, then we have that $F_1(x) \in L_1(R^d)$. Then for sufficiently large values of $\lambda > 0$

$$\frac{\lambda}{4} m\{x \in R^d : \ |F_1(x)| > \frac{\lambda}{4}\} \leq \int_{\{x \in R^d : \, |F_1(x)| > \lambda/4\}} |F_1(x)| dx < \frac{\varepsilon}{8}. \tag{6}$$

On the other hand, for any $x \in R^d \backslash B(0;2r)$ we have

$$|R_j([f]_r^n)(x)| \le \gamma_{(d)} \int_{B(0;r)} \frac{|x_j - y_j|}{|x-y|^{d+1}} \cdot |[f]_r^n(y)| dy$$

$$\le \frac{\gamma_{(d)}}{r^d} \int_{B(0;r)} |[f]_r^n(y)| dy = \frac{\gamma_{(d)}}{r^d} \|[f]_r^n\|_{L_1} \le \frac{\gamma_{(d)}}{r^d} \|f\|_{L_1}.$$

This shows that the function $F_2(x)$ is bounded. Then it follows from (6) that for sufficiently large values of $\lambda > 0$

$$m\{x \in R^d: \ |R_j([f]_r^n)(x)| > \frac{\lambda}{2}\} \le m\{x \in R^d: \ |F_1(x)| > \frac{\lambda}{4}\} < \frac{\varepsilon}{2\lambda}. \tag{7}$$

It follows from (5) and (7) that for sufficiently large values of $\lambda > 0$

$$m\{x \in R^d: \ |(R_j f)(x)| > \lambda\}$$

$$\le m\{x \in R^d : |R_j([f]_r^n)(x)| > \frac{\lambda}{2}\} + m\{x \in R^d : |R_j(f - [f]_r^n)(x)| > \frac{\lambda}{2}\} < \frac{\varepsilon}{\lambda}.$$

This shows that the equation (3) holds. Theorem 1 is proved.

3. Asymptotic behavior of the distribution function of the Riesz transform as $\lambda \to 0+$

In this section we studying the asymptotic behavior of the distribution function of the Riesz transform as $\lambda \to 0+$.

Theorem 2. Let $f \in L_1(R^d)$. Then the equation

$$\lim_{\lambda \to 0+} \lambda m\{x \in R^d : |(R_j f)(x)| > \lambda\} = \gamma_{(d)} \theta_{(d)} \left| \int_{R^d} f(x) dx \right| \tag{8}$$

holds, where $\theta_{(d)} = \frac{2^d}{d(d-1)!!} (\frac{\pi}{2})^{[\frac{d-1}{2}]}$ and $[\frac{d-1}{2}]$ - integer part of a number $\frac{d-1}{2}$.

At first we prove the auxiliary lemma.

Lemma 1. If $f \in L_1(R^d)$ and $\int_{R^d} f(x)dx = 0$, then the equation

$$m\{x \in R^d : |(R_j f)(x)| > \lambda\} = o(1/\lambda), \lambda \to 0+ \tag{9}$$

holds.

Proof of Lemma 1. At first assume that the function f is concentrated on some ball $B(0;r) \subset R^d$. In this case, for values of $|x| > 2r$ from the equality

$$(R_j f)(x) = \gamma_{(d)} \int_{B(0;r)} \frac{x_j - y_j}{|x-y|^{d+1}} f(y) dy$$

$$= \gamma_{(d)} \int_{B(0;r)} \frac{x_j - y_j}{|x-y|^{d+1}} f(y) dy - \gamma_{(d)} \int_{B(0;r)} \frac{x_j}{|x|^{d+1}} f(y) dy$$

$$= \gamma_{(d)} \int_{B(0;r)} \left[\frac{x_j - y_j}{|x-y|^{d+1}} - \frac{x_j}{|x|^{d+1}} \right] f(y)dy,$$

we get that

$$|(R_j f)(x)| \le \gamma_{(d)} \int_{B(0;r)} \left[|x_j| \frac{||x|^{d+1} - |x-y|^{d+1}|}{|x-y|^{d+1} \cdot |x|^{d+1}} + \frac{|y_j|}{|x-y|^{d+1}} \right] |f(y)|dy$$

$$\le \gamma_{(d)} \int_{B(0;r)} \left[|x_j| \cdot |y| \cdot \sum_{k=1}^{d+1} \frac{1}{|x|^k \cdot |x-y|^{d+2-k}} + \frac{|y_j|}{|x-y|^{d+1}} \right] |f(y)|dy$$

$$\le \frac{c_0}{|x|^{d+1}},$$

where $c_0 = \gamma_{(d)} r(d+2)2^{d+1}\|f\|_{L_1}$. Then it follows that

$$m\{x \in R^d : |(R_j f)(x)| > \lambda\} \le m\{x \in R^d : |x| \le 2r\} + m\{x \in R^d : \frac{c_0}{|x|^{d+1}} > \lambda\}$$

$$= \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2}+1)} \cdot (2r)^d + m\left\{x \in R^d : |x| < \sqrt[d+1]{\frac{c_0}{\lambda}}\right\} = \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2}+1)} \cdot \left[(2r)^d + \left(\frac{c_0}{\lambda}\right)^{\frac{d}{d+1}}\right],$$

whence it follows asymptotic equality (9).

Now let's consider the general case. From the condition $\int_{R^d} f(x)dx = 0$ it follows that for any $\varepsilon > 0$ there exist the functions f_1 and f_2 satisfying the condition: $f = f_1 + f_2$, the function f_1 is concentrated on some ball $B(0;r) \subset R^d$ and $\int_{R^d} f_1(x)dx = 0$, the function f_2 satisfies the inequality $\|f_2\|_{L_1} < \frac{\varepsilon}{4C_1}$, where C_1 is a constant in estimation (2). Since the function f_1 is concentrated on the ball $B(0;r) \subset R^d$ and $\int_{R^d} f_1(x)dx = 0$, then for the function f_1 equality (9) is satisfied, and therefore there exists $\lambda(\varepsilon) > 0$ such that for $0 < \lambda < \lambda(\varepsilon)$ the inequality

$$\lambda m\{x \in R^d : |(R_j f_1)(x)| > \frac{\lambda}{2}\} < \frac{\varepsilon}{2} \tag{10}$$

holds. On the other hand, from the inequality (2) it follows that

$$\lambda m\{x \in R^d : |(R_j f_2)(x)| > \frac{\lambda}{2}\} \le 2C_1\|f_2\|_{L_1} < \frac{\varepsilon}{2} \tag{11}$$

for any $\lambda > 0$. From inequalities (10), (11) we get

$$\lambda m\{x \in R^d : |(R_j f)(x)| > \lambda\}$$

$$\le \lambda m\{x \in R^d : |(R_j f_1)(x)| > \frac{\lambda}{2}\} + \lambda m\{x \in R^d : |(R_j f_2)(x)| > \frac{\lambda}{2}\} < \varepsilon$$

for $0 < \lambda < \lambda(\varepsilon)$. This shows that equality (9) was satisfied for all functions $f \in L_1(R^d)$, satisfying the condition $\int_{R^d} f(x)dx = 0$. This completes the Proof of the Lemma 1.

Proof of Theorem 2. In the case $\int_{R^d} f(x)dx = 0$ the assertion of the Theorem follows from Lemma 1. Let's consider the case $\int_{R^d} f(x)dx = \alpha \neq 0$. Denote by $f_1(x) = \alpha\eta_{(d)}\chi(B(0;\,1))(x)$, where $\eta_{(d)} = \frac{\Gamma(\frac{d}{2}+1)}{\pi^{\frac{d}{2}}}$, $\chi(B(0;\,1))$ is a characteristic function on the unit circle $B(0;\,1)$ and $f_2(x) = f(x) - f_1(x)$. Then $\int_{R^d} f_2(x)dx = 0$, and from Lemma 1

$$m\{x \in R^d:\ |(R_j f_2)(x)| > \lambda\} = o\left(\frac{1}{\lambda}\right), \lambda \to 0+. \tag{12}$$

Since for any $x \in \{x \in R^d : x_j > 2\}$

$$|(R_j f_1)(x)| = \eta_{(d)}\gamma_{(d)}|\alpha| \left| \int_{B(0;\,1)} \frac{x_j - y_j}{|x-y|^{d+1}} dy \right|$$

$$\leq \eta_{(d)}\gamma_{(d)}|\alpha| \int_{B(0;\,1)} \frac{|x_j|+1}{||x|-1|^{d+1}} dy = \gamma_{(d)}|\alpha| \frac{|x_j|+1}{||x|-1|^{d+1}}$$

$$\leq \gamma_{(d)}|\alpha| \frac{|x_j|}{||x|-1|^{d+1}} + \gamma_{(d)}|\alpha| \frac{2^d}{|x|^{d+1}},$$

$$|(R_j f_1)(x)| \geq \gamma_{(d)}|\alpha| \frac{|x_j|}{||x|+1|^{d+1}} - \gamma_{(d)}|\alpha| \frac{2^d}{|x|^{d+1}},$$

and for any $\lambda > 0$

$$m\{x \in R^d:\ \frac{|x_j|}{|x|^{d+1}} > \lambda\} = \int_{\{x \in R^d:\ |x_j| > \lambda |x|^{d+1}\}} dx = \frac{\theta_{(d)}}{\lambda},$$

$$m\{x \in R^d:\ \frac{1}{|x|^{d+1}} > \lambda\} = m\left\{x \in R^d:\ |x| < \left(\frac{1}{\lambda}\right)^{\frac{1}{d+1}}\right\} = \frac{1}{\eta_{(d)}}\left(\frac{1}{\lambda}\right)^{\frac{d}{d+1}},$$

then we get that

$$\limsup_{\lambda \to 0+} \lambda m\{x \in R^d:\ |(R_j f_1)(x)| > \lambda\} \leq \gamma_{(d)}\theta_d|\alpha|, \tag{13}$$

$$\liminf_{\lambda \to 0+} \lambda m\{x \in R^d:\ |(R_j f_1)(x)| > \lambda\} \geq \gamma_{(d)}\theta_d|\alpha|. \tag{14}$$

It follows from (13), (14) that

$$\lim_{\lambda \to 0+} \lambda m\{x \in R^d : |(R_j f_1)(x)| > \lambda\} = \gamma_{(d)}\theta_d|\alpha|. \tag{15}$$

For any $0 < \varepsilon < 1$, by the inclusions

$$\{x \in R^d:\ |(R_j f_1)(x)| > (1+\varepsilon)\lambda\} \backslash \{x \in R^d:\ |(R_j f_2)(x)| > \varepsilon\lambda\} \subset$$

$$\subset \{x \in R^d:\ |(R_j f)(x)| > \lambda\} \subset$$

$$\subset \{x \in R^d : |(R_j f_2)(x)| > \varepsilon\lambda\} \cup \{x \in R^d : |(R_j f_1)(x)| > (1-\varepsilon)\lambda\}$$

and equalities (12), (15) we have

$$\limsup_{\lambda\to 0+} \lambda m\{x \in R^d : |(R_j f)(x)| > \lambda\} \le \frac{\gamma_{(d)}\theta_d|\alpha|}{1-\varepsilon},$$

$$\liminf_{\lambda\to 0+} \lambda m\{x \in R^d : |(R_j f)(x)| > \lambda\} \ge \frac{\gamma_{(d)}\theta_d|\alpha|}{1+\varepsilon},$$

This implies the equation (8) and completes the proof of the Theorem 2.

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