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# Estimations of Functions from Lizorkin-Triebel Spaces Reduced by Corresponding Polynomials

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**Abstract.** In this paper the integral inequalities as estimation of the norms of functions reduced by polynomials, are proved.

**Key Words and Phrases**: integral representation, the space Lizorkin-Triebel, flexible $\varphi$-horn, generalized derivatives.

**2010 Mathematics Subject Classifications**: Primary 26A33, 42B35, 46E30

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## 1. Introduction

In this paper, with help method of integral representation, we estimate the norms of functions from the Lizorkin-Triebel spaces $F^l_{p,\theta}(G)$, where $l \in (0,\infty)^n$ , $p, \theta \in (1,\infty)$, $G \subset R^n$ (introduced and studied from point of view of embedding theory in papers [1]), reduced by polynomials, determined in $n$-dimensional domains satisfying the flexible $\varphi$-horn condition. In other words, we prove inequality type

$$\left\| D^v \left(f - P_{l-1}(f,x)\right)\right\|_{q,G} \le C \left|\tilde{A}(1)\right| \|f\|_{\tilde{F}_{p,\theta}(G_\varphi)}, \tag{1}$$

where

$$\tilde{A}(1) = \max_i A^i(1) = \int_0^1 \prod_{j=1}^n (\varphi_j(t))^{-v_j - \frac{1}{p} + \frac{1}{q}} \frac{\varphi^1_i(t)}{\varphi_i(t)} dt,$$

$$\|f\|_{\tilde{F}_{p,\theta}(G_\varphi)} = \sum_{i=1}^n \left\| \left\{ \int_0^{t_0} \left[ \frac{\delta_i^{m_i}\left(\varphi_i(t), G_{\varphi(t)}\right) D_i^{k_i} f(\cdot)}{(\varphi_i(t))^{l_i - k_i}} \right]^\theta \frac{d\varphi_i(t)}{\varphi_i(t)} \right\}^{\frac{1}{\theta}} \right\|_p . \tag{2}$$

Note that the normed linear space $F^l_{p,\theta}(G_\varphi)$ of functions $f \in L^{loc}(G)$ with the finite norm, defined in paper [5]

$$\|f\|_{F^l_{p,\theta}(G_\varphi)} = \|f\|_{p,G} +$$

 

$$+\sum_{i=1}^{n}\left\|\left\{\int_0^{t_0}\left[\frac{\delta_i^{m_i}\left(\varphi_i(t),G_{\varphi(t)}\right)D_i^{k_i}f\left(\cdot\right)}{\left(\varphi_i(t)\right)^{l_i-k_i}}\right]^{\theta}\frac{d\varphi_i(t)}{\varphi_i(t)}\right\}^{\frac{1}{\theta}}\right\|_p, \tag{3}$$

where

$$\|f\|_{p,G}=\|f\|_{L_p(G)}=\left(\int_G|f(x)|^p\,dx\right)^{\frac{1}{p}},$$

$$\delta_i^{m_i}\left(\varphi_i(t)\right)f(x)=\int_{-1}^{1}\left|\Delta_i^{m_i}\left(\varphi_i(t),G_{\varphi(t)}\right),f(x)\right|dt,$$

$$\Delta_i^{m_i}\left(\varphi_i(t),G_{\varphi(t)}\right)f(x)=\begin{cases}\Delta_i^{m_i}\left(\varphi_i(t)\right)f(x), & \text{for}\quad [x,x+m_i\varphi_i(t)e_i]\subset G_{\varphi(t)},\\ 0, & \text{for}\quad [x,x+m_i\varphi_i(t)e_i]\not\subseteq G_{\varphi(t)},\end{cases}$$

$$\Delta_i^{m_i}\left(\varphi_i(t)\right)f(x)=\sum_{j=0}^{m_i}(-1)^{m_i-j}C_{m_i}^{j}f\left(x+j\varphi_i(t)e_i\right),\quad e_i=\{0,...,0,1,0,...,0\},$$

and let $G\subset R^n$; $l\in(0,\infty)^n$; $m_i\in N$; $k_i\in N_0$; $1<p,\theta<\infty$; $\varphi(t)=(\varphi_1(t),...,\varphi_n(t))$, $\varphi_j(t)>0\,(t>0)$ be continuously-differentiable functions, $\lim\limits_{t\to+0}\varphi_j(t)=0$, $\lim\limits_{t\to+0}\varphi_j(t)= = L_j\le\infty$, $j=1,2,...,n$. We denote the set of such vector functions by $A$. For any $x\in R^n$ we assume

$$G_{\varphi(t)}(x)=G\cap I_{\varphi(t)}(x)=G\cap\left\{y:|y_j-x_j|<\frac{1}{2}\varphi_j(t),\ j=1,2,...,n\right\}.$$

In case $\varphi_j(t)=t^{\lambda_j},\lambda_j>0$, $j=1,2,...,n$ the spaces $F^l_{p,\theta}(G)$, was introduced and studied view of theory embedding in monograph [1].

Was proved that [6] for $f\in F^l_{p,\theta}(G_\varphi)$, $p,\theta\in(1,\infty)$, $l\in(0,\infty)^n$, if

$$A^i(T)=\int_0^T\prod_{j=1}^{n}\left(\varphi_j(t)\right)^{-t_j}\frac{\varphi_i^1(t)}{\left(\varphi_i(t)\right)^{1-l_i}}dt<\infty,\ \ i=1,2,...,n,$$

then there exists $D^vf\in L_p(G)$ and the following identity is valid [5]

$$D^vf(x)=f^{(v)}_{\varphi(T)}(x)+\sum_{i=1}^{n}\int_0^T\int_{R^n}L_i^{(v)}\left(\frac{y}{\varphi(t)},\frac{\rho\left(\varphi(t),x\right)}{\varphi(t)}\right)\times$$

$$\times f_i(x+y,t)\prod_{j=1}^{n}\left(\varphi_j(t)\right)^{-1-v_j}\frac{\varphi_i^1(t)}{\varphi_i(t)}dtdy, \tag{4}$$

$$f^{(v)}_{\varphi(T)}(x)=\prod_{j=1}^{n}\left(\varphi_j(T)\right)^{-1-v_j}\int_{R^n}\Omega^{(v)}\left(\frac{y}{\varphi(T)},\frac{\rho\left(\varphi(T),x\right)}{3\varphi(T)}\right)f(x+y)dy, \tag{5}$$

where

$$|f_i(x,t)| \leq \int\limits_{-1}^{1} |\delta_i^{m_i}(\varphi_i(t))\, f(x+u\varphi_i(t))|\, du.$$

## 2. Main results.

**Theorem 1.** *Let $G \subset R^n$ satisfy the condition of flexible $\varphi$-horn $1 < p < q \leq \infty$, $v = (v_1, ..., v_n)$, $v_j \geq 0$ be entire $j = 1, ..., n$; $A^i(1) < \infty$ $(i = 1, ..., n)$ and let $f \in F^l_{p,\theta}(G_\varphi)$. Then*

$$\|D^v (f - P_{l-1}(f,x))\|_{q,G} \leq C\, |A(1)|\, \|f\|_{\tilde{F}^l_{p,\theta}(G_\varphi)},$$

*where $A(1) = \max\limits_i A^i(1)$, $A^i(1) = \int\limits_0^1 \prod_{j=1} (\varphi_j(t))^{-v_j - \frac{1}{p} + \frac{1}{q}} \frac{\varphi_i^1(t)}{\varphi_i(t)^{1-l_i}} dt$ and $C$ the constant independent of $f$.*

*Proof.* Under the conditions of our theorem, there exist generalized derivatives $D^v f \in L_p(G)$ and for almost each point $x \in G$ the integral representation (4) and (5) with the kernels is valid. In (4) and (5) if $\rho(\varphi(t), x) = -x\varphi(t)$, $0 < t \leq T = 1$ we get identity

$$D^v f(x) = P_{l-1}^{(v)} + \sum_{i=1}^{n} \int\limits_0^1 \int\limits_{R^n} L_i^{(v)} \left( \frac{y}{\varphi(t)}, \frac{p(\varphi(t), x)}{\varphi(t)} \right) f_i(x+y,t) \times$$

$$\times \prod_{j=1}^{n} (\varphi_j(t))^{-1-v_j} \frac{\varphi_i^1(t)}{\varphi_i(t)} dt, \tag{6}$$

where

$$|f_i(x,t)| \leq \int\limits_{-1}^{1} |\delta_i^{m_i}(\varphi_i(t))|\, f(x+u\varphi_i(t))\, u,$$

the support of this identity (6) is contained in the flexible $\varphi$ horn

$$x + V(\varphi) = x + \bigcup_{0<t\leq T\leq 1} \left\{ y : \left( \frac{y}{\varphi(t)} \right) \in S(L_i),\, i = 1, ..., n \right\},$$

where $L_i(\cdot, y) \in C_0^\infty(R^n)$ $(i = 1, ..., n)$, and let

$$S(L_i) = \sup p L_i \subset I_{\varphi(1)} = \left\{ x : |x_j| < \frac{1}{2}\varphi_j(1), j = 1, ..., n \right\}.$$

Let $U$ be is open set contained in the domain $G$; hence forth we always assume that $U + V(\varphi) \subset G$.

Hence, by the Minkowski inequality, we have:

$$\left\| D^{v} \left( f - p_{l-1}(f,x) \right) \right\|_{q,U} \leq \sum_{i=1}^{n} \left\| F_i \left( \cdot, t \right) \right\|_{q,U}, \tag{7}$$

here

$$F_i(x,t) = \int_0^1 \int_{R^n} L_i^{(v)} \left( \frac{y}{\varphi(t)}, \frac{p\left(\varphi(t), x\right)}{\varphi(t)} \right) f_i(x+y,t) \times$$
$$\times \prod_{j=1}^{n} \left( \varphi_j(t)^{-1-v_j} \frac{\varphi_i(t)}{\varphi_i(t)} \right) dtdy. \tag{8}$$

Applying generalized Minkowski inequality (8) for $F_i(x,t)$, we get

$$\left\| F_i \left( \cdot, t \right) \right\|_{q,U} \leq \int_0^1 \left\| E_i \left( \cdot, t \right) \right\|_{q,U} \left| \prod_{j=1}^{n} \left( \varphi_j(t) \right)^{-1-v_j} \frac{\varphi_i^1(t)}{\left( \varphi_i(t) \right)^{1-l_j}} \right| dt, \tag{9}$$

here

$$E_i(x,t) = \int_{R^n} L_i^{(v)} \left( \frac{y}{\varphi(t)}, \frac{\rho\left(\varphi(t), x\right)}{\varphi(t)} \right) f_i(x+y,t)dy. \tag{10}$$

From the Holder inequality ($q \leq r \leq \infty$) we have

$$\left\| E_i \left( \cdot, t \right) \right\|_{q,U} \leq \left\| E_i \left( \cdot, t \right) \right\|_{r,U} \left( mesU \right)^{\frac{1}{q} - \frac{1}{r}}. \tag{11}$$

Now estimate the norm $\left\| E_i \left( \cdot, t \right) \right\|_{r,U}$. Let $\chi$ be a characteristic function of the set $S(L_i)$. Again applying the Holder inequality for representing the function in the form (10) in the case $1 < p < r \leq \infty$, $s \leq r$ as $\left( \frac{1}{s} = 1 - \frac{1}{p} + \frac{1}{r} \right)$, we get

$$\left\| E_i \left( \cdot, t \right) \right\|_{r,U} \leq \sup_{x \in U} \left( \int_{R^n} \left| f_i(x+y) \right|^p \chi \left( \frac{y}{\varphi(t)} \right) \right)^{\frac{1}{p} - \frac{1}{r}} \times$$
$$\times \sup_{y \in V} \left( \int_U \left| f_i(x+y,t) \right|^p dx \right)^{\frac{1}{r}} \left( \int_U \left| \tilde{L}_i \left( \frac{y}{\varphi(t)} \right) \right|^s dy \right)^{\frac{1}{s}}. \tag{12}$$

It is assumed that $\left| L_i(x,y,z) \right| \leq C \left| \tilde{L}_i(x) \right|$, and $\tilde{L}_i \in C_0^{\infty} \left( R^n \right)$.

For any $x \in U$ we have

$$\int_{R^n} \left| f_i(x+y,t) \right|^p \chi \left( \frac{y}{\varphi(t)} \right) dy \leq$$

$$\leq \int\limits_{U+V} |f_i(x+y,t)|^p\,dy \leq (\varphi_i(t))^{pl_i} \left\|(\varphi_i(t))^{-l_i}\,\delta_i^{m_i}\,(\varphi_i(t))\,f\right\|_{p,C}^p . \tag{13}$$

For $y \in V$

$$\int\limits_U |f_i(x+y,t)|^p\,dx \leq \int\limits_{U+V} |f_i(x,t)|^p\,dx \leq (\varphi_i(t))^{l_i} \left\|(\varphi_i(t))^{-l_i}\,\delta_i^{m_i}\,(\varphi_i(t))\,f\right\|_{p,G}, \tag{14}$$

$$\int\limits_{R^n} \left|\widetilde{L}_i\left(\frac{y}{\varphi(t)}\right)\right|^s dy = \left\|\widetilde{L}_i\right\|_s^s \prod_{j=1}^n \varphi_j(t). \tag{15}$$

From inequalities (12)-(15) it follows that

$$\|E_i(\cdot,t)\|_{r,U} \leq C_1 \left\|\tilde{L}_i\right\|_s \prod_{j=1}^n (\varphi_j(t))^{\frac{1}{s}} (\varphi_i(t))^{l_i} \times$$

$$\times \left\|(\varphi_i(t))^{-l_i}\,\delta_i^{m_i}\,(\varphi_i(t))\right\|_{p,G}, \tag{16}$$

and by the inequality (12) we have

$$\|E_i(\cdot,t)\|_{q,U} \leq C_2 \prod_{j=1}^n (\varphi_j(t))^{\frac{1}{s}} (\varphi_i(t))^{l_i} \times$$

$$\times \left\|(\varphi_i(t))^{-l_i}\,\delta_i^{m_i}\,(\varphi_i(t))\,f\right\|_{p,G}. \tag{17}$$

From inequalities (7) and (9) for $(r = q)$ we have

$$\|D^v(f - P_{l-1}(f,x))\|_{q,G} \leq C\left|\tilde{A}(1)\right| \|f\|_{F_{p,\theta}(G_\varphi)}.$$

This completes the poof of Theorem 2.

The following theorem is proved analogously to Theorem 1.

**Theorem 2.** *Let all the conditions of Theorem 2.1 be fulfilled. Furthermore, let* $l^1 \in (0,\infty)^n$, $1 < \theta < \theta_1 < \infty$, *if*

$$A^{i,1}(1) = \int\limits_0^1 \prod_{j=1}^n (\varphi_j(t))^{-v_j - l_j^1 - \frac{1}{p} + \frac{1}{q}} \frac{\varphi_i^1(t)}{(\varphi_i(t))^{1-l_i}}dt < \infty$$

$i = 1,2,...,n,$ *then*

$$\|D^v(f - P_{l-1}(f,x))\|_{F^{l^1}_{q,\theta}(G_\varphi)} \leq C\left|A^1(1)\right| \|f\|_{F^l_{p,\theta}(G_\varphi)},$$

*where* $A^{(1)}(1) = \max\limits_i A^{i,1}(1)$ *and* $C$ *the constant independent of* $f$.

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