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On Pouncare Type Inequalities for Functions from Sobolev-Morrey Type Spaces with Dominant Mixed Derivatives

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Abstract. In this paper, are proved help of method integral representation are proved Pouncare type inequalities for functions from Sobolev-Morrey spaces with dominant mixed derivatives.

Key Words and Phrases: Sobolev-Morrey spaces with dominant mixed derivatives, integral representation, flexible φ-horn.

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1. Introduction

Let $G \subset R^n$, $1 \le p < \infty$; $\varphi(t) = (\varphi_1(t_1), \varphi_2(t_2), ..., \varphi_n(t_n))$, $\varphi_j(t_j) > 0\,(t_j > 0)$ is Lebesgue measurable functions, $\lim\limits_{t_j \to +0} \varphi_j(t_j) = 0$, $\lim\limits_{t_j \to +\infty} \varphi_j(t_j) \le K \le \infty$, we denote the set of vector-functions $\varphi(t)$ by A. Let $e_n = \{1, 2, ..., n\}$, $e \subseteq e_n$; and $l = (l_1, l_2, ..., l_n)$, $l_j > 0$ are integers $(j \in e_n)$; and $l^e = (l_1^e, ..., l_n^e)$, where $l_j^e = l_j$ for $j \in e$; $l_j^e = 0$ for $j \in e_n \backslash e = e'$;

For any $x \in R^n$ put

$$G_{\varphi(t)}(x) = G \cap I_{\varphi(t)}(x) = G \cap \left\{ y : |y_j - x_j| < \frac{1}{2}\varphi_j(t_j), j \in e_n \right\},$$

and

$$\int\limits_{a^e}^{b^e} f(x)dx = \left(\prod_{j \in e} \int\limits_{a_j}^{b_j} dx_j \right) f(x),$$

i.e. integration is carried and only with respect to the variables x_j whose indices belong to e.

Definition 1. *The Sobolev-Morrey spaces with dominant mixed derivatives $S^l_{p,\varphi,\beta}W(G)$ of locally summable functions f on G having the generalized derivatives $D^{l^e}f (e \subseteq e_n)$ on G with the finite norm*

$$\|f\|_{S^l_{p,\varphi,\beta}W(G)} = \sum_{e \subseteq e_n} \left\| D^{l^e} f \right\|_{p,\varphi,\beta;G}, \tag{1}$$

where

$$\|f\|_{p,\varphi,\beta;G} = \|f\|_{L_{p,\varphi,\beta}(G)} = \sup_{\substack{x\in G,\\ t>0}} \left(|\varphi([t]_1)|^{-\beta} \|f\|_{p,G_{\varphi(t)}(x)} \right), \tag{2}$$

$$|(\varphi([t]_1))|^{-\beta} = \prod_{j\in e_n} \left(\varphi_j\left([t_j]_1\right)\right)^{-\beta_j}, \ \ \beta_j \in [0,1],\ [t_j]_1 = \min\{1, t_j\},\ j \in e_n.$$

Note that the space $S^l_{p,\varphi,\beta}W(G)$ in the case $\beta_j = 0\,(j \in e_n)$ coincides with Sobolev space with dominant mixed derivatives $S^l_p W(G)$ which were introduced and investigated by S.M. Nikolskii and were further considered in the papers [2]-[5].

For any $t_j > 0$ $(j \in e_n)$, there exists a positive constant $C > 0$ such that $|\varphi([t]_1)| \le C$, then the embeddings $L_{p,\varphi,\beta}(G) \to L_p(G),\ S^l_{p,\varphi,\beta}W9G) \to S^l_p W(G)$, hold i.e.

$$\|f\|_{p,G} \le C\,\|f\|_{p,\varphi,\beta,G},\ \ \|f\|_{S^l_pW(G)} \le C\,\|f\|_{S^l_{p,\varphi,\beta}W(G)}. \tag{3}$$

In this paper, with method of integral representation, we estimate the norms of functions from Sobolev-Morrey spaces with dominant mixed derivatives $S^l_{p,\varphi,\beta}W(G)$ reduced by polynomials, determined in n-dimensional domains satisfying the flexible φ-horn condition.

In order words, we prove the inequalities type Pouncare as

$$\|D^\nu(f - P_{l-1}(f,x))\|_{q,\infty} \le C\,|\varphi(1)|\ \|f\|_{S^l_{p,\varphi,\beta}\omega(G)},$$

where

$$\|f\|_{S^l_{p,\varphi,\beta}\omega(G)} = \sum_{\emptyset\ne e\subseteq e_n} \left\|D^{l^e} f\right\|_{p,\varphi,\beta,G},$$

$$\varphi(1) = \max_{e\subseteq e_n} \prod_{j\in e_n} (\varphi_j(1))^{s_e},\ \ s_{e,j} = \begin{cases} \mu_j, & j \in e \\ -\nu_j - (1-\beta_j p)\left(\frac{1}{p} - \frac{1}{q}\right), & j \in e'. \end{cases}$$

Such problems in different spaces were studied in papers [1,9,10].

Let $M_e(\cdot, y, z) \in C_0^\infty(R^n)$, $e \subseteq e_n$, and by such that

$$S(M_e) = \sup pL_e \subset I_{\varphi(T)} = \left\{x : |x_j| < \frac{1}{2}\varphi_j(T_j),\ j \in e_n\right\}.$$

Assume that for any $0 < T_j \le 1$ $(j \in e_n)$

$$V = \underset{0<t_j\le T_j}{U} \left\{y : \frac{y}{\varphi\left(t^e + T^{e'}\right)} \in S(M_e)\right\},$$

where $\left(t^e + T^{e'}\right) = t_j,\ j \in e;\ \left(t^e + T^{e'}\right) = T_j,\ j \in e'$.

It is clear that $V \subset I_{\varphi(T)}$, and suppose $U + V \subset G$, where U is an open set, contained in the domain G. Let

$$G_{\varphi(T)}(U) = \left(U + I_{\varphi(T)}(x)\right) \cap G = Z,$$

and let $U+V\subset G_{\varphi(T)}(U)$.

Assuming that $\varphi_j(t)$ $(j\in e_n)$ are also differentiable on $[0,T_j]$, $j\in e_n$, and is obtained that in for $f\in S_p^lW(G)$ determined in n-dimensional domains, satisfying the condition of flexible φ-horn, it holds the following integral representation $(\forall x\in U\subset G)$ [8]

$$D^v f(x)=\sum_{e\leq e_n}(-1)^{|v|+|l^e|}\prod_{j\in e'}(\varphi_j(T_j))^{-1-v_j}\times$$

$$\times\int\limits_{O^e}^{T^e}\int\limits_{R^n}M_e^{(v)}\left(\frac{y}{\varphi\left(t^e+T^{e'}\right)},\ \frac{\rho\left(\varphi\left(t^e+T^{e'}\right),x\right)}{\varphi\left(t^e+T^{e'}\right)},\ \rho'\left(\varphi\left(t^e+T^{e'}\right),x\right)\right)\times$$

$$\times\prod_{j\in e}(\varphi_j(t_j))^{l_j-v_j-2}\prod_{j\in e}\varphi_j'(t_j)dt^e dy. \tag{4}$$

2. Main results

Theorem 1. *Let $G\subset R^n$ satisfy the condition of flexible φ-horn [5], $1\leq p\leq q\leq\infty$, $v=(v_1,v_2,...,v_n)$, $v_j\geq 0$ be entire $(\in e_n)$, $f\in S_{p,\varphi,\beta}^lW(G)$, and let*

$$\mu_j=l_j-v_j-(1-\beta_j p)\left(\frac{1}{p}-\frac{1}{q}\right)>0,\ j\in e_n. \tag{5}$$

Then exists polynomials $P_{l-1}(f,x)$

$$\|D^v(f-P_{l-1}(f,x))\|_{q,G}\leq C_1\varphi(1)\sum_{\emptyset\neq e\subseteq e_n}\left\|D^{l^e}f\right\|_{p,\varphi,\beta,G}, \tag{6}$$

where $\varphi(1)=\max\limits_{e\subseteq e_n}\prod_{j\in e_n}(\varphi_j(1))^{S_{e,j}}$,

$$S_{e,j}=\begin{cases}\mu_j,\ j\in e\\ -v_j-(1-\beta_j p)\left(\frac{1}{p}-\frac{1}{q}\right),\ j\in e',\end{cases}$$

and C_1 is constant independent of f.

Proof. At first note that in the conditions of our theorem there exists generalized derivatives $D^v f$ in G. Indeed, if $\mu_j>0$ $(j\in e_n)$, $p\leq q$, it follows that for $f\in S_{p,\varphi,\beta}^l\to f\in S_p^lW(G)$. Then there exists generalized $D^v f\in L_p(G)$, it holds the following integral representation (**??**). Note that in case

$$\rho\left(\varphi\left(t^e+T^{e'}\right),x\right)=$$

$$=-x\rho\varphi\left(t^e+T^{e'}\right),\ 0<t_j\leq T_j=1\ \ (j\in e_n)$$

is valid:

$$D^v f(x) = P_{l-1}(f,x) + \sum_{\emptyset\neq e\subseteq e_n} (-1)^{|v|+|l^e|} \int\limits_{0^e}^{1^e} \int\limits_{R^n} M_e^{(v)} \times$$

$$\times \left(\frac{y}{\varphi\left(t^e+1^{e'}\right)}, \frac{\rho\left(\varphi\left(t^e+1^{e'}\right),x\right)}{\varphi\left(t^e+1^{e'}\right)},\ \rho'\left(\varphi\left(t^e+1^{e'}\right),x\right)\right) \times$$

$$\times D^{l^e} f(x+y) \prod_{j\in e} (\varphi_j(t_j))^{l_j-v_j-2} \prod_{j\in e} \varphi_j'(t_j) dt^e dy.$$

The support of this identity is contained in the flexible φ-horn $x+V\subset G$. Hence, by the Minkowski inequality, we have

$$\|D^v\left(f-P_{l-1}\left(f,\cdot\right)\right)\|_{q,U} \leq \sum_{\emptyset\neq e\subseteq e_n} \left\|K_e\left(\cdot,t^e+1^{e'}\right)\right\|_{q,U} \tag{7}$$

here

$$K_e\left(x,t^e+1^{e'}\right) = \int\limits_{0^e}^{1^e} \int\limits_{R^n} M_e^{(v)} \times$$

$$\times \left(\frac{y}{\varphi\left(t^e+1^{e'}\right)}, \frac{\rho\left(\varphi\left(t^e+1^{e'}\right),x\right)}{\varphi\left(t^e+1^{e'}\right)},\ \rho'\left(\varphi\left(t^e+1^{e'}\right),x\right)\right) \times$$

$$\times D^{l^e} f(x+y) \prod_{j\in e} (\varphi_j(t_j))^{l_j-v_j-2} \prod_{j\in e} \varphi_j'(t_j) dt^e dy. \tag{8}$$

Applying generalized Minkowski inequality (7) for $K_e(x,t)$ defined by equality (8) we get

$$\left\|K_e(\cdot,t^e+1^{e'})\right\|_{q,U} \leq C_1 \int\limits_{0^e}^{1^e} \prod_{j\in e} (\varphi_j(t_j))^{l_j-v_j-2} \times$$

$$\times \prod_{j\in e} \varphi_j'(t_j) \left\|R_e\left(\cdot,t^e+1^{e'}\right)\right\| dt^e dy. \tag{9}$$

where

$$R_e\left(x,t^e+1^{e'}\right) = \int\limits_{R^n} M_e^{(v)} \left(\frac{y}{\varphi\left(t^e+1^{e'}\right)}, \frac{\rho\left(\varphi\left(t^e+1^{e'}\right),x\right)}{\varphi\left(t^e+1^{e'}\right)}, \right. \times$$

$$\left. \times \rho'\left(\varphi\left(t^e+1^{e'}\right),x\right) \right) D^{l^e} f(x+y) dy.$$

Help of inequality ($q\leq r\leq\infty$) we get

$$\left\|R_e\left(\cdot,t^e+1^{e'}\right)\right\|_{q,U} \leq \left\|R_e\left(\cdot,t^e+1^{e'}\right)\right\|_{q,U} (mesU)^{\frac{1}{q}-\frac{1}{r}}, \tag{10}$$

let $1 \le p \le r \le \infty,\ s \le r \left(\frac{1}{s} = 1 - \frac{1}{p} + \frac{1}{r}\right)$,

$$\left|M_e^{(v)} D^{l^e} f\right| = \left(\left|M_e^{(v)}\right|^p \left|D^{l^e} f\right|^s\right)^{\frac{1}{p}} \left(\left|D^{l^e} f\right|^p X\right)^{\frac{1}{p}-\frac{1}{r}} \left(\left|M_e^{(v)}\right|^s\right)^{\frac{1}{s}-\frac{1}{r}}$$

and apply Hölder inequality for $\left(\frac{1}{r} + \left(\frac{1}{p} - \frac{1}{r}\right) + \left(\frac{1}{s} - \frac{1}{r}\right) = 1\right)$, we have

$$\left|R_e\left(x, t^e + 1^{e'}\right)\right| \le \left(\int\limits_{R^n} \left|D^{l^e} f(x+y)\right|^p M_e^{(v)} \left(\frac{y}{\varphi(t^e + 1^{e'})},\right.\right.$$

$$\left.\left.\frac{\rho\left(\varphi\left(t^e + 1^{e'}, x\right)\right)}{\varphi(t^e + 1^{e'})}, \rho'\left(\varphi\left(t^e + 1^{e'}\right), x\right)\right) dy^s\right)^{\frac{1}{r}} \times$$

$$\times \left(\int\limits_{R^n} \left|D^{l^e} f(x+y)\right|^p \chi\left(\frac{y}{\varphi(t^e + 1^{e'})}\right) dy\right)^{\frac{1}{p}-\frac{1}{r}} \times$$

$$\left.\times \int\limits_{R^n} \left|M_e^{(v)}\left(\frac{y}{\varphi(t^e + 1^{e'})}, \frac{\rho\left(\varphi\left(t^e + 1^{e'}, x\right)\right)}{\varphi(t^e + 1^{e'})}, \rho'\left(\varphi\left(t^e + 1^{e'}\right), x\right)\right|^s dy\right)^{\frac{1}{s}-\frac{1}{r}}.$$

Further, we will assume that there exists a function $\tilde{M}_e(x)$ such that $|M_e(x,y,z)| \le$ $\le C\left|\tilde{M}_e(x)\right|$, for all $(y,z) \in R^{2n}$.

Let χ be a characteristic function of the set $S(M_e)$. Then, we have

$$\left\|R_e\left(\cdot, t^e + 1^{e'}\right)\right\|_{r, U_\varphi(t^e + 1^e)} \le$$

$$\le \sup_{x\in U}\left(\int\limits_{R^n} \left|D^{l^e} f(x+y)\right|^p \chi\left(\frac{y}{\varphi(t^e + 1^{e'})}\right) dy\right)^{\frac{1}{p}-\frac{1}{r}} \times$$

$$\times \sup_{y\in V}\left(\int\limits_{U} \left|D^{l^e} f(x+y)\right|^p dx\right)^{\frac{1}{r}} \times$$

$$\times \left(\int\limits_{R^n} \left|\tilde{M}_e^{(v)}\left(\frac{y}{\varphi(t^e + 1^{e'})}\right)\right|^s dy\right)^{\frac{1}{s}}. \tag{11}$$

For any $x \in U$ we have

$$\int\limits_{R^n} \left|D^{l^e} f(x+y)\right|^p \chi\left(\frac{y}{\varphi(t^e + 1^{e'})}\right) dy \le \int\limits_{(U+V)_{\varphi(t^e + 1^e)}} \left|D^{l^e} f(y)\right|^p dy \le$$

$$\leq \int\limits_{Z} \left|D^{l^e} f(y)\right|^p dy \leq \left\|D^{l^e} f\right\|_{p,\varphi,\beta,z} \prod_{j\in e_n} (\varphi_j(1))^{\beta_j p}, \tag{12}$$

for $y \in V$

$$\int\limits_{U_{\varphi\left(t^e+1^{e'}\right)}} \left|D^{l^e} f(x+y)\right|^p dx \leq$$

$$\leq \int\limits_{Z} \left|D^{l^e} f(x)\right|^p dx \leq \left\|D^{l^e} f\right\|_{p,\varphi,\beta,z} \prod_{j\in e_n} \varphi_j(1)^{\beta_j p}, \tag{13}$$

and

$$\int\limits_{R^n} \left|\tilde{M}_e\left(\frac{y}{\varphi\left(t^e+1^{e'}\right)}\right)\right|^s dy = \left\|\tilde{M}_e\right\|_s^s \prod_{j\in e_n} \varphi_j(1). \tag{14}$$

$$\left\|R_e\left(\cdot, t^e+1^{e'}\right)\right\|_{r,U_{\varphi\left(t^e+1^e\right)}} \leq C_2 \left\|M_e^{(v)}\right\|_s \prod_{j\in e_n} (\varphi_j(1))^{\frac{1}{s}+\beta_j p\left(\frac{1}{p}-\frac{1}{r}\right)} \times$$

$$\times \left\|D^{l^e} f\right\|_{p,\varphi,\beta,z}. \tag{15}$$

By means inequalities (9), (10) and (15) for $r = q$ we have

$$\left\|K_e\left(\cdot, t^e+1^{e'}\right)\right\|_{q,U} \leq C\varphi(1) \left\|D^{l^e} f\right\|_{p,\varphi,\beta,z}. \tag{16}$$

Substituting the inequality (**??**) in (**??**), we obtain the inequality (**??**).

This completes the proof of Theorem 2.1.

Theorem 2. *Let all the conditions of Theorem 2.1 be fulfilled. Furthermore, let* $l^1 \in N$ *and if*

$$\mu_j^1 = l_j - v_j - (1-\beta_j p)\left(\frac{1}{p}-\frac{1}{q}\right) = l_j^1 > 0,\ j \in e_n.$$

Then

$$\left\|D^v\left(f - P_{l-1}(f,x)\right)\right\|_{S_q^{l^1}W(G)} \leq C_2\varphi_1(1) \sum_{\emptyset\neq e\subseteq e_n} \left\|D^{l^e} f\right\|_{p,\varphi,\beta;G}, \tag{17}$$

where $\varphi_1(1) = \max\limits_{e\subseteq e_n} \prod_{j\in e_n} (\varphi_1(1))^{S_{e,j}^1}$.

$$s_{e,j}^1 = \begin{cases} \mu_j^1, & j \in e \\ -v_j - (1-\beta_j p)\left(\frac{1}{p}-\frac{1}{q}\right) - l_j^1, & j \in e'. \end{cases}$$

C_2 *is constant independent an function on* f. *The theorem is proved analogously to Theorem 2.1.*

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