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On the Solvability of One Inverse Boundary Value Problem for the Linearized Benny–Luc Equation with Non-self-adjoint Boundary Conditions

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Abstract. An inverse problem is investigated for the linearized Benny-Luc equation with non-self-adjoint boundary conditions. First, the original problem is reduced to an equivalent problem (in a certain sense), for which the existence and uniqueness theorem is proved. Further, on the basis of these facts, the existence and uniqueness of the classical solution to the original problem are proved.

Key Words and Phrases: inverse boundary value problem, Benny–Luc equation, existence, uniqueness of classical solution.

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1. Introduction

Many problems in mathematical physics and continuum mechanics are boundary value problems that reduce to the integration of a differential equation or a system of partial differential equations for given boundary and initial conditions. Many problems in gas dynamics, the theory of elasticity, the theory of plates and shells are reduced to the consideration of high-order partial differential equations [1]. Differential equations of the fourth order are of great interest from the point of view of applications (see, for example, [2, 3]). Partial differential equations of Benny – Luc type have applications in mathematical physics (see [3]).

Problems in which, together with the solution of a particular differential equation, it is also required to determine the coefficient (coefficients) of the equation itself, or the right side of the equation, in mathematics and in mathematical modeling are called inverse problems. The theory of inverse problems for differential equations is a dynamically developing branch of modern science. Recently, inverse problems have arisen in various fields of human activity, such as seismology, mineral exploration, biology, medicine, quality control of industrial products, etc., which puts them in a number of urgent problems of modern mathematics. Various inverse problems for certain types of partial differential equations have been studied in many works. Let us first of all note here the works of A.N. Tikhonov

[4], M.M. Lavrent'ev [5, 6], V.K. Ivanov [7] and their students. More details can be found in the monograph by A.M. Denisov [8].

The theory of inverse boundary value problems for fourth-order equations is still understudied. The works [9–12] are devoted to inverse boundary value problems for the Benny – Luc equation.

The aim of this work is to prove the existence and uniqueness of solutions to the inverse boundary value problem for the Benny-Luc equation with non-self-adjoint boundary conditions.

2. Statement of the problem and its reduction to an equivalent problem

Let $D_T = \{(x,t):\ \ 0 \le x \le 1, \quad 0 \le t \le T\}$. Consider the following inverse boundary value problem in a rectangle D_T: find a pair $\{u(x,t)\,,\ a(t)\}$ of functions $u(x,t)$, $a(t)$ satisfying the equation [3]

$$u_{tt}(x,t) - u_{xx}(x,t) + \alpha u_{xxxx}(x,t) - \beta u_{xxtt}(x,t) = \ a(t)u(x,t) + f(x,t) \ \ (x,t) \in D_T, \quad (1)$$

with initial conditions

$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x) \quad (0 \le x \le 1), \tag{2}$$

with non-self-adjoint boundary conditions

$$u(1,t) = 0, u_x(0,t) = u_x(1,t) \ \ , \ \ u_{xx}(1,t) \ = 0, \ u_{xxx}(0,t) = u_{xxx}(1,t) \ \ (0 \le t \le T) \tag{3}$$

and with the additional condition

$$u(0,t) = h(t) \qquad (0 \le t \le T), \tag{4}$$

where $\alpha > 0$, $\beta > 0$ - are fixed numbers, $f(x,t)$, $\varphi(x)$, $\psi(x)$, $h(t)$ - are given functions.

Denote

$$\tilde{C}^{4,2}(D_T) = \left\{u(x,t):\ u(x,t) \in C^2(D_T)\ , u_{ttx}(x,t), \right.$$
$$\left. u_{ttxx}(x,t), u_{xxx}(x,t), u_{xxxx}(x,t) \in C(D_T)\right\}.$$

Definition 1. *By the classical solution of the inverse boundary value problem* (1) - (4) *we mean a pair* $\{u(x,t)\,,\ a(t)\}$ *of functions* $u(x,t) \in \tilde{C}^{4,2}(D_T)$, $a(t) \in C[0,T]$, *satisfying equation* (1) *and conditions* (2) - (4) *in the usual sense.*

Similarly to [13], the following theorem is proved.

Theorem 1. *Let* $\varphi(x),\ \ \psi(x) \in C[0,1],\ \ h(t) \in C^2[0,T]\ ,\ h(t) \ne 0 \ \ (0 \le t \le T),\ \ f(x,t) \in C(D_T)$ *and the conditions of consistency are hold*

$$\varphi(0) = h(0)\ , \quad \psi(0) = h'0). \tag{5}$$

Then the problem of finding a classical solution to problem (1) - (4) *is equivalent to the problem of determining the functions* $u(x,t) \in \tilde{C}^{4,2}(D_T)$ *and* $a(t) \in C[0,T]$ *from relations* (1) - (3) *and the condition*

$$h''(t) - u_{xx}(0,t) + \alpha u_{xxxx}(0,t) - \beta u_{xxtt}(0,t) = \ a(t)h(t) + f(0,t) \ \ (\,0 \le t \le T\,)\,. \tag{6}$$

3. Solvability of the inverse boundary value problem

It is known that [14], function sequences

$$X_0(x) = 2(1-x),\ \ X_{2k-1}(x) = 4(1-x)\cos\lambda_k x,\ \ X_{2k}(x) = 4\sin\lambda_k x\ (k = 1, 2, ...), \quad (7)$$

$$Y_0(x) = 1,\ \ Y_{2k-1}(x) = \cos\lambda_k x,\ \ Y_{2k}(x) = x\sin\lambda_k x\ (k = 1, 2, ...) \quad (8)$$

form a biorthogonal system, and system (7) forms a Riesz basis for $L_2(0,1)$, where $\lambda_k = 2k\pi\,(k = 1, 2, ...)$. Then an arbitrary function $\vartheta(x) \in L_2(0,1)$ is expanded into a biorthogonal series:

$$\vartheta(x) = \vartheta_0\, X_0(x) + \sum_{k=1}^{\infty}\vartheta_{2k-1}\ X_{2k-1}(x)\ +\ \sum_{k=1}^{\infty}\vartheta_{2k}\ X_{2k}(x)\,,$$

where

$$\vartheta_0\ = \int_0^1 \vartheta_0\, Y_0\,(x)dx\,,\ \ \vartheta_{2k-1}\ = \int_0^1 \vartheta_{2k-1}\, Y_{2k-1}\,(x)dx\,,\ \ \vartheta_{2k-1}\ = \int_0^1 \vartheta_{2k-1}\, Y_{2k-1}\,(x)dx\,.$$

It is known that [15],

$$\vartheta(x) \in C^{2i-1}[0,1],\quad \vartheta^{(2i)}(x) \in L_2(0,1)\,,$$

$$\vartheta^{(2s)}(1) = 0,\ \ \vartheta^{(2s+1)}(0) = \vartheta^{(2s)}(1)\ \ (s = \overline{0, i-1}\ ),$$

then

$$\sum_{k=1}^{\infty}\left(\lambda_k^{2i}\vartheta_{2k-1}\ \right)^2 \le \frac{1}{2}\left\|\vartheta^{(2i)}(x)\right\|^2_{L_2(0,1)}\,,$$

$$\sum_{k=1}^{\infty}\left(\lambda_k^{2i}\vartheta_{2k}\ \right)^2 \le \frac{1}{2}\left\|\vartheta^{(2i)}(x)x + 2i\vartheta^{(2i-1)}(x)\right\|^2_{L_2(0,1)}\,. \quad (9)$$

Under assumptions

$$\vartheta(x) \in C^{2i}[0,1],\quad \vartheta^{(2i+1)}(x) \in L_2(0,1)\,,$$

$$\vartheta^{(2s)}(1) = 0,\ \ \vartheta^{(2s-1)}(0) = \vartheta^{(2s-1)}(1)\ \ (i \ge 1,\ s = \overline{0, i}\ ),$$

the validity of estimates [15]:

$$\sum_{k=1}^{\infty}\left(\lambda_k^{2i+1}\vartheta_{2k-1}\ \right)^2 \le \frac{1}{2}\left\|\vartheta^{(2i+1)}(x)\right\|^2_{L_2(0,1)}\,,$$

$$\sum_{k=1}^{\infty}\left(\lambda_k^{2i+1}\vartheta_{2k}\ \right)^2 \le \frac{1}{2}\left\|\vartheta^{(2i+1)}(x)x + (2i+1)\vartheta^{(2i)}(x)\right\|^2_{L_2(0,1)}\,. \quad (10)$$

is established. In order to study problem (1) - (3), (6), consider the following space.

Denote by $B_{2,T}^5$ [15] the collection of all functions $u\,(x,t)$ of the form

$$u\,(x,t)=\sum_{k=0}^{\infty}u_k\,(t)\,X_k\,(x)\,,$$

considered on D_T, for which all functions $u_k\,(t)\in C[0,T]$ and

$$J_T(u)\equiv\|u_0(t)\|_{C[0,T]}+$$

$$+\left(\sum_{k=1}^{\infty}\left(\lambda_k^5\,\|u_{2k-1}(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}+\left(\sum_{k=1}^{\infty}\left(\lambda_k^5\,\|u_{2k}(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}},$$

where the function $X_k\,(x)\ \ (k=0,1,2,...)$ are defined by (7).

The norm in this set is defined as follows: $\|u(x,t)\|_{B_{2,T}^5}=J_{"}(u)$.

Let E_T^5 denote the space of vector functions $\{u(x,t),a(t)\}$ such that $u\,(x,t)\in B_{2,T}^5$, $a\,(t)\in C[0,T]$. Equip this space with a norm

$$\|z\|_{E_T^5}=\|u(x,t)\|_{B_{2,T}^5}+\|a(t)\|_{C[0,T]}\,.$$

It is clear that $B_{2,T}^5$ and E_T^5 are Banach spaces.

Since system (7) forms a Riesz basis in $L_2\,(0,1)$ and system (7) and (8) forms biorthogonal to the system of functions in $L_2\,(0,1)$, then the first component $u\,(x,t)$ of the solution $\{u(x,t),a(t)\}$ of problem (1) - (3), (6) will be sought in the form

$$u\,(x,t)=u_0\,(t)\,X_0\,(x)+\sum_{k=1}^{\infty}u_{2k-1}\,(t)\,X_{2k-1}\,(x)+\sum_{k=1}^{\infty}u_{2k}\,(t)\,X_{2k}\,(x)\,, \tag{11}$$

where

$$u_0(t)=\int_0^1 u(x,t)Y_0(x)dx,$$

$$u_{2k-1}(t)=\int_0^1 u(x,t)Y_{2k-1}(x)dx,\ \ u_{2k}(t)=\int_0^1 u(x,t)Y_{2k}(x)dx\ \ (k=1,2,...)\,, \tag{12}$$

is the solution of the following problem:

$$u_0''\,(t)=F_0\,(t;u,a)\quad(0\le t\le T), \tag{13}$$

$$u_{2k-1}''(t)+\beta_k^2u_{2k-1}(t)=\frac{1}{1+\beta\lambda_k^2}F_{2k-1}\,(t;u,a)\quad(0\le t\le T,\ k=1,2,...), \tag{14}$$

$$u_{2k}''\,(t)+\beta_k^2u_{2k}\,(t)=\frac{1}{1+\beta\lambda_k^2}F_{2k}\,(t;u,a)+$$

$$+\frac{2\lambda_k(1+2\alpha\lambda_k^2)}{1+\beta\lambda_k^2}u_{2k-1}\,(t)+\frac{2\beta\lambda_k}{1+\beta\lambda_k^2}\,u_{2k-1}''\,(t)\ \ (0\le t\le T,\ \ k=1,2,...), \tag{15}$$

$$u_k(0)=\varphi_k,\quad u_k'(0)=\psi_k\quad (k=0,1,2,...)\,, \tag{16}$$

moreover

$$\beta_k^2=\frac{\lambda_k^2(1+\alpha\lambda_k^2)}{1+\beta\lambda_k^2},\,F_k(t;u,a)=a(t)u_k(t)+f_k(t),\quad f_k(t)=\int_0^1 f(x,t)Y_k(x)dx,$$

$$\varphi_k=\int_0^1\varphi(x)Y_k(x)dx,\quad \psi_k=\int_0^1\psi(x)Y_k(x)dx\ \ (k=0,1,...).$$

Solving problem (13) - (16) we find:

$$u_0(t)=\varphi_0+\psi_0t+\int_0^t(t-\tau)F_0(\tau;u,a)d\tau, \tag{17}$$

$$u_{2k-1}(t)\;=\;\varphi_{2k-1}\cos\beta_kt+\frac{1}{\beta_k}\psi_{2k-1}\sin\beta_kt+\frac{1}{\beta_k(1\;+\;\beta\lambda_k^2)}\int_0^t F_{2k-1}(\tau;u,a)\sin\beta_k(t-\tau)d\tau, \tag{18}$$

$$u_{2k}(t)\;=\;\varphi_{2k}\cos\beta_kt\;+\;\frac{1}{\beta_k}\psi_{2k}\sin\beta_kt\;+\;\frac{1}{\beta_k(1\;+\;\beta\lambda_k^2)}\int_0^t F_{2k}(\tau;u,a)\sin\beta_k(t\;-\;\tau)d\tau+$$

$$+\frac{\lambda_k(1+2\alpha\lambda_k^2+\alpha\beta\lambda_k^4)}{(1+\beta\lambda_k^2)^3}\left[t\varphi_{2k-1}\sin\beta_kt+\left(\frac{1}{\beta_k}\sin\beta_kt-t\cos\beta_kt\right)\frac{1}{\beta_k}\psi_{2k-1}+\right.$$

$$\left.+\frac{1}{\beta_k(1+\beta\lambda_k^2)}\int_0^t\left(\int_0^\tau F_{2k-1}\left(\xi;u,a\right)\sin\beta_k\left(t-\xi\right)d\xi\right)\sin\beta_k\left(t-\tau\right)d\tau\right]+$$

$$+\frac{2\beta\lambda_k}{\beta_k(1+\beta\lambda_k^2)^2}\int_0^t F_{2k-1}(\tau;u,a)\sin\lambda_k(t-\tau)d\tau. \tag{19}$$

After substituting the expression $u_k(t)$ $(k\;=\;0,1,...)$ in (11), to determine the component $u(x,t)$ of the solution to problem (1) - (3), (6), we obtain:

$$u(x,t)=\left(\varphi_0+\psi_0t+\int_0^t(t-\tau)F_0(\tau;u,a)d\tau\right)X_0(x)+$$

$$+\left\{\varphi_{2k-1}\cos\beta_kt\;+\;\frac{1}{\beta_k}\psi_{2k-1}\sin\beta_kt+\right.$$

$$\left.\frac{1}{\beta_k(1\;+\;\beta\lambda_k^2)}\int_0^t F_{2k-1}(\tau;u,a)\sin\beta_k(t\;-\;\tau)d\tau\right\}X_{2k-1}(x)+$$

$$+\sum_{k=1}^{\infty}\left\{\varphi_{2k}\cos\beta_kt\;+\;\frac{1}{\beta_k}\psi_{2k}\sin\beta_kt\;+\;\frac{1}{\beta_k(1\;+\;\beta\lambda_k^2)}\int_0^t F_{2k}(\tau;u,a)\sin\beta_k(t\;-\;\tau)d\tau+\right.$$

$$+\frac{\lambda_k(1+2\alpha\lambda_k^2+\alpha\beta\lambda_k^4)}{(1+\beta\lambda_k^2)^3}\left[t\varphi_{2k-1}\sin\beta_kt+\left(\frac{1}{\beta_k}\sin\beta_kt-t\cos\beta_kt\right)\frac{1}{\beta_k}\psi_{2k-1}+\right.$$

$$+\frac{1}{\beta_k(1+\beta\lambda_k^2)}\int_0^t\left(\int_0^\tau F_{2k-1}\left(\xi;u,a\right)\sin\beta_k\left(t-\xi\right)d\xi\right)\sin\beta_k\left(t-\tau\right)d\tau\Bigg]+$$

$$+\frac{2\beta\lambda_k}{\beta_k(1+\beta\lambda_k^2)^2}\int_0^t F_{2k-1}(\tau;u,a)\sin\lambda_k(t-\tau)d\tau\Bigg\}X_{2k}(x). \tag{20}$$

Now, from (6), taking into account (11), we have:

$$a(t)=[h(t)]^{-1}\left\{h''(t)-f(0,t)+4\sum_{k=1}^{\infty}((\lambda_k^2+\alpha\lambda_k^4)u_{2k-1}(t)+\beta\lambda_k^2u''_{2k-1}(t))\right\}. \tag{21}$$

Further, from (14), taking into account (18), we obtain:

$$(\lambda_k^2+\alpha\lambda_k^4)u_{2k-1}(t)+\beta\lambda_k^2u''_{2k-1}(t)=$$

$$=F_{2k-1}(t;u,a)-u''_{2k-1}(t)=\frac{\beta\lambda_k^2}{1+\beta\lambda_k^2}F_{2k-1}(t;u,a)-\beta_k^2u_{2k-1}(t)=$$

$$=\frac{\beta\lambda_k^2}{1+\beta\lambda_k^2}F_{2k-1}(t;u,a)-\beta_k^2\Bigg(\varphi_{2k-1}\cos\beta_kt+\frac{1}{\beta_k}\psi_{2k-1}\sin\beta_kt+$$

$$+\frac{1}{\beta_k(1+\beta\lambda_k^2)}\int_0^t F_{2k-1}(\tau;u,a)\sin\beta_k(t-\tau)d\tau\Bigg). \tag{22}$$

In order to obtain an equation for the second component $a(t)$ of the solution $\{u(x,t),a(t)\}$ to problem (1) - (3), (6), we substitute expression (22) into (21):

$$a(t)=[h(t)]^{-1}\Bigg\{h''(t)-f(0,t)+4\sum_{k=1}^{\infty}\Bigg[\frac{\beta\lambda_k^2}{1+\beta\lambda_k^2}F_{2k-1}(t;u,a)-$$

$$-\beta_k^2\left(\varphi_{2k-1}\cos\beta_kt+\frac{1}{\beta_k}\psi_{2k-1}\sin\beta_kt+\frac{1}{\beta_k(1+\beta\lambda_k^2)}\int_0^t F_{2k-1}(\tau;u,a)\sin\beta_k(t-\tau)d\tau\right)\Bigg]\Bigg\}. \tag{23}$$

Thus, the solution of problem (1) - (3), (6) is reduced to the solution of system (20), (23) with respect to unknown functions $u(x,t)$ and $a(t)$.

To study the question of uniqueness of the solution of problem (1) - (3), (6), the following lemma plays an important role.

Lemma 1. *If $\{u(x,t),a(t)\}$ is any solution to problem* (1) - (3), (6), *then the functions $u_k(t)$ $(k=0,1,2,\ldots)$ defined by relation* (12) *satisfy the counting system* (17), (18) *and* (19) *on* $[0,T]$.

Obviously, if $u_k(t)=\int_0^1 u(x,t)Y_k(x)dx \quad (k=0,1,...)$ is a solution to system (17), (18) and (19), then a pair $\{u(x,t),a(t)\}$ of functions $u(x,t)=\sum_{k=0}^{\infty}u_k(t)X_k(x)$ and $a(t)$ is a solution to system (20), (23).

Lemma 1 has the following

Corollary 1. *Let system* (20), (23) *have a unique solution. Then problem* (1) - (3), (6) *cannot have more than one solution, i.e. if problem* (1) - (3), (6) *has a solution, then it is unique.*

Now consider the following operator in space E_T^5

$$\Phi(u,a)=\{\Phi_1(u,a),\Phi_2(u,a)\}\ ,$$

where

$$\Phi_1(u,a)=\tilde{u}(x,t)=\sum_{k=0}^{\infty}\tilde{u}_k(t)X_k(x),\Phi_2(u,a)=\tilde{a}(t),$$

and $\tilde{u}_0(t),\ \tilde{u}_{2k-1}(t),\ \tilde{u}_{2k}(t)$ and $\tilde{a}(t)$ equal corresponding to the right side (17), (18) , (19) and (23).

It is easy to see that

$$1+\beta\lambda_k^2>\beta\lambda_k^2,\quad \frac{1}{1+\beta\lambda_k^2}<\frac{1}{\beta\lambda_k^2},$$

$$\sqrt{\frac{\alpha}{1+\beta}}\,\lambda_k\le\beta_k\le\sqrt{\frac{1+\alpha}{\beta}}\,\lambda_k,\ \ \sqrt{\frac{\beta}{1+\alpha}}\,\frac{1}{\lambda_k}\le\frac{1}{\beta_k}\le\sqrt{\frac{1+\beta}{\alpha}}\,\frac{1}{\lambda_k},$$

Taking these relations into account, we find:

$$\|\tilde{u}_0\,(t)\|_{C[0,T]}\le|\varphi_0|+T\,|\psi_0|+T\sqrt{T}\left(\int_0^T|f_0\,(\tau)|^2\,d\tau\right)^{\frac{1}{2}}+T^2\,\|a(t)\|_{C[0,T]}\,\|u_0(t)\|_{C[0,T]}\,, \tag{24}$$

$$\left(\sum_{k=1}^{\infty}(\lambda_k^5\,\|\tilde{u}_{2k-1}(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}\le$$

$$\le 2\left(\sum_{k=1}^{\infty}(\lambda_k^5\,|\varphi_{2k-1}|)^2\right)^{\frac{1}{2}}+\,2\sqrt{\frac{1+\beta}{\alpha}}\,\left(\sum_{k=1}^{\infty}(\lambda_k^4\,|\,\psi_{2k-1}\,|)^2\right)\,+\,\frac{2}{\beta}\sqrt{\frac{1+\beta}{\alpha}}$$

$$\left[\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k^2\,|\,f_{2k-1}\,(\tau)\,|)^2d\tau\right)^{\frac{1}{2}}+\,T\,\|a(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^5\,\|u_{2k-1}(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}\right], \tag{25}$$

$$\left(\sum_{k=1}^{\infty}(\lambda_k^5\,\|\tilde{u}_{2k}(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}\le$$

$$\le 3\left(\sum_{k=1}^{\infty}(\lambda_k^5\,|\varphi_{2k}|)^2\right)^{\frac{1}{2}}+3\,\sqrt{\frac{1+\beta}{\alpha}}\,\left(\sum_{k=1}^{\infty}(\lambda_k^4\,|\,\psi_{2k}\,|)^2\right)^{\frac{1}{2}}\,+\,\frac{3}{\beta}\sqrt{\frac{1+\beta}{\alpha}}$$

$$\left[\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k^2\mid f_{2k}(\tau)\mid)^2 d\tau\right)^{\frac{1}{2}}+T\left\|a(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^5\left\|u_{2k}(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}\right]+$$

$$+\frac{3(1+2\alpha+\alpha\beta)}{\beta^3}\left[T\left(\sum_{k=1}^{\infty}(\lambda_k^5\left|\varphi_{2k-1}\right|)^2\right)^{\frac{1}{2}}+\left(\sqrt{\frac{1+\beta}{\alpha}}+T\right)\sqrt{\frac{1+\beta}{\alpha}}\left(\sum_{k=1}^{\infty}(\lambda_k^4\left|\psi_{2k-1}\right|)^2\right)^{\frac{1}{2}}+\right.$$

$$+\frac{1}{\beta}\sqrt{\frac{1+\beta}{\alpha}}\left(T\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k^2\mid f_{2k-1}(\tau)\mid)^2 d\tau\right)^{\frac{1}{2}}+\right.$$

$$\left.\left.+T^2\left\|a(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^5\left\|u_{2k-1}(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}\right)\right]+$$

$$+\frac{6}{\beta}\sqrt{\frac{1+\beta}{\alpha}}\left[\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k^2\mid f_{2k-1}(\tau)\mid)^2 d\tau\right)^{\frac{1}{2}}+\right.$$

$$\left.+T\left\|a(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^5\left\|u_{2k-1}(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}\right], \tag{26}$$

$$\left\|\tilde{a}(t)\right\|_{C[0,T]}\leq\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left\{\left\|h''(t)-f(0,t)\right\|_{C[0,T]}+\right.$$

$$+4\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\left\{\frac{1+\alpha}{\beta}\left[\left(\sum_{k=1}^{\infty}(\lambda_k^5\left|\varphi_{2k-1}\right|)^2\right)^{\frac{1}{2}}+\sqrt{\frac{1+\beta}{\alpha}}\left(\sum_{k=1}^{\infty}(\lambda_k^4\mid\psi_{2k-1}\mid)^2\right)+\right.\right.$$

$$+\frac{1}{\beta}\sqrt{\frac{1+\beta}{\alpha}}\left[\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k^2\mid f_{2k-1}(\tau)\mid)^2 d\tau\right)^{\frac{1}{2}}+\right.$$

$$\left.\left.+T\left\|a(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^5\left\|u_{2k-1}(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}\right]\right]+$$

$$\left.\left.+\left(\sum_{k=1}^{\infty}(\lambda_k^2\left\|f_{2k}(t)\right\|_{C[0,T]}\mid\right)^{\frac{1}{2}}+\left\|a(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^5\left\|u_{2k-1}(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}\right\}\right\}. \tag{27}$$

Suppose that the data of problem (1) - (3), (6) satisfy the following conditions:
1.$\alpha>0,\beta>0,h(t)\in C^2[0,T],\ h(t)\neq 0\ (0\leq t\leq T)$.
2.$\varphi(x)\in C^4[0,1],\ \varphi^{(5)}(x)\in L_2(0,1),\ \varphi(1)=0,\varphi'(0)=\varphi'(1)$,

$\varphi''(1)=0, \varphi'''(0)=\varphi'''(1),\ \varphi^{(4)}(1)=0.$
3.$\psi(x)\in C^3[0,1],\ \psi^{(4)}(x)\in L_2(0,1),\ \psi(1)=0,\ \psi'(0)=\psi'(1),\ \psi''(1)=0, \psi'''(0)=\psi'''(1).$
4.$f(x,t), f_x(x,t)\in C(D_T),\ f_{xx}(x,t)\in L_2(D_T),\ f(1,t)=0,\ \ f_x(0,t)=f_x(1,t)\ \ (0\le t\le T).$

Then from (24)- (27) we find:

$$\|\tilde{u}(x,t)\|_{B^5_{2,T}}=A_1(T)+B_1(T)\,\|a(t)\|_{C[0,T]}\,\|u(x,t)\|_{B^5_{2,T}}\,, \tag{28}$$

$$\|\tilde{a}(t)\|_{C[0,T]}=A_2(T)+B_2(T)\,\|a(t)\|_{C[0,T]}\,\|u(x,t)\|_{B^5_{2,T}}\,, \tag{29}$$

where

$$A_1(T)=\|\varphi(x)\|_{L_2(0,1)}+T\,\|\psi(x)\|_{L_2(0,1)}+T\sqrt{T}\,\|f(x,t)\|_{L_2(D_T)}+\sqrt{2}\left\|\varphi^{(5)}(x)\right\|_{L_2(0,1)}+$$
$$+\sqrt{\frac{2(1+\beta)}{\alpha}}\left\|\psi^{(4)}(x)\right\|_{L_2(0,1)}+\sqrt{\frac{2(1+\beta)}{\alpha}}\,\|f_{xx}(x,t)\|_{L_2(D_T)}+\frac{3}{\sqrt{2}}\left\|\varphi^{(5)}(x)+4\varphi^{(3)}(x)\right\|_{L_2(0,1)}+$$
$$+\frac{3}{\sqrt{2}}\sqrt{\frac{1+\beta}{\alpha}}\left\|\psi^{(4)}(x)+3\psi^{(3)}(x)\right\|_{L_2(0,1)}+\frac{3}{\beta}\sqrt{\frac{T(1+\beta)}{2\alpha}}\,\|f_{xx}(x,t)+2f_x(x,t)\|_{L_2(D_T)}+$$
$$+\frac{3(1+2\alpha+\alpha\beta)}{\beta^3}\left(\frac{T}{\sqrt{2}}\left\|\varphi^{(5)}(x)\right\|_{L_2(0,1)}+\left(\sqrt{\frac{1+\beta}{\alpha}}+T\right)\sqrt{\frac{1+\beta}{2\alpha}}\left\|\psi^{(4)}(x)\right\|_{L_2(0,1)}+\right.$$
$$\left.+\frac{T}{\beta}\sqrt{\frac{T(1+\beta)}{2\alpha}}\,\|f_{xx}(x,t)\|_{L_2(D_T)}\right)+\frac{6}{\beta}\sqrt{\frac{T(1+\beta)}{\alpha}}\,\|f_{xx}(x,t)\|_{L_2(D_T)}\,,$$

$$B_1(T)=T^2+\frac{11T}{\beta}\sqrt{\frac{1+\beta}{\alpha}}\left(1+\frac{3(1+2\alpha+\alpha\beta)}{\beta^3}T\right),$$

$$A_2(T)=\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left\{\left\|h''(t)-f(0,t)\right\|_{C[0,T]}+\right.$$
$$+2\sqrt{2}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\left\{\frac{1+\alpha}{\beta}\left[\left\|\varphi^{(5)}(x)\right\|_{L_2(0,1)}+\sqrt{\frac{1+\beta}{\alpha}}\ \left\|\psi^{(4)}(x)\right\|_{L_2(0,1)}+\right.\right.$$
$$\left.\left.\left.+\frac{1}{\beta}\sqrt{\frac{T(1+\beta)}{\alpha}}\,\|f_{xx}(x,t)\|_{L_2(D_T)}\right]+\left\|\|f_{xx}(x,t)\|_{C[0,T]}\right\|_{L_2(0,1)}\right\},$$

$$B_2(T)=2\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left(\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\frac{1+\alpha}{\beta^2}\sqrt{\frac{1+\beta}{\alpha}}T+1\right).$$

From inequalities (27), (28) we conclude:

$$\|\tilde{u}(x,t)\|_{B^5_{2,T}}+\|\tilde{a}(t)\|_{C[0,T]}\le A(T)+B(T)\,\|a(t)\|_{C[0,T]}\,\|u(x,t)\|_{B^5_{2,T}}\,, \tag{30}$$

where

$$A(T) = A_1(T) + A_2(T), \quad B(T) = B_1(T) + B_2(T).$$

So, the following theorem is proved.

Theorem 2. *Let conditions 1-4 be satisfied and*

$$B(T)(A(T)+2)^2 < 1. \tag{31}$$

Then problem (1)-(3)*, (6) has a unique solution in the ball* $K = K_R(||\, z\, ||_{E_T^5} \leq R = A(T)+2)$ *from* E_T^5*.*

Proof. In the space E_T^5, consider the equation

$$z = \$z, \tag{32}$$

where $z = \{u, a\}$, the components $\$_i(u,a)\,(i=1,2)$ of the operator $\$(u,a)$ are defined by the right-hand sides of equations (20), (23), respectively.

Consider an operator $\$(u,a)$ in a ball $K = K_R$ of E_T^5. Similarly, from (30) we obtain that for any z, z_1, $z_2 \in K_R$ the following estimates are valid:

$$\|\$z\|_{E_T^5} \leq A(T) + B(T)\,\|a(t)\|_{C[0,T]}\,\|u(x,t)\|_{B_{2,T}^5} \leq A(T) + B(T)(A(T)+2)^2, \tag{33}$$

$$\|\$z_1 - \$z_2\|_{E_T^5} \leq B(T)R\left(\|a_1(t)-a_2(t)\|_{C[0,T]} + \|u_1(x,t)-u_2(x,t)\|_{B_{2,T}^5}\right). \tag{34}$$

Then, taking into account (31), it follows from estimates (33), (34) that the operator \$ acts in the ball $K = K_R$ and is contracting. Therefore, in the ball $K = K_R$, the operator \$ has a unique fixed point $\{u, a\}$, which is a solution to equation (32), that is, is the only solution in the ball $K = K_R$ to system (20), (23).

A function $u(x,t)$ as an element of space $B_{2,T}^5$, has continuous derivatives $u(x,t)$, $u_x(x,t)$, $u_{xx}(x,t)$, $u_{xxx}(x,t)$, $u_{xxxx}(x,t)$ in D_T.

Similarly to [10], one can show that $u_t(x,t)$, $u_{tt}(x,t)$, $u_{tt}(x,t)$, $u_{ttx}(x,t)$, $u_{ttxx}(x,t)$ are continuous in D_T.

It is easy to check that equation (2) and conditions (2), (3) and (6) are satisfied in the usual sense. Hence, $\{u(x,t), a(t)\}$ is a solution to problem (1) - (3), (6), and by virtue of the corollary to Lemma 1, it is unique. ◄

Using Theorem 1, we prove the following

Theorem 3. *Let all conditions of Theorem 2 be satisfied and the conditions of consistency*

$$\varphi(0) = h(0)\ , \quad \psi(0) = h'0).$$

Then problem (1) - (4) *has a unique classical solution in the ball* $K = K_R(||\, z\, ||_{E_T^5} \leq R = A(T)+2)$ *from* E_T^5*.*

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