Caspian Journal of Applied Mathematics, Ecology and Economics
V. 8, No 2, 2020, December
ISSN 1560-4055

An Application of Generalized Distribution Series on Certain Classes of Univalent Functions

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Abstract. The purpose of the present paper is to obtain some sufficient conditions for generalized distribution series belonging to the classes $\mathcal{S}^*(\alpha,\beta,\gamma)$, $\mathcal{K}(\alpha,\beta,\gamma)$ and inclusion relation of these subclasses by $\Re^\tau(A,B)$. Finally, we obtain some necessary and sufficient conditions of an integral operator associated with the generalized distribution series.

Key Words and Phrases: generalized distribution, Analytic, univalent functions, convex function and starlike functions.

2010 Mathematics Subject Classifications: 30C45

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1. Introduction

Let $\mathcal{A}$ denote the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z : z \in C \text{ and } |z| < 1\}$. As usual, by $\mathcal{S}$ we shall represent the class of all functions in $\mathcal{A}$ which are univalent in $\mathbb{U}$ and further, we denote $\mathcal{T}$ be the subclass of $\mathcal{S}$ consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n. \tag{2}$$

The convolution (or Hadamard product) of two series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ is defined as the power series

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n.$$

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A function $f \in \mathcal{A}$ is said to be starlike of order α $(0 \leq \alpha < 1)$, if and only if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \quad (z \in \mathbb{U}).$$

This function class is denoted by $\mathcal{S}^*(\alpha)$. We also write $\mathcal{S}^*(0) =: \mathcal{S}^*$, where $\mathcal{S}^*$ denotes the class of functions $f \in \mathcal{A}$ that $f(\mathbb{U})$ is starlike with respect to the origin.

A function $f \in \mathcal{A}$ is said to be convex of order α $(0 \leq \alpha < 1)$, if and only if

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \quad (z \in \mathbb{U}).$$

This class is denoted by $\mathcal{K}(\alpha)$. Further, $\mathcal{K} = \mathcal{K}(0)$, the well-known standard class of convex functions.

It is an established fact that $f \in \mathcal{K}(\alpha) \Longleftrightarrow zf' \in \mathcal{S}^*(\alpha)$.

We recall the class $\mathcal{S}^*(\alpha, \beta, \gamma)$ defined and studied by Kulkarni [10].

Let $\mathcal{S}^*(\alpha, \beta, \gamma)$ the subclass of $\mathcal{T}$ consisting of functions which satisfy the condition

$$\left|\frac{f'(z) - 1}{2\gamma(f'(z) - \alpha) - (f'(z) - 1)}\right| < \beta, \quad (z \in \mathbb{U}). \tag{3}$$

where $0 < \beta \leqq 1, 0 \leqq \alpha < \frac{1}{2\gamma}$ and $\frac{1}{2} \leqq \gamma \leqq 1$

Recently, some conditions of hypergeometric functions on the class $\mathcal{S}^*(\alpha, \beta, \gamma)$ have been studied by Joshi *et al.* [9].

Now, we define a new class $\mathcal{K}(\alpha, \beta, \gamma)$ be the subclass of $\mathcal{T}$ consisting of functions which satisfy the condition

$$\left|\frac{zf''(z) + f'(z) - 1}{2\gamma(zf''(z) + f'(z) - \alpha) - (zf''(z) + f'(z) - 1)}\right| < \beta, \quad (z \in \mathbb{U}). \tag{4}$$

where $0 < \beta \leqq 1, 0 \leqq \alpha < \frac{1}{2\gamma}$ and $\frac{1}{2} \leqq \gamma \leqq 1$.

By using (3) and (4) we note that

$$f(z) \in \mathcal{K}(\alpha, \beta, \gamma) \Leftrightarrow zf'(z) \in \mathcal{S}^*(\alpha, \beta, \gamma).$$

A function $f \in \mathcal{A}$ is said to be in the class $f \in \Re^\tau(A, B)$ $(\tau \in \mathbb{C}\backslash\{0\},\ -1 \leq B < A \leq 1)$, if it satisfies the inequality

$$\left|\frac{f'(z) - 1}{(A - B)\tau - B[f'(z) - 1]}\right| < 1, \quad (z \in \mathbb{U}).$$

The class $\Re^\tau(A, B)$ was introduced earlier by Dixit and Pal [7].

The applications of hypergeometric functions ([9], [20]), confluent hypergeometric functions [5], generalized hypergeometric functions [8], Wright function [17], Fox-Wright function [6], generalized Bessel functions ([4], [15]) are interesting topics of research in Geometric Function Theory. In 2014, Porwal [13] (see also [2], [11]) introduced Poisson distribution series and obtain necessary and sufficient conditions for certain classes of univalent functions and co-relates probability density function with Geometric Function Theory. After the appearance of this paper several researchers introduced hypergeometric distribution series [1], confluent hypergeometric distribution series [16], Binomial distribution series [12], Mittag-Leffler type Poisson distribution series [3] and obtain some interesting properties of various classes of univalent functions. Recently Porwal [14] introduced generalized distribution series and obtain some necessary and sufficient conditions belonging to the certain classes of univalent functions. Now, we recall the definition of generalized distribution. Let the series $\sum_{n=0}^{\infty} t_n$, where $t_n \geq 0, \forall n \in N$ is convergent and its sum is denoted by S, i.e.

$$S = \sum_{n=0}^{\infty} t_n. \tag{5}$$

Now, we introduce the generalized discrete probability distribution whose probability mass function is

$$p(n) = \frac{t_n}{S}, \quad n = 0, 1, 2, \ldots \tag{6}$$

Obviously $p(n)$ is a probability mass function because $p(n) \geq 0$ and $\sum_n p_n = 1$.

Now, we introduce the series

$$\phi(x) = \sum_{n=0}^{\infty} t_n x^n. \tag{7}$$

From (5) it is easy to see that the series given by (7) is convergent for $|x| < 1$ and for $x = 1$ it is also convergent.

Now, we introduce a power series whose coefficients are probabilities of the generalized distribution

$$K_\phi(z) = z + \sum_{n=2}^{\infty} \frac{t_{n-1}}{S} z^n. \tag{8}$$

Further, we define the function

$$TK_\phi(z) = z - \sum_{n=2}^{\infty} \frac{t_{n-1}}{S} z^n. \tag{9}$$

Next, we introduce the convolution operator $TK_\phi(f, z)$ for functions f of the form (2) as follows

$$TK_\phi(f, z) = K_\phi(z) * f(z) = z - \sum_{n=2}^{\infty} |a_n| \frac{t_{n-1}}{S} z^n. \tag{10}$$

In the present paper, motivated with the above mentioned work, we obtain necessary and sufficient conditions for generalized distribution series belonging to the classes $\mathcal{K}(\alpha, \beta, \gamma)$, $\mathcal{S}^*(\alpha, \beta, \gamma)$ and inclusion relation of these subclasses by $\Re^\tau(A, B)$.

2. Main Results

To establish our main results we shall require the following lemmas.

Lemma 1. *([9]) A function $f \in \mathcal{A}$ and of the form* (1) *belongs to the class $\mathcal{S}^*(\alpha,\beta,\gamma)$ if*

$$\sum_{n=2}^{\infty} n[1+\beta(1-2\gamma)]|a_n| \leq 2\beta\gamma(1-\alpha). \tag{11}$$

Our next lemma is a direct consequences of definition (4).

Lemma 2. *A function $f \in \mathcal{A}$ and of the form* (1) *belongs to the class $\mathcal{K}(\alpha,\beta,\gamma)$ if*

$$\sum_{n=2}^{\infty} n^2[1+\beta(1-2\gamma)]|a_n| \leq 2\beta\gamma(1-\alpha).$$

Lemma 3. *[7] A function $f \in \Re^\tau(A,B)$ is of form* (1)*, then*

$$|a_n| \leq (A-B)\frac{|\tau|}{n}, \quad n \in \mathbb{N}\backslash\{1\}. \tag{12}$$

The bound given in (12) *is sharp.*

Theorem 1. *If $f \in \mathcal{A}$ is of the form* (1) *and the inequality*

$$(1+\beta(1-2\gamma))\left[\phi''(1)+3\phi'(1)+(\phi(1)-\phi(0))\right] \leq 2\beta\gamma(1-\alpha)S \tag{13}$$

is satisfied, then $K_\phi(z)$ is of the form (8) *is in the class $\mathcal{K}(\alpha,\beta,\gamma)$.*

Proof. To prove that $K_\phi(z) \in \mathcal{K}(\alpha,\beta,\gamma)$from Lemma 2 it suffices to prove that

$$\sum_{n=2}^{\infty} n^2[1+\beta(1-2\gamma)]\frac{t_{n-1}}{S} \leq 2\beta\gamma(1-\alpha).$$

Now

$$\begin{aligned}
&\sum_{n=2}^{\infty} n^2[1+\beta(1-2\gamma)]\frac{t_{n-1}}{S} \\
&= \frac{1+\beta(1-2\gamma)}{S}\left[\sum_{n=2}^{\infty} n^2\right] t_{n-1} \\
&= \frac{1+\beta(1-2\gamma)}{S}\sum_{n=2}^{\infty}\left[(n-1)(n-2)+3(n-1)+1\right] t_{n-1} \\
&= \frac{1+\beta(1-2\gamma)}{S}\sum_{n=1}^{\infty}\left[n(n-1)t_n+3nt_n+t_n\right]
\end{aligned}$$

$$= \frac{1+\beta(1-2\gamma)}{S}\left[\phi''(1) + 3\phi'(1) + (\phi(1)-\phi(0))\right]$$
$$\le 2\beta\gamma(1-\alpha).$$

This completes the proof of Theorem 1. ◄

Theorem 2. *If $f \in \mathcal{A}$ is of the form* (1) *and the inequality*

$$[1+\beta(1-2\gamma)]\left[\phi'(1) + (\phi(1)-\phi(0))\right] \le 2\beta\gamma(1-\alpha)S. \tag{14}$$

is satisfied, then $K_\phi(z)$ is of the form (8) *is in the class $\mathcal{S}^*(\alpha,\beta,\gamma)$.*

Proof. To prove that $K_\phi(z) \in \mathcal{S}^*(\alpha,\beta,\gamma)$ from Lemma 1 it suffices to prove that

$$\sum_{n=2}^{\infty} n[1+\beta(1-2\gamma)]\frac{t_{n-1}}{S} \le 2\beta\gamma(1-\alpha).$$

Now

$$\sum_{n=2}^{\infty} n[1+\beta(1-2\gamma)]\frac{t_{n-1}}{S}$$
$$= \frac{[1+\beta(1-2\gamma)]}{S}\left[\sum_{n=2}^{\infty}(n-1)+1\right]t_{n-1}$$
$$= \frac{[1+\beta(1-2\gamma)]}{S}\sum_{n=1}^{\infty}[nt_n + t_n]$$
$$= \frac{[1+\beta(1-2\gamma)]}{S}\left[\phi'(1) + (\phi(1)-\phi(0))\right]$$
$$\le 2\beta\gamma(1-\alpha).$$

Thus the proof of Theorem 2 is established. ◄

Remark 1. *The conditions* (13) *and* (14) *are also necessary for the distribution series $TK_\phi(z)$ defined by* (9)*.*

Theorem 3. *If $f \in R^\tau(A,B)$ is of the form* (2) *and the operator $TK_\phi(f,z)$ defined by* (10) *is in the class $\mathcal{K}(\alpha,\beta,\gamma)$, if and only if*

$$\frac{(A-B)[1+\beta(1-2\gamma)]|\tau|}{S}\left[\phi'(1) + (\phi(1)-\phi(0))\right] \le 2\beta\gamma(1-\alpha). \tag{15}$$

Proof. To prove $TK_\phi(f,z) \in \mathcal{K}(\alpha,\beta,\gamma)$, from Lemma 2, it suffices to prove that

$$P_1 = \sum_{n=2}^{\infty} n^2[1+\beta(1-2\gamma)]|a_n| \le 2\beta\gamma(1-\alpha).$$

Since $f \in R^{\tau}(A,B)$ then by using Lemma 3 we have

$$|a_n| \leq \frac{(A-B)|\tau|}{n}.$$

Hence

$$\begin{aligned}
P_1 \leq & \frac{(A-B)|\tau|}{S} \sum_{n=2}^{\infty} n[1+\beta(1-2\gamma)]t_{n-1} \\
= & \frac{(A-B)[1+\beta(1-2\gamma)]|\tau|}{S} \left[\sum_{n=2}^{\infty}(n-1)+1 \right] t_{n-1} \\
= & \frac{(A-B)[1+\beta(1-2\gamma)]|\tau|}{S} \sum_{n=1}^{\infty} [nt_n + t_n] \\
= & \frac{(A-B)[1+\beta(1-2\gamma)]|\tau|}{S} \left[\phi'(1) + (\phi(1)-\phi(0)) \right] \\
\leq & 2\beta\gamma(1-\alpha).
\end{aligned}$$

Thus the proof of Theorem 3 is established. ◄

Theorem 4. *If $f \in R^{\tau}(A,B)$ is of the form* (2) *and the operator $TK_{\phi}(f,z)$ defined by* (10) *is in the class $\mathcal{S}^*(\alpha,\beta,\gamma)$, if and only if*

$$\frac{(A-B)[1+\beta(1-2\gamma)]|\tau|}{S} [\phi(1)-\phi(0)] \leq 2\beta\gamma(1-\alpha). \tag{16}$$

Proof. The proof of above theorem is similar to that of Theorem 3. Therefore we omit the details involved. ◄

3. An Integral Operator

In this section, we introduce an integral operator $TG_{\phi}(z)$ as follows

$$TG_{\phi}(z) = \int_0^z \frac{TK_{\phi(t)}}{t} dt, \tag{17}$$

and obtain a necessary and sufficient condition for $TG_{\phi}(z)$ belonging to the classes $\mathcal{S}^*(\alpha,\beta,\gamma)$ and $\mathcal{K}(\alpha,\beta,\gamma)$.

Theorem 5. *If $TK_{\phi}(z)$ defined by* (10)*, then $TG_{\phi}(z)$ defined by* (17) *is in the class $\mathcal{K}(\alpha,\beta,\gamma)$, if and only if* (14) *satisfies.*

Proof. Since

$$TG_{\phi}(z) = z - \sum_{n=2}^{\infty} \frac{t_{n-1}}{nS} z^n$$

by Lemma 2, we have to prove that

$$\sum_{n=2}^{\infty} n^2[1+\beta(1-2\gamma)]\frac{t_{n-1}}{nS} \leq 2\beta\gamma(1-\alpha).$$

Now

$$\begin{aligned}
&\sum_{n=2}^{\infty} n^2[1+\beta(1-2\gamma)]\frac{t_{n-1}}{nS} \\
=&\frac{1}{S}\sum_{n=2}^{\infty} n[1+\beta(1-2\gamma)]t_{n-1} \\
&= \frac{[1+\beta(1-2\gamma)]}{S}\sum_{n=1}^{\infty}[(n+1)-1]\,t_n \\
&= \frac{[1+\beta(1-2\gamma)]}{S}\sum_{n=1}^{\infty}[n+1]\,t_n \\
&= \frac{[1+\beta(1-2\gamma)]}{S}\left[\phi'(1)+[\phi(1)-\phi(0)]\right] \\
&\leq 2\beta\gamma(1-\alpha).
\end{aligned}$$

This completes the proof of Theorem 5. ◄

Theorem 6. *If* $TK_\phi(z)$ *defined by* (10)*, then* $TG_\phi(z)$ *defined by* (17) *is in the class* $\mathcal{S}^*(\alpha,\beta,\gamma)$*, if and only if*

$$\frac{[1+\beta(1-2\gamma)]}{S}\,[\phi(1)-\phi(0)] \leq 2\beta\gamma(1-\alpha).$$

satisfies.

Proof. The proof of above theorem is much akin to that of Theorem 5. Therefore we omit the details involved. ◄

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Received 03 July 2020
Accepted 23 December 2020