Caspian Journal of Applied Mathematics, Ecology and Economics
V. 9, No 1, 2021, July
ISSN 1560-4055

On Properties of Ahlfors-Beurling Transform of Finite Complex Measures

Khanim I. Nebiyeva

Abstract. In the present paper we study asymptotic behavior of the distribution function of the Ahlfors-Beurling transform of the finite complex atomic discrete measure, and using the notion of Q-integration introduced by Titchmarsh we prove that the restricted Ahlfors-Beurling transform of the finite complex measure, singular part which is concentrated at a finite number of points is Q-integrable and an analogue of the Riesz equality holds.

Key Words and Phrases: Ahlfors-Beurling transform, distribution function, asymptotic behavior, complex measure.

2010 Mathematics Subject Classifications: 44A15, 42B20.

1. Introduction

The Ahlfors–Beurling transform of a function $f \in L_p(C)$, $1 \le p < \infty$, is defined as the following singular integral

$$(Bf)(z) = -\frac{1}{\pi} \lim_{\varepsilon \to 0+} \int_{\{w \in C\,:\,|z-w|>\varepsilon\}} \frac{f(w)}{(z-w)^2} dm(w).$$

Let Ω be a bounded domain in the complex plane and $f \in L_1(\Omega)$. Namely, the restricted Ahlfors–Beurling transform B_Ω is defined as

$$(B_\Omega f)(z) = B(\chi_\Omega f)(z) = -\frac{1}{\pi} \lim_{\varepsilon \to 0+} \int_{\{w \in \Omega\,:\,|z-w|>\varepsilon\}} \frac{f(w)}{(z-w)^2} dm(w), \qquad z \in \Omega.$$

The Ahlfors–Beurling transform is one of the important operators in complex analysis. It is the "Hilbert transform" on complex plane. It has been shown in [1, 11,13, 22, 29, 31] that this transform plays an essential role in applications to the theory of quasiconformal mappings and to the Beltrami equation with discontinuous coefficients.

From the theory of singular integrals (see [26]) it is known that the Ahlfors–Beurling transform is a bounded operator in the space $L_p(\Omega)$, $1 < p < \infty$, that is, if $f \in L_p(\Omega)$, then $B_\Omega(f) \in L_p(\Omega)$ and

$$\|B_\Omega f\|_{L_p} \le C_p \|f\|_{L_p}. \qquad (1)$$

In the case $f \in L_1(\Omega)$ only the weak inequality holds,

$$m\{z \in \Omega: \ |(B_\Omega f)(z)| > \lambda\} \leq \frac{C_1}{\lambda}\|f\|_{L_1}, \tag{2}$$

where m stands for the Lebesgue measure, C_p, C_1 are constants independent of f. From inequalities (1), (2) it follows that the Ahlfors–Beurling transform of the function $f \in L_1(\Omega)$ satisfies the condition

$$m\{z \in \Omega: \ |(B_\Omega f)(z)| > \lambda\} = o(1/\lambda), \lambda \to +\infty.$$

In [13-16, 20, 21, 23, 27, 30] the boundedness of the operator B in other function spaces (in the spaces of Sobolev, Besov, Campanato, Morrey, etc.) was studied.

Note that the Ahlfors–Beurling transform of a function $f \in L_1(\Omega)$ is not Lebesgue integrable. In [7] the authors shows that the Ahlfors–Beurling transform of a function $f \in L_1(\Omega)$ is A-integrable on Ω and an analogue of the Riesz equality holds.

In the present paper we study asymptotic behavior of the distribution function of the Ahlfors-Beurling transform of the finite complex atomic discrete measure, and using the notion of Q-integration introduced by Titchmarsh we prove that the restricted Ahlfors-Beurling transform of the finite complex measure, singular part which is concentrated at a finite number of points is Q-integrable and an analogue of the Riesz equality holds.

2. Asymptotic behavior of the distribution function of the Ahlfors-Beurling transform of the finite complex atomic discrete measure

In this section we studying the asymptotic behavior of the distribution function of the Ahlfors-Beurling transform of the finite complex atomic discrete measure.

Definition 1. Let the set $X \in \mathbb{C}$ consist of a finite or countable number of elements. If there exist $\delta > 0$ such that for every $x, y \in X$ in the inequality $|x - y| \geq \delta$ holds, then the set X is called an atomic set.

Definition 2. If the measure ν is concentrated on an atomic set, then the measure ν is called an atomic discrete measure.

Let ν is atomic discrete measure on the complex plane. The function

$$(B\nu)(z) = -\frac{1}{\pi}\lim_{\varepsilon\to 0+}\int_{\{w\in C\,:\,|z-w|>\varepsilon}\frac{d\nu(w)}{(z-w)^2}, \quad z \in \mathbb{C}$$

is called the Ahlfors-Beurling transform of the measure ν.

It's obvious that the Ahlfors-Beurling transform of the atomic discrete measure ν is finite for any $z \in \mathbb{C}$, and if

$$\mathrm{supp}\nu = X = \{z_j\}_{j\in J}, \qquad \nu(z_j) = \alpha_j, \quad j \in J,$$

then

$$(B\nu)(z) = -\frac{1}{\pi}\sum_{j\in J}\frac{\alpha_j}{(z_j - z)^2}, \quad z \notin X,$$

$$(B\nu)(z) = -\frac{1}{\pi} \sum_{j\in J,\ j\neq j_0} \frac{\alpha_j}{(z_j - z)^2}, \quad z = z_{j_0} \in X.$$

Theorem 1. Let ν is atomic discrete measure on the complex plane. Then the equation

$$\lim_{\lambda\to+\infty} \lambda m\{z \in \mathbb{C} : \ |(B\nu)(z)| > \lambda\} = \|\nu\|$$

holds, where $\|\nu\|$ is a total variation of the measure ν.

Proof: Let supp$\nu = \{z_j\}_{j\in J}$ and $\nu(z_j) = \alpha_j, \ \ j \in J$. Denote

$$\delta_0 = \inf\{\rho(z_i, z_j) : \ z_i, z_j \in \text{supp}\nu\} > 0,$$

$$K_0 = \bigcup_{j\in J} U\left(z_j; \frac{\delta_0}{4}\right),$$

where $U(z;\epsilon) = \{w \in \mathbb{C} : |w - z| < \epsilon\}$. Then for every $z \in \mathbb{C} \setminus K_0$ we have

$$|(B\nu)(z)| = \frac{1}{\pi}\left|\sum_{j\in J} \frac{\alpha_j}{(z_j - z)^2}\right| \leq \frac{1}{\pi}\sum_{j\in J} \frac{|\alpha_j|}{|z_j - z|^2} \leq \frac{16}{\pi\delta_0^2}\|\nu\|.$$

This show that for any $\lambda > \frac{16}{\pi\delta_0^2}\|\nu\|$ the set $\{z \in \mathbb{C} \setminus K_0 : |(B\nu)(z)| > \lambda\}$ is empty, and therefore

$$m\{z \in \mathbb{C} \setminus K_0 : |(B\nu)(z)| > \lambda\} = 0. \tag{3}$$

For every $z \in U\left(z_{j_0}; \frac{\delta_0}{4}\right) \setminus \{z_{j_0}\}$ it follows from the inequality

$$\frac{1}{\pi}\left|\sum_{j\in J,\ j\neq j_0} \frac{\alpha_j}{(z_j - z)^2}\right| \leq \frac{1}{\pi}\sum_{j\in J,\ j\neq j_0} \frac{|\alpha_j|}{|z_j - z|^2} \leq \frac{16}{\pi\delta_0^2}\|\nu\|$$

that for any $j_0 \in J$

$$\lim_{\lambda\to+\infty} \lambda m\{z \in U\left(z_{j_0}; \frac{\delta_0}{4}\right) : \ |(B\nu)(z)| > \lambda\}$$

$$= \lim_{\lambda\to+\infty} \lambda m\{z \in U\left(z_{j_0}; \frac{\delta_0}{4}\right) : \ \frac{1}{\pi}|\frac{|\alpha_{j_0}|}{|z_{j_0} - z|^2}| > \lambda\} = |\alpha_{j_0}|. \tag{4}$$

From equations (3) and (4) we obtain that

$$\lim_{\lambda\to+\infty} \lambda m\{z \in \mathbb{C} : \ |(B\nu)(z)| > \lambda\}$$

$$= \lim_{\lambda\to+\infty} \sum_{j_0\in J} \lambda m\{z \in U\left(z_{j_0}; \frac{\delta_0}{4}\right) : \ |(B\nu)(z)| > \lambda\} = \sum_{j_0\in J} |\alpha_{j_0}| = \|\nu\|.$$

This completes the proof of the Theorem 1.

Corollary 1. Let the measure ν be concentrated at a finite number of points in the domain $\Omega \in \mathbb{C}$. Then the equation

$$\lim_{\lambda\to+\infty} \lambda m\{z \in \Omega : |(B\nu)(z)| > \lambda\} = \lim_{\lambda\to+\infty} \lambda m\{z \in \mathbb{C} : |(B\nu)(z)| > \lambda\} = \|\nu\|$$

holds.

3. On the additivity of the Q-integral and Q'-integral

For a measurable complex function $f(z)$ on domain Ω we set

$$[f(z)]_n = [f(z)]^n = f(z) \text{ for } |f(z)| \le n$$

$$[f(z)]_n = n \cdot \operatorname{sgn} f(z), \ [f(z)]^n = 0 \text{ for } |f(z)| > n, \ n \in N,$$

where $\operatorname{sgn} w = \frac{w}{|w|}$ for $w \ne 0$ and $\operatorname{sgn} 0 = 0$.

In 1928, Titchmarsh [28] introduced the notions of Q- and Q'-integrals of a function measurable on Ω.

Definition 3. If the finite limit $\lim_{n\to\infty} \int_\Omega [f(z)]_n \, dm(z)$ ($\lim_{n\to\infty} \int_\Omega [f(z)]^n \, dm(z)$, respectively) exists, then f is said to be Q-*integrable* (Q'-*integrable*, respectively) on Ω; that is, $f \in Q(\Omega)$ ($f \in Q'(\Omega)$). The value of this limit is referred to as the Q-*integral* (Q'-*integral*) of this function and is denoted by $(Q)\int_\Omega f(z)\, dm(z)$ $\left((Q')\int_\Omega f(z)\, dm(z)\right)$.

In the same paper, Titchmarsh when studying properties of trigonometric series conjugate to Fourier series of Lebesgue integrable functions, established that the Q-integrability leads to a series of natural results. A very uncomfortable fact impeding the application of Q-integrals and Q'-integrals when dealing with diverse problems of function theory is the absence of the additivity property; that is, the Q-integrability (Q'-integrability) of two functions does not imply the Q-integrability (Q'-integrability) of their sum. If one adds the condition

$$m\{z \in \Omega : |f(z)| > \lambda\} = o(1/\lambda), \lambda \to +\infty \quad (5)$$

to the definition of Q-integrability (Q'-integrability) of a function f, then the Q-integral and Q'-integral coincide ($Q(\Omega) = Q'(\Omega)$), and these integrals become additive.

Definition 4. If $f \in Q'(\Omega)$ (or $f \in Q(\Omega)$) and condition (5) holds, then f is said to be A-*integrable* on Ω, $f \in A(\Omega)$, and the limit $\lim_{n\to\infty} \int_\Omega [f(z)]_n dm(z)$ (or the limit $\lim_{n\to\infty} \int_\Omega [f(z)]^n dm(z)$) is denoted in this case by $(A)\int_\Omega f(z)dm(z)$.

The properties of Q- and Q'-integrals were investigated in [3, 4, 10, 17, 18, 28]; for the applications of A-, Q- and Q'-integrals in the theory of functions of real and complex variables we refer the reader to [2, 5-10, 24, 25, 32, 33].

Let us note some properties of the Q and Q'-integrals that we will need.

Theorem 2 [10]. If $f \in Q(\Omega)$ and $g \in L(\Omega)$ (that is, g is Lebesgue integrable on the domain Ω). Then $f + g \in Q(\Omega)$ and

$$(Q)\int_\Omega [f(x) + g(x)]dx = (Q)\int_\Omega f(x)dx + (L)\int_\Omega g(x)dx.$$

Theorem 3 [10]. Let $f \in Q'(\Omega)$. Then $f \in Q(\Omega)$ and the following equation holds:

$$(Q)\int_{\Omega} f(x)dx = (Q')\int_{\Omega} f(x)dx. \tag{6}$$

Definition 5. We denote by $M(\Omega;\mathbb{C})$ the class of measurable functions f on the domain Ω for which a finite limit $\lim\limits_{\lambda\to+\infty} \lambda m\{z\in\Omega : |f(z)|>\lambda\}$ exists.

Theorem 4 [10]. The Q-integral and the Q'-integral coincide on the function class $M(\Omega;\mathbb{C})$, that is if $f \in M(\Omega;\mathbb{C})$, then for the existence of the integral $(Q)\int_\Omega f(x)dx$ it is necessary and sufficient that the integral $(Q')\int_\Omega f(x)dx$ exist, and in that case equation (6) holds.

Theorem 5 [10]. If a function $f \in M(\Omega;\mathbb{C})$ is Q' -integrable on the domain Ω and $g \in A(\Omega)$, then the sum $f+g \in M(\Omega;\mathbb{C})$ is Q' integrable on Ω, and the following equation holds:

$$(Q')\int_{\Omega}[f(x)+g(x)]dx = (Q')\int_{\Omega} f(x)dx + (A)\int_{\Omega} g(x)dx.$$

Corollary 2. If a function $f \in M(\Omega;\mathbb{C})$ is Q -integrable on the domain Ω and $g \in A(\Omega)$, then the sum $f+g \in M(\Omega;\mathbb{C})$ is Q integrable on Ω, and the following equation holds:

$$(Q)\int_{\Omega}[f(x)+g(x)]dx = (Q)\int_{\Omega} f(x)dx + (A)\int_{\Omega} g(x)dx.$$

4. Riesz's equality for the Ahlfors-Beurling transform of the finite complex measure, singular part which is concentrated at a finite number of points

From the properties of singular integrals it follows that (see [26]) if $f \in L_p(\Omega)$, $p>1$ and $g \in L_q(\Omega)$, $q>1$, $1/p+1/q=1$, then

$$\int_{\Omega} g(z)(B_\Omega f)(z)dm(z)$$

$$= -\frac{1}{\pi}\lim_{\varepsilon\to 0}\iint_{\{w,z\in\Omega\,:\,|z-w|>\varepsilon\}} \frac{f(w)g(z)}{(z-w)^2}dm(w)dm(z)$$

$$= \int_{\Omega} f(z)(B_\Omega g)(z)dm(z). \tag{7}$$

In [7] the authors shows that the Ahlfors–Beurling transform of a function $f \in L_1(\Omega)$ is A-integrable on Ω and put forward an analogue of (7):

Theorem 6 [7]. Let $f \in L_1(\Omega)$ and $g(z)$ be a bounded function on Ω with bounded $(B_\Omega g)(z)$ on Ω. Then the function $g(z)\cdot(B_\Omega f)(z)$ is A-integrable on Ω and

$$(A)\int_{\Omega} g(z)(B_\Omega f)(z)dm(z) = \int_{\Omega} f(z)(B_\Omega g)(z)dm(z). \tag{8}$$

In this section we prove that the restricted Ahlfors-Beurling transform of the finite complex measure, singular part which is concentrated at a finite number of points is Q-integrable and an analogue of (7) and (8) holds.

Theorem 7. Let ν be a finite complex measure on the bounded domain Ω, singular part which is concentrated at a finite number of points, and the function g is Hölder continuous on the closure of the domain Ω. Then the function $(B_\Omega \nu)(z)g(z)$ is Q-integrability on the domain Ω, and the equation

$$(Q)\int_\Omega g(z)(B_\Omega \nu)(z)dm(z) = \int_\Omega (B_\Omega g)(z)d\nu(z)$$

holds.

Proof of Theorem 7. Let

$$d\nu(z) = f(z)dm(z) + d\nu_s(z),$$

where ν_s is singular part of the measure ν and $f(z)$ is the Radon-Nikodym derivative of the absolutely continuous part of the measure ν. At first we prove that

$$(Q')\int_\Omega g(z)(B_\Omega \nu_s)(z)dm(z) = \int_\Omega (B_\Omega g)(z)d\nu_s(z). \tag{9}$$

Since the function g is continuous on the closure of the domain Ω, then it is bounded on it, that is there exists a number $M > 0$ such that for any $z \in \Omega$ the inequality $|g(z)| \le M$ holds.

Let $\mathrm{supp}\nu_s = \{z_k\}_{k=1}^n$ and $\nu_s(z_k) = \alpha_k, \quad k = \overline{1,n}$. Denote

$$G(z) = -\pi g(z)(B_\Omega \nu_s)(z) = \sum_{k=1}^n \frac{\alpha_k}{(z_k - z)^2} g(z),$$

$$\delta_0 = \inf\{\rho(z_i, z_j) : \ z_i, z_j \in \mathrm{supp}\nu_s\} > 0, \quad \delta_1 = \frac{16nM\|\nu_s\|}{\delta_0^2}.$$

For any $\lambda > \delta_1$ denote

$$K_{1,\lambda} = \bigcup_{k=1}^n U\left(z_k; \sqrt{\frac{nM|\alpha_k|}{\lambda}}\right), \quad K_{2,\lambda} = \bigcup_{k=1}^n U\left(z_k; \sqrt{\frac{|\alpha_k||g(z_k)|}{\lambda+\delta_1}}\right).$$

Then for any $\lambda > \delta_1$ it follows from the inclusion

$$\Omega \setminus K_{1,\lambda} \subset \{z \in \Omega : \ |G(z)| \le \lambda\} \subset \Omega \setminus K_{2,\lambda}$$

and from the inequality

$$\int_{K_1 \setminus K_2} |G(z)|dm(z) \le m(K_1 \setminus K_2)\left(\delta_1 + (\lambda + \delta_1)\sum_{k:g(z_k)\ne 0} \frac{M}{|g(z_k)|}\right)$$

$$+\sum_{k:g(z_k)=0}\int_{|z-z_k|\le\sqrt{\frac{nM|\alpha_k|}{\lambda}}}\frac{|g(z)-g(z_k|}{|z-z_K|^2}|\alpha_k|dm(z)\to 0 \quad at \quad \lambda\to+\infty$$

that there exist $(Q)\int_\Omega G(z)dm(z)$ and

$$(Q)\int_\Omega G(z)dm(z)=\lim_{\lambda\to+\infty}\int_{\{z\in\Omega:\ |G(z)|\le\lambda\}}G(z)dm(z)$$

$$=\sum_{k=1}^n\lim_{\epsilon\to 0+}\int_{\{z\in\Omega:\ |z-z_k|>\epsilon\}}\frac{\alpha_k}{(z-z_k)^2}g(z)dm(z)$$

$$=-\pi\sum_{k=1}^n\alpha_k B_\Omega g(z_k)=-\pi\int_\Omega(B_\Omega g)(z)d\nu_s(z),$$

that is the equation (8) holds. Then it follows from the Theorems 1, 4, 5 and from (8) and (9) that

$$(Q)\int_\Omega g(z)(B_\Omega\nu)(z)dm(z)$$

$$=(Q)\int_\Omega g(z)(B_\Omega\nu_s)(z)dm(z)+(A)\int_\Omega g(z)(B_\Omega f)(z)dm(z)$$

$$=(Q')\int_\Omega g(z)(B_\Omega\nu_s)(z)dm(z)+(A)\int_\Omega g(z)(B_\Omega f)(z)dm(z)$$

$$=\int_\Omega(B_\Omega g)(z)d\nu_s(z)+\int_\Omega f(z)(B_\Omega g)(z)dm(z)=\int_\Omega(B_\Omega g)(z)d\nu(z).$$

This iscompletes the proof of the Theorem 7.

Corollary 3. Let ν be a finite complex measure on the bounded domain Ω, singular part which is concentrated at a finite number of points, and the function g is Hölder continuous on the closure of the domain Ω. Then the function $(B_\Omega\nu)(z)g(z)$ is Q'-integrability on the domain Ω, and the equation

$$(Q')\int_\Omega g(z)(B_\Omega\nu)(z)dm(z)=\int_\Omega(B_\Omega g)(z)d\nu(z)$$

holds.

References

[1] Ahlfors LV. Lectures on Quasiconformal Mappings. 2nd ed., University Lecture Series, v. 38. AMS, Providence, RI; 2006.

[2] Aleksandrov AB. *A*-integrability of the boundary values of harmonic functions. Math Notes. 1981;30(1):515–523.

[3] Aliev RA. On properties of Hilbert transform of finite complex measures. Complex analysis and operator theory. 2016;10(1):171–185.

[4] Aliev RA. Riesz's equality for the Hilbert transform of the finite complex measures. Azer J Math. 2016;6(1):126–135.

[5] Aliev RA. On Laurent coefficients of Cauchy type integrals of finite complex measures. Proc Inst Math Mech NAS Azer. 2016;42(2):292-303.

[6] Aliev RA. Representability of Cauchy-type integrals of finite complex measures on the real axis in terms of their boundary values. Complex Variables and Elliptic Eq. 2017;62(4):536-553.

[7] Aliev R.A., Nabiyeva Kh.I., *The A-integral and restricted Ahlfors-Beurling transform*, Integral transforms and Special functions, **29:10**, 2018, 820-830.

[8] Aliev R.A., Nabiyeva Kh.I., *The A-integral and restricted complex Riesz transform*, Azerbaijan Journal of Mathematics, **10:1**, 2020, 209-221.

[9] Aliev R.A., Nabiyeva Kh.I., *The A-integral and restricted Riesz transform*, Constructive Mathematical Analysis, **3:3**, 2020, 104-112.

[10] Aliev RA. $N^{\pm}$–integrals and boundary values of Cauchy-type integrals of finite measures. Sbornik: Mathematics. 2014;205(7):913-935.

[11] Astala K, Iwaniec T, Martin G. Elliptic partial differential equations and quasiconformal mappings in the plane. Princeton: University Press; 2009.

[12] Banuelos R, Janakiraman P. Sobolev regularity of the Beurling transform on planar domains. Trans Amer Math Sos. 2008;360(7):3603-3612.

[13] Cruz V, Mateu J, Orobitg J. Beltrami equation with coefficient in Sobolev and Besov spaces. Canadian J Math. 2013;65:1217-1235.

[14] Cruz V, Tolsa X. Smoothness of the Beurling transform in Lipschitz domains. J Func Anal. 2012;262:4423–4457.

[15] Doubtsov E, Vasin AV. Restricted Beurling transforms on Campanato spaces. Complex Variables and Elliptic Eq. 2017;62(3):333-346.

[16] Dragicevic O. Weighted estimates for powers of the Ahlfors–Beurling operators. Proc Amer Math Sos. 2011;139:2113-2120.

[17] Efimova MP. On the properties of the Q-integral. Math Notes. 2011;90(3-4):322-332.

[18] Efimova MP. The sufficient condition for integrability of a generalized Q-integral and points of integrability. Moscow Univ Math Bul. 2015;70(4):181-184.

[19] Evans LC, Gariepy RF. Measure theory and fine properties of functions. Boca Raton: CRC Press; 1992.

[20] Iwaniec T. The best constant in a BMO-inequality for the Beurling-Ahlfors transform. Michigan Math J. 1987;34:407-434.

[21] Kwok-Pun H. The Ahlfors–Beurling transform on Morrey spaces with variable exponents. Integral Transforms and Special Functions, 2018;29(3):207-220.

[22] Mateu J, Orobitg J, Verdera J. Extra cancellation of even Calderon-Zygmund operators and quasiconformal mappings. J Math Pures et Appl. 2009;91(4):402-431.

[23] Prats M. L^p-bounds for the Beurling-Ahlfors transform. Publicacions Mat. 2017;61(2):291-336.

[24] Salimov TS. The A-integral and boundary values of analytic functions. Math USSR-Sbornik. 1989;64(1):23–40.

[25] Skvortsov VA. A-integrable martingale sequences and Walsh series. Izvestia: Math. 2001;65(3):607–616.

[26] Stein EM. Singular Integrals and Differentiability Properties of Functions. Princeton: University Press; 1970.

[27] Tolsa X. Regularity of C^1 and Lipschitz domains in terms of the Beurling transform. J Math Pures et Appl. 2013;100(2);137-165.

[28] Titchmarsh EC. On conjugate functions. Proc London Math Soc. 1928;29(2):49–80.

[29] Tumanov A. Commutators of singular integrals, the Bergman projection, and boundary regularity of elliptic equations in the plane. Math Research Letters. 2016;23(4):1221-1246.

[30] Vasin AV. Regularity of the Beurling Transform in Smooth Domains. J Math Sciences. 2016;215(5):577-584.

[31] Vekua IN. Generalized analytic functions. Pergamon Press; 1962.

[32] Ul'yanov PL. The A-integral and conjugate functions. Mathematics. vol.7. Uch Zap Mosk Gos Univ. Moscow: University Press. 1956;181:139-157 (in Russian).

[33] Ul'yanov PL. Integrals of Cauchy type. Twelve Papers on Approximations and Integrals. Amer Math Soc Trans. 1965;2/44:129–150.

Khanim I. Nebiyeva
Khazar University, Baku, AZ 1096, Azerbaijan
E-mail: xanim.nebiyeva@gmail.com

Received 12 May 2021
Accepted 07 June 2021