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Basicity of Linear Phase Exponential System in Grand-Sobolev Spaces

Seadet A.Nurieva

Abstract. We define a separable $MW_{p)}{}^{1}(a,b)$ subspace in grand-Sobolev spaces. Then we show that this subspace is isomorphic to the direct sum of some subspace of grand-Lebesgue space and complex plane and so the system $1 \cup \left\{e^{i(n+\alpha signn)t}\right\}_{n\epsilon Z}$ forms a basis for the space $MW_{p)}{}^{1}(-\pi,\pi)$, where $\alpha \in C$ is a complex parameter.

Key Words and Phrases: basicity, grand-Lebesgue space, grand-Sobolev space.

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Lately in mathematics, there has been an upsurge of interest in non-standard spaces (see [17, 18, 19, 20, 21, 22]). The study of differential equations in non-standard Sobolev spaces requires the knowledge of basicity properties of trigonometric systems in corresponding non-standard function spaces. Basicity properties of some trigonometric systems in such spaces have been treated in [23, 24, 25, 26, 27, 28, 29].

$$\left\{e^{i(n+\alpha signn)t}\right\}_{n\epsilon Z}, \tag{1}$$

$$1 \cup \left\{e^{i(n+\alpha signn)t}\right\}_{n\neq 0}. \tag{2}$$

The study of basicity properties of the systems (1) and (2) in Lebesgue function space probably dates back to Paley-Wiener [6] and N. Levinson [7]. Riesz basicity of (1)-type systems was studied in L_2 by M.I.Kadets [8], and in L_p by A.M.Sedletski [9] and E.I.Moiseyev [10, 11]. This field was further developed by B.T. Bilalov [12, 13, 14, 15].

Grand-Lebesgue spaces $L^{p)}$ have been introduced in [17] in the study of Jacobian in an open set. These are the functional Banach spaces, and they have wide applications in the theory of partial differential equations, theory of interpolation, etc. The study of some problems of harmonic analysis in these spaces is of special interest.

As these spaces are not separable, basis and approximation-related problems remained unsolved in them. In [25], some $M^{p)}$ subspace was constructed, interesting from the point of view of the theory of differential equations. In [26, 27], basicity properties of the systems (1) and (2) have been studied in this subspace.

Grand-Sobolev spaces have been studied in many works, including [17]. In this work, we explore the basicity of one exponential system for a subspace $MW_{p)}{}^{1}(-\pi,\pi)$ of grand-Sobolev space.

So, let $1<p<\infty$. A space $L^{p)}(a,b)$ of measurable functions satisfying the condition

$$\|f\|_{p)}=\sup_{0<\varepsilon<p-1}\left(\frac{\varepsilon}{b-a}\int_a^b|f|^{p-\varepsilon}dt\right)^{\frac{1}{p-\varepsilon}}<\infty \tag{3}$$

in the interval $(a,b)\subset R$ is called a grand-Lebesgue space.

Denote by $\tilde{M}^{p)}(a,b)$ the set of all functions satisfying the condition $\left\|\hat{f}(\cdot+\delta)-\hat{f}(\delta)\right\|_{p)}\to 0$ as $\delta\to 0$ and belonging to $L^{p)}(a,b)$, where

$$\hat{f}(t)=\begin{cases} f(t)\,, t\in(a,b)\,,\\ 0,\ t\notin(a,b)\ . \end{cases}$$

It is clear that the set $\tilde{M}^{p)}(a,b)$ is a manifold in $L^{p)}(a,b)$. Denote by $M^{p)}(a,b)$ the closure of $\tilde{M}^{p)}(a,b)$ with respect to the norm (3).

Denote by $W_{p)}{}^{1}(a,b)$ the space of functions which belong to $L^{p)}(a,b)$ together with their derivatives equipped with the norm

$$\|f\|_{W_{p)}}=\|f\|_{p)}+\|f'\|_{p)}. \tag{4}$$

We will call this space a grand-Sobolev space:

$$W_{p)}{}^{1}(a,b)=\left\{f\setminus f,f'\in L^{p)}(a,b),\|f\|_{p)}+\|f'\|_{p)}<\infty\right\}.$$

It is easy to prove that this is a Banach space. As is known, $L^{p)}(a,b)$ is not separable. Therefore, $W_{p)}{}^{1}(a,b)$ is also not a separable space. Denote by $\tilde{M}W_{p)}{}^{1}(a,b)$ the set of all functions which satisfy the condition $\left\|\hat{f}'(\cdot+\delta)-\hat{f}'(\delta)\right\|_{p)}\to 0$ as $\delta\to 0$ and belong to $W_{p)}{}^{1}(a,b)$, where

$$\hat{f}(t)=\begin{cases} f(t)\,, t\in(a,b)\,,\\ 0,\ t\notin(a,b)\,. \end{cases}$$

It is clear that the set $\tilde{M}W_{p)}{}^{1}(a,b)$ is a manifold in $W_{p)}{}^{1}(a,b)$. Denote by $MW_{p)}{}^{1}(a,b)$ the closure of $\tilde{M}W_{p)}{}^{1}(a,b)$ with respect to the norm (4).

The following lemma is true.

Lemma 1. *The operator $A(f,\lambda)=\lambda+\int_a^t f(\tau)\,d\tau$ creates an isomorphism between the spaces $M^{p)}(a,b)\oplus\mathbf{C}$ and $MW_{p)}{}^{1}(a,b)$, where $\mathbf{C}$ is a complex plane, $1<p<\infty$.*

Proof. Let $f\in M^{p)}(a,b)$. Then

$$\left\| \lambda + \int_a^t f(\tau)\, d\tau \right\|_{W_{p)}} = \left\| \lambda + \int_a^t f(\tau)\, d\tau \right\|_{p)} + \|f\|_{p)} \le \|\lambda\|_{p)} +$$

$$+ \left\| \int_a^t f(\tau)\, d\tau \right\|_{p)} + \|f\|_{p)}.$$

Obviously, $\|\lambda\|_{p)} \le K_1 |\lambda|$, $\left\| \int_a^t f(\tau)\, d\tau \right\|_{p)} \le K_2 \|f\|_{L^1} \le K_3 \|f\|_{L^{p-\varepsilon}} \le K_4 \|f\|_{p)}$, because $L^p \subset L^1$, $L^p \subset L^{p)} \subset L^{p-\varepsilon}$ (K_1, K_2, K_3, K_4 are constants). Thus, $\|A(f, \lambda)\|_{W_{p)}} \le K(|\lambda| + \|f\|_{p)})$, i.e. A is a bounded operator. For $v = \lambda + \int_a^t f(\tau)\, d\tau$ we have $v' = f(t)$. Then $v \in MW_{p)}{}^1(a, b)$.

Let's show that $ker A = \{0\}$. Assume $A(u, \lambda) = 0$, i.e. $\lambda + \int_a^t f(\tau)\, d\tau = 0$. Differentiating both sides, we get $f(t) = 0$ a.e. Consequently, $\lambda = 0$. Let $\tilde{v} = \left(v', v(a)\right)$ for $\forall v \in MW_{p)}{}^1(a, b)$. Then $\tilde{v} \in M^{p)}(a, b) \oplus \mathbf{C}$ and $A(\tilde{v}) = v$. This means $R_A = MW_{p)}{}^1(a, b)$, where R_A is a range of the operator A. By Banach inverse operator theorem, the inverse of the operator A exists and is continuous. The lemma is proved.

We will significantly use the following theorem.

Theorem 1. *([26]) Let $-2Re\alpha + \frac{1}{p} \notin Z$,$1 < p < \infty$. Then the system* (1) *forms a basis for the space $M^{p)}(-\pi, \pi)$,$1 < p < \infty$, if and only if $d = \left[-2Re\alpha + \frac{1}{p}\right] = 0$ ($[\alpha]$ denotes the integer part of α). The defect of the system* (1) *is $d = \left[-2Re\alpha + \frac{1}{p}\right]$. When $d < 0$, the system* (1) *is not complete, but minimal in $M^{p)}(-\pi, \pi)$. When $d > 0$, the system* (1) *is complete , but not minimal in $M^{p)}(-\pi, \pi)$.*

So the following theorem is true.

Theorem 2. *Let $-2Re\alpha + \frac{1}{p} \notin Z$, $1 < p < \infty$. Then the system*

$$1 \cup \left\{ e^{i(n + \alpha signn)t} \right\}_{n \epsilon Z} \tag{5}$$

forms a basis for the space $MW_{p)}{}^1(-\pi, \pi)$, $1 < p < \infty$, if and only if $\left[-2Re\alpha + \frac{1}{p}\right] = 0$.

Proof. Let $\left[-2Re\alpha + \frac{1}{p}\right] = 0$. Let's first prove that the system $\hat{u}_{-1} = \binom{0}{1}$, $\hat{u}_0 = \binom{i\alpha e^{i\alpha t}}{e^{-i\pi\alpha}}$, $\hat{u}_n^{\pm} = \binom{i(n+\alpha signn)e^{i(n+\alpha signn)t}}{e^{-i\pi(n+\alpha signn)}}$, $n = \pm 1, \pm 2, \ldots$, forms a basis for the space $M^{p)}(-\pi, \pi) \oplus \mathbf{C}$. To do so, it suffices to show that $\forall \hat{u} = \binom{u}{\lambda} \in M^{p)}(-\pi, \pi) \oplus \mathbf{C}$ the expansion

$$\hat{u} = c_{-1}\hat{u}_{-1} + c_0 \hat{u}_0 + \sum_{n \neq 0} c_n^{\pm} \hat{u}_n{}^{\pm} \tag{6}$$

exists and is unique. This expansion is equivalent to two following expansions:

$$u\left(t\right)=\mathrm{c}_0 i\alpha e^{i\alpha t}+\sum_{n\neq 0}c_n^{\pm}i(n+\alpha signn)e^{i(n+\alpha signn)t}, \tag{7}$$

$$\lambda=-\pi\mathrm{c}_{-1}+\mathrm{c}_0 e^{-i\pi\alpha}+\sum_{n\neq 0}c_n^{\pm}e^{-i\pi(n+\alpha signn)}. \tag{8}$$

By Theorem 1 ([26]), the expansion (7) exists and is unique. As $\forall\varepsilon\in(0,p-1)$, $L^{p)}\subset L^{p-\varepsilon}$ and $\left[-2Re\alpha+\frac{1}{p}\right]=0$, by [16], Hausdorff-Young inequality is true for the system (1) in grand-Lebesgue space $L^{p)}$, too. That is, if $1<p\leq 2$, then

$$\left(\left|c_0\right|^q+\sum_{n\neq 0}\left|c_n^{\pm}n\right|^q\right)^{1/q}\leq M\|u\|_{p-\varepsilon}\leq M\|u\|_{p)},$$

where $p-\varepsilon$ and q are mutually conjugate numbers: $\frac{1}{p-\varepsilon}+\frac{1}{q}=1$.

Using Hölder's inequality, we obtain

$$|c_0|+\sum_{n\neq 0}\left|c_n^{\pm}\right|=|c_0|+\sum_{n\neq 0}\frac{1}{|n|}\left|c_n^{\pm}n\right|\leq|c_0|+\left(\sum_{n\neq 0}\frac{1}{|n|^p}\right)^{\frac{1}{p}}\left(\sum_{n\neq 0}\left|c_n^{\pm}n\right|^q\right)^{\frac{1}{q}}<\infty.$$

When $2<p$, we can find $\varepsilon>0$ such that $2<p-\ \varepsilon$. Therefore,

$$L^{p)}\subset L^{p-\varepsilon}\subset L^2.$$

Similarly we have

$$|c_0|+\sum_{n\neq 0}\left|c_n^{\pm}\right|=|c_0|+\sum_{n\neq 0}\frac{1}{|n|}\left|c_n^{\pm}n\right|\leq|c_0|+\left(\sum_{n\neq 0}\frac{1}{|n|^2}\right)^{\frac{1}{2}}\left(\sum_{n\neq 0}\left|c_n^{\pm}n\right|^2\right)^{\frac{1}{2}}<\infty.$$

So, the series $\sum_{n\neq 0}|c_n^{\pm}|$ is convergent. Therefore, the expansion (8) also exists and is unique. This implies the existence and uniqueness of the expansion (6), i.e. the system

$$\hat{u}_{-1}\cup\hat{u}_0\cup\left\{\hat{u}_n^{\pm}\right\}\ ,n=\pm1,\pm2,\ldots$$

forms a basis for the space $M^{p)}\left(-\pi,\pi\right)\oplus\mathbf{C}$. As the operator A is an isomorphism, the system

$$\left\{A\hat{u}_{-1}\right\}\cup\left\{A\hat{u}_0\right\}\cup\left\{A\hat{u}_n^{\pm}\right\},n=\pm1,\pm2,\ldots$$

must form a basis for the space $MW_{p)}{}^1\left(-\pi,\pi\right)$. Simple calculations show that

$$A\hat{u}_{-1}=1,\ \ A\hat{u}_0=e^{i\alpha t},$$

$$A\hat{u}_n^{\pm}=e^{i(n+\alpha signn)t},n=\pm1,\pm2,\ldots$$

That is, the system $1 \cup \left\{e^{i(n+\alpha signn)t}\right\}_{n\epsilon Z}$ forms a basis for the space $MW_{p)}{}^{1}(-\pi,\pi)$.

Now let $\left[-2Re\alpha+\frac{1}{p}\right]>0$. For certainty, we assume $\left[-2Re\alpha+\frac{1}{p}\right]=1$, i.e. $1<-2Re\alpha+\frac{1}{p}<2$.

Let's rewrite the system (5) as $1 \cup \left\{e^{int}e^{i\alpha t}; e^{-ikt}e^{-i\alpha t}\right\}_{n\geq 0,k\geq 1}$ and multiply every term of it by $e^{-it/2}$. After making some transformations, we obtain:

$$e^{-it/2} \cup \left\{e^{int}e^{i(\alpha-\frac{1}{2})t}; e^{-ikt}e^{-i(\alpha+\frac{1}{2})t}\right\}_{n\geq 0,k\geq 1} \equiv$$

$$\equiv e^{-it/2} \cup \left\{e^{it}e^{i(n-1)t}e^{i(\alpha-\frac{1}{2})t}; e^{-ikt}e^{-i(\alpha+\frac{1}{2})t}\right\}_{n\geq 0,k\geq 1} \equiv$$

$$\equiv e^{-it/2} \cup \left\{e^{int}e^{i(\alpha+\frac{1}{2})t}; e^{-ikt}e^{-i(\alpha+\frac{1}{2})t}\right\}_{n\geq -1,k\geq 1}.$$

Denoting $\alpha^{'}=\alpha+\frac{1}{2}$, we can rewrite the last system as

$$e^{-it/2} \cup \left\{e^{int}e^{i\alpha^{'}t}; e^{-ikt}e^{-i\alpha^{'}t}\right\}_{n\geq -1,k\geq 1}. \tag{9}$$

As $-2Re\alpha^{'}+\frac{1}{p}=-2Re\alpha+\frac{1}{p}-1$, we have $0<-2Re\alpha^{'}+\frac{1}{p}<1$. In this case, due to the fact we have proved above, the system

$$1 \cup \left\{e^{int}e^{i\alpha^{'}t}; e^{-ikt}e^{-i\alpha^{'}t}\right\}_{n\geq 0,k\geq 1}, \tag{10}$$

forms a basis for $MW_{p)}{}^{1}(-\pi,\pi)$. It is clear that if we remove $\{1\}$ from (10) and add the functions $e^{-it/2}$ and $e^{i(\alpha^{'}-1)t}$, we obtain the system (9). It is known from the theory of bases that in this case the system (8) cannot be a basis.

Note that the basicity properties of the systems (9) and (5) are absolutely identical. Because it is easy to verify that the operator of multiplying by $e^{-it/2}$ is an automorphism in $MW_{p)}{}^{1}(-\pi,\pi)$. So, in case $\left[-2Re\alpha+\frac{1}{p}\right]=1$ the system (5) does not form a basis for $MW_{p)}{}^{1}(-\pi,\pi)$. The case of $\left[-2Re\alpha+\frac{1}{p}\right]>1$ can be treated similarly.

Let $\left[-2Re\alpha+\frac{1}{p}\right]<0$. For certainty, assume $\left[-2Re\alpha+\frac{1}{p}\right]=-1$, i.e.$-1<-2Re\alpha+\frac{1}{p}<0$.

Let's rewrite the system (5) as $1 \cup \left\{e^{int}e^{i\alpha t}; e^{-ikt}e^{-i\alpha t}\right\}_{n\geq 0,k\geq 1}$ and multiply every term of it by $e^{it/2}$. Once again, after making some transformations, we obtain:

$$e^{it/2} \cup \left\{e^{int}e^{i(\alpha+\frac{1}{2})t}; e^{-ikt}e^{-i(\alpha-\frac{1}{2})t}\right\}_{n\geq 0,k\geq 1} \equiv$$

$$\equiv e^{it/2} \cup \left\{e^{-it}e^{i(n+1)t}e^{i(\alpha+\frac{1}{2})t}; e^{-ikt}e^{-i(\alpha-\frac{1}{2})t}\right\}_{n\geq 0,k\geq 1} \equiv$$

$$\equiv e^{it/2} \cup \left\{e^{int}e^{i(\alpha-\frac{1}{2})t}; e^{-ikt}e^{-i(\alpha-\frac{1}{2})t}\right\}_{n\geq 1,k\geq 1}.$$

Denoting $\alpha^{''}=\alpha-\frac{1}{2}$, we can rewrite the last system as

$$e^{it/2}\cup\left\{e^{int}e^{i\alpha^{''}t};e^{-ikt}e^{-i\alpha^{''}t}\right\}_{n\geq1,k\geq1}. \tag{11}$$

As $-2Re\alpha^{''}+\frac{1}{p}=-2Re\alpha+\frac{1}{p}+1$, we have $0<-2Re\alpha^{''}+\frac{1}{p}<1$. In this case, due to the fact we have proved above, the system

$$1\cup\left\{e^{int}e^{i\alpha^{''}t};e^{-ikt}e^{-i\alpha^{''}t}\right\}_{n\geq0,k\geq1} \tag{12}$$

forms a basis for $MW_{p)}{}^{1}(-\pi,\pi)$. It is clear that if we remove $\{1\}$ and $e^{i\alpha^{''}t}$ from (12) and add the function $e^{it/2}$, we obtain the system (11). It is known from the theory of bases that in this case the system (11) cannot be a basis.

Note that the basicity properties of the systems (11) and (5) are absolutely identical. Because it is easy to verify that the operator of multiplying by $e^{it/2}$ is an automorphism in $MW_{p)}{}^{1}(-\pi,\pi)$. So, in case $\left[-2Re\alpha+\frac{1}{p}\right]=-1$ the system (5) does not form a basis for $MW_{p)}{}^{1}(-\pi,\pi)$. The case of $\left[-2Re\alpha+\frac{1}{p}\right]<-1$ can be treated similarly. Thus, if the condition $\left[-2Re\alpha+\frac{1}{p}\right]=0$ is not satisfied, then the system (5) cannot form a basis.

The theorem is proved.

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Seadet A. Nurieva
Azerbaijan Tourism and Management University, Baku, Azerbaijan
E-mail: sada.nuriyeva@inbox.ru

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