Caspian Journal of Applied Mathematics, Ecology and Economics
V. 9, No 2, 2021, December
ISSN 1560-4055

Completeness of the Perturbed Trigonometric System in Generalized Weighted Lebesgue Spaces

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Abstract. A double exponential system with complex-valued complex coefficients is considered in generalized weighted Lebesgue spaces. Completeness of this system in $L_{p(\cdot);\rho}$ spaces is studied.

Key Words and Phrases: exponential system, basicity, variable exponent, generalized Lebesgue space

2010 Mathematics Subject Classifications: 30B60; 42C15; 46A35

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1. Introduction

In the context of applications to some problems of mechanics and mathematical physics, since recently there arose great interest in the study of different problems in generalized Lebesgue spaces $L_{p(\cdot)}$ of variable summability rate $p(\cdot)$. Some fundamental results of classical harmonic analysis have been extended to the case of $L_{p(\cdot)}$ (for more details see [9-12]). Note that the use of Fourier method in solving some problems for partial differential equations in generalized Sobolev classes requires the study of approximative properties of perturbed exponential systems in generalized Lebesgue spaces. Some approximation problems in these spaces have been studied by I.I. Sharapudinov (see, e.g., [11]).

In this work, we consider the completeness of a double exponential system with complex-valued complex coefficients in the spaces $L_{p(\cdot);\rho}$. The completeness is reduced to trivial solvability of the corresponding homogeneous Riemann problem in the classes $H^+_{q(\cdot);\rho} \times_{-1} H^-_{q(\cdot);\rho}$, where $q\,(t)$ is a conjugate function of $p\,(t)$. Note that when considering the basicity of such systems in $L_{p(\cdot);\rho}$, unlike in the case of completeness, the solvability of corresponding Riemann problem is studied in the classes $H^+_{p(\cdot);\rho} \times_{-1} H^-_{p(\cdot);\rho}$. That's why we treat the completeness separately. The scheme we use is not new. We just follow the works [2;4].

2. Needful Information

Let $\omega \equiv \{z : |z| < 1\}$ be a unit ball in the complex plane and $\Gamma = \partial\omega$ be a unit circumference. Let $p : [-\pi, \pi] \to [1, +\infty)$ be some Lebesgue measurable function. The class of all Lebesgue measurable functions on $[-\pi, \pi]$ is denoted by L_0. Denote

$$I_p(f) \overset{def}{\equiv} \int_{-\pi}^{\pi} |f(t)|^{p(t)}\, dt.$$

Let

$$\mathrm{L} \equiv \{f \in \mathrm{L}_0 : I_p(f) < +\infty\}.$$

For $p^+ = \sup\underset{[-\pi,\pi]}{vrai}\, p(t) < +\infty$, L becomes a linear space with the usual linear operations of addition of functions and multiplication by a number. Equipped with the norm

$$\|f\|_{p(\cdot)} \overset{def}{\equiv} \inf\left\{\lambda > 0 : I_p\left(\frac{f}{\lambda}\right) \le 1\right\},$$

L becomes a Banach space which we denote by $L_{p(\cdot)}$. Let

$$WL \overset{def}{\equiv} \{p : p(-\pi) = p(\pi); \exists C > 0, \quad \forall t_1, t_2 \in [-\pi, \pi] : |t_1 - t_2| \le \tfrac{1}{2} \Rightarrow$$
$$\Rightarrow |p(t_1) - p(t_2)| \le \tfrac{C}{-\ln|t_1 - t_2|}\Big\}.$$

Throughout this work, $q(\cdot)$ denotes a conjugate function of $p(\cdot)$: $\frac{1}{p(t)} + \frac{1}{q(t)} \equiv 1$. Denote$p^- = \inf\underset{[-\pi,\pi]}{vrai}\, p(t)$.

The following generalized Hölder inequality is true:

$$\int_{-\pi}^{\pi} |f(t)\, g(t)|\, dt \le c\left(p^-; p^+\right) \|f\|_{p(\cdot)} \|g\|_{q(\cdot)},$$

where

$$c\left(p^-; p^+\right) = 1 + \frac{1}{p^-} - \frac{1}{p^+}.$$

We will significantly use the following easy-to-prove:

Statement 1. *Suppose*

$$p \in WL,\ p(t) > 0, \forall t \in [-\pi, \pi];\ \{\alpha_i\}_0^m \subset R.$$

The weight function

$$\rho(t) = |t|^{\alpha_0} \prod_{i=1}^{m} |t - \tau_i|^{\alpha_i} \tag{1}$$

belongs to the space $L_{p(\cdot)}$ *if the following inequalities are true:*

$$\alpha_i > -\frac{1}{p(\tau_i)}, \forall i = \overline{0, m};$$

where $-\pi = \tau_1 < \tau_2 < ... < \tau_m = \pi,\ \tau_0 = 0,\ \tau_i \ne 0, \forall i = \overline{1, m}.$

To obtain our main results, we will also use the following important fact:

Property B. *If* $p(t): 1 < p^- \le p^+ < +\infty$, *then the class* $C_0^\infty(-\pi,\pi)$ *(class of finite, infinitely differentiable functions on* $(-\pi,\pi)$*) is everywhere dense in* $L_{p(\cdot)}$.

Define the weighted class $h_{p(\cdot),\rho}$ of functions which are harmonic inside the unit circle ω with the variable summability rate $p(\cdot)$, where the weight function $\rho(\cdot)$ is defined by (1).

Denote

$$h_{p(\cdot),\rho} \equiv \left\{ u : \Delta u = 0 \ in \ \omega \ and \ \|u\|_{p(\cdot),\rho} = \sup_{0<r<1} \left\| u\left(re^{it}\right)\right\|_{p(\cdot),\rho} < +\infty \right\}.$$

We will need the following

Lemma 1. *Let* $p \in WL, p^- \ge 1$, *and the weight* $\rho(\cdot)$ *satisfy the condition*

$$-\frac{1}{p(\tau_k)} < \alpha_k < \frac{1}{q(\tau_k)}, \ k = \overline{0,m}. \tag{2}$$

If $f \in L_{p(\cdot),\rho}$, *then* $\exists p_0 \ge 1 : f \in L_{p_0}$.

The following lemma is also true:

Lemma 2. *Let* $p \in WL, p^- \ge 1$, *and the weight* $\rho(\cdot)$ *satisfy the condition* (2). *If* $u \in h_{p(\cdot),\rho}$, *then* $\exists p_0 \in [1,+\infty] : u \in h_{p_0}$.

Using these lemmas, one can prove the following theorem:

Theorem 1. *Let* $p \in WL, p^- > 1$, *and the inequalities* (2) *be fulfilled. If* $u \in h_{p(\cdot),\rho}$, *then* $\exists f \in L_{p(\cdot),\rho}$:

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t) f(t)dt, \tag{3}$$

where

$P_r(\alpha) = \frac{1-r^2}{1+r^2-2r\cos\alpha}$ *is a Poisson kernel.*

On the contrary, if $f \in L_{p(\cdot),\rho}$, *then the function* u *defined by* (3) *belongs to the class* $h_{p(\cdot),\rho}$.

Similarly we define the weighted Hardy classes $H^{\pm}_{p(\cdot),\rho}$. By $H^+_{p_0}$ we denote the usual Hardy class, where $p_0 \in [1,+\infty)$ is some number. Let

$$H^{\pm}_{p(\cdot),\rho} \equiv \left\{ f \in H_1^+ : f^+ \in L_{p(\cdot),\rho}(\partial\omega) \right\},$$

where f^+ are nontangential boundary values of $f(\cdot)$ on $\partial\omega$.

It is absolutely clear that $f(\cdot)$ belongs to the space $H^+_{p(\cdot),\rho}$ only when Ref and Imf belong to the space $h_{p(\cdot),\rho}$. Therefore, many properties of the functions from $h_{p(\cdot),\rho}$ are transferred to the functions from $H^+_{p(\cdot),\rho}$. Taking into account the relationship between the Poisson kernel $P_r(\alpha)$ and the Cauchy kernel $K_z(t) = \frac{e^{it}}{e^{it}-z}$, it is easy to derive from Theorem 2.1 the validity of the following:

Theorem 2. *Let $p \in WL, p^- > 1$, and the inequalities* (2) *be fulfilled. If $F \in H^+_{p(\cdot),\rho}$, then $F^+ \in L_{p(\cdot),\rho}$:*

$$F(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{F^+(t)dt}{1-ze^{-it}} = \frac{1}{2\pi}\int_{-\pi}^{\pi}K_z(t)F^+(t)dt. \tag{4}$$

On the contrary, if $F^+ \in L_{p(\cdot),\rho}$, then the function F defined by (4) *belongs to the class $H^+_{p(\cdot),\rho}$, where $F^+(\cdot)$ are nontangential boundary values of $F(\cdot)$ on $\partial\omega$.*

Following the classics, we define the weighted Hardy class ${}_mH^-_{p(\cdot),\rho}$ of analytic functions on $C\backslash\bar{\omega}$ of order $k \leq m$ at infinity. Let $f(z)$ be an analytic function on $C\backslash\bar{\omega}$ of finite order $k \leq m$ at infinity, i.e.

$$f(z) = f_1(z) + f_2(z),$$

where $f_1(z)$ is a polynomial of degree $k \leq m$, $f_2(z)$ is the principal part of Laurent decomposition of the function $f(z)$ at infinity. If the function $\varphi(z) \equiv \overline{f_2\left(\frac{1}{\bar{z}}\right)}$ belongs to the class $H^+_{p(\cdot),\rho}$, then we will say that the function $f(z)$ belongs to the class ${}_mH^-_{p(\cdot),\rho}$.

Absolutely similar to the classical case, one can prove the following:

Theorem 3. *Let $p \in WL, p^- > 1$, and the inequalities* (2) *be fulfilled. If $f \in H^+_{p(\cdot),\rho}$, then*

$$\left\|f(re^{it}) - f^+(e^{it})\right\|_{p(\cdot),\rho} \to 0,\ r \to 1-0,$$

$$\left\|f\left(re^{it}\right)\right\|_{p(\cdot),\rho} \to \left\|f^+\left(e^{it}\right)\right\|_{p(\cdot),\rho},\ r \to 1-0,$$

where f^+ are nontangential boundary values of f on $\partial\omega$.

The similar fact is true also in ${}_mH^-_{p(\cdot),\rho}$ classes.

Theorem 4. *Let $p \in WL, p^- > 1$, and the inequalities* (2) *be fulfilled. If $f \in {}_mH^-_{p(\cdot),\rho}$, then*

$$\left\|f(re^{it}) - f^-(e^{it})\right\|_{p(\cdot),\rho} \to 0,\ r \to 1+0,$$

$$\left\|f\left(re^{it}\right)\right\|_{p(\cdot),\rho} \to \left\|f^-\left(e^{it}\right)\right\|_{p(\cdot),\rho},\ r \to 1+0,$$

where f^- are nontangential boundary values of $\theta(t) \equiv \arg G\left(e^{it}\right)$ on $\partial\omega$ from outside ω.

The following analog of the classical Smirnov theorem is valid:

Theorem 5. *Let $p \in WL, p^- > 1$, and the inequalities* (2) *be fulfilled. If $u \in H^+_1$ and $L_{p(\cdot),\rho}$, then $u \in H^+_{p(\cdot),\rho}$.*

Denote the restrictions of the classes $H^+_{p(\cdot),\rho}$,${}_mH^-_{p(\cdot),\rho}$ to $\partial\omega$ by $L^+_{p(\cdot),\rho}$ and ${}_mL^-_{p(\cdot),\rho}$, respectively, i.e.

$$L^+_{p(\cdot),\rho} = H^+_{p(\cdot),\rho}/_{\partial\omega}\,;\ {}_mL^-_{p(\cdot),\rho} = {}_mH^-_{p(\cdot),\rho}/_{\partial\omega}.$$

We will need the following result:

Theorem 6. *Let $p \in WL, p^- > 1$, and the inequalities* (2) *be fulfilled. Then the system $E^{(0)}_+ = \left\{e^{int}\right\}_{n\geq 0}$($E^{(m)}_- = \left\{e^{-int}\right\}_{n\geq m}$) forms a basis for $L^+_{p(\cdot),\rho}$(${}_mL^-_{p(\cdot),\rho}$),$1 < p < +\infty$.*

We will also need the following easy-to-prove lemma, which is derived immediately from the definition of weighted space $L_{p(\cdot),\rho}$.

Lemma 3. *Let*
$p \in C[-\pi,\pi]$ *and* $p(t) > 0,\ \forall t \in [-\pi,\pi]$.
Then the function $\xi(t) = |t-c|^{\alpha}$ *belongs to* $L_{p(\cdot),\rho}$*, if*
$\alpha > -\frac{1}{p(c)}$*, for* $c \neq \tau_k,\ \forall\, k = \overline{1,m}$,
and
$\alpha + \alpha_{k_0} > -\frac{1}{p(c)}$*, for* $c = \tau_{k_0}$.

3. Main Assumptions and Riemann Problem Statement

Let's state the Riemann problem in the classes $H^{\pm}_{p(\cdot);\rho}$. Let the complex-valued function $G(t)$ on $[-\pi,\pi]$ satisfy the following conditions:
i) Function $|G(t)|$ *belongs to the space* $L_{r(\cdot)}$ *for some* $r:\ 0 < r^- \le r^+ < +\infty$*, and* $|G(t)|^{-1} \in L_{\omega(\cdot)}$ *for* $\omega:\ 0 < \omega^- \le \omega^+ < +\infty$.
ii) Argument $\theta(t) \equiv \arg G(t)$ *has a following decomposition:*

$$\theta(t) = \theta_0(t) + \theta_1(t),$$

where $\theta_0(t)$ *is a continuous function on* $[-\pi,\pi]$ *and* $\theta_1(t)$ *is a function of bounded variation on* $[-\pi,\pi]$.

It is required to find a piecewise analytic function $F^{\pm}(z)$ on the complex plane with a cut $\partial\omega$ which satisfies the following conditions:

a) $F^{+}(z) \in H^{+}_{p(\cdot)} : 0 < p^- \le p^+ < +\infty$;

b) $F^{-}(z) \in {}_mH^{-}_{\nu(\cdot)}; 0 < \nu^- \le \nu^+ < +\infty$;

c) nontangential boundary values on the unit circumference $\partial\omega$ *satisfy the relation*
$F^{+}\left(e^{it}\right) - G(t)\,F^{-}\left(e^{it}\right) = g(t)$*, for a.e.* $t \in (-\pi,\pi)$,
where $g \in L_{\rho(\cdot)} : 0 < \rho^- \le \rho^+ < +\infty$ *is some given function.*

Note that in the case of constant summability rate, the theory of such problems has been well studied (see [3]).

Consider the following homogeneous Riemann problem in the classes $H^{+}_{p(\cdot),\rho} \times {}_mH^{-}_{p(\cdot),\rho}$:

$$F^{+}(z) - G(z)F^{-}(z) = 0,\ z \in \partial\omega. \tag{5}$$

By the solution of the problem (5) we mean a pair of analytic functions

$$\left(F^{+}(z); F^{-}(z)\right) \in H^{+}_{p(\cdot),\rho} \times {}_mH^{-}_{p(\cdot),\rho},$$

whose boundary values satisfy a.e. the equation (5). Introduce the following functions $X_i(z)$ analytic inside (with the sign "+") and outside (with the sign "-") the unit circle:

$$X_1(z) \equiv \exp\left\{\frac{1}{4\pi}\int_{-\pi}^{\pi} \ln\left|G\left(e^{it}\right)\right| \frac{e^{it}+z}{e^{it}-z}dt\right\},$$

$$X_2(z) \equiv \exp\left\{\frac{i}{4\pi}\int_{-\pi}^{\pi}\theta(t)\frac{e^{it}+z}{e^{it}-z}dt\right\},$$

where $\theta(t) \equiv \arg G\left(e^{it}\right)$. Define

$$Z_i(z) \equiv \begin{cases} X_i(z), \; |z|<1, \\ \left[X_i(z)\right]^{-1}, \; |z|>1, \quad i=1,2 \;\; and \;\; Z(z_i) = Z_1(z)\times Z_2(z) \end{cases}$$

Let $\{s_k\}_1^r : -\pi < s_1 < ...s_r < \pi$ be points of discontinuity of the function $\theta(t)$ and

$$\{h_k\}_1^r : \; h_k = \theta(s_k+0) - \theta(s_k-0), k=\overline{1,r},$$

be the corresponding jumps of this function at these points. Denote

$$h_0 = \theta(-\pi) - \theta(\pi); h_0^{(0)} = \theta_0(\pi) - \theta_0(-\pi).$$

Let

$$u_0(t) \equiv \left\{\sin\left|\frac{t-\pi}{2}\right|\right\}^{-\frac{h_0^{(0)}}{2\pi}} \exp\left\{-\frac{1}{4\pi}\int_{-\pi}^{\pi}\theta_0(\tau)\, ctg\frac{t-\tau}{2}d\tau\right\}$$

and

$u(t) = \prod_{k=0}^{r}\left\{\sin\left|\frac{t-s_k}{2}\right|\right\}^{\frac{h_k}{2\pi}}$, where $s_0 = \pi$.

As is known, (see [3]), the boundary values $\left|Z_2^-(\tau)\right|$ are defined by the formula

$$\left|Z_2^-\left(e^{it}\right)\right| = u_0(t)\left[u(t)\right]^{-1}\left\{\sin\left|\frac{t-\pi}{2}\right|\right\}^{-\frac{h_0}{2\pi}},$$

i.e.

$$\left|Z_2^-\left(e^{it}\right)\right| = u_0(t)\prod_{k=0}^{r}\left\{\sin\left|\frac{t-s_k}{2}\right|\right\}^{-\frac{h_k}{2\pi}}.$$

It follows directly from Sokhotskii-Plemelj formula that

$$\sup_{(-\pi,\pi)} vrai\left\{\left|Z_1^-\left(e^{it}\right)\right|^{\pm 1}\right\} < +\infty.$$

Thus, for $\left|Z^-\left(e^{it}\right)\right|^{-1}$ we have the representation

$$\left|Z^-\left(e^{it}\right)\right|^{-1} = \left|Z_1^-\left(e^{it}\right)\right|^{-1}\left|u_0(t)\right|^{-1}\prod_{k=0}^{r}\left\{\sin\left|\frac{t-s_k}{2}\right|\right\}^{\frac{h_k}{2\pi}}. \tag{6}$$

Represent the product $\left|Z^-\rho\right|^{-1}$ as follows:

$$\left|Z^-\rho\right|^{-1} \approx \left|Z_1^-\right|^{-1}\left|u_0\right|^{-1}\prod_{k=0}^{l}\left|t-t_k\right|^{\beta_k},$$

where
$\{t_k\}_{k=0}^{l} \equiv \{\tau_k\}_{k=1}^{m} \bigcup \{s_k\}_{k=0}^{r}$, and β_k's are defined by

$$\beta_k = -\sum_{i=1}^{m} \alpha_i \chi_{\{t_k\}}(\tau_i) + \frac{1}{2\pi}\sum_{i=0}^{r} h_i \chi_{\{t_k\}}(s_i), \ \ k = \overline{0,l}. \tag{7}$$

By virtue of Lemma 2.3, we obtain that if the inequalities

$$\beta_k > -\frac{1}{q(t_k)}, \ \ k = \overline{0,r}, \tag{8}$$

are true, then the product $|Z^-\rho|^{-1}$ belongs to the space $L_{q(\cdot)}$, i.e. $|Z^-|^{-1} \in L_{q(\cdot),\rho^{-1}}$. So, if the inequalities (8) are true, then the function $\Phi(z) = \frac{F(z)}{Z(z)}$ belongs to the classes $H_1^{\pm}$. Then, according to [3], $\Phi(z)$ is a polynomial $P_{m_0}(z)$ of degree $m_0 \leq m$. Thus,

$$F^-(z) = P_{m_0}(z) Z^-(z).$$

Let's find out under which conditions the function $F^-(z)$ belongs to the space $H^-_{p(\cdot),\rho}$. We have

$$|Z^-\rho| \approx |Z_1|\,|u_0| \prod_{k=0}^{l} |t - t_k|^{-\beta_k}.$$

Consequently, if the inequalities

$$\beta_k < \frac{1}{p(t_k)}, \ k = \overline{0,r},$$

are true, then it is clear that $F^-(\tau) \in L_{p(\cdot),\rho}$, and hence $F^- \in {}_mH^-_{p(\cdot),\rho}$. So, if the inequalities

$$-\frac{1}{q(t_k)} < \beta_k < \frac{1}{p(t_k)}, \ k = \overline{0,r}, \tag{9}$$

are true, then the general solution of homogeneous problem

$$F_0^+(\tau) = G_1(\tau) F_0^-(\tau), \quad \tau \in \partial\omega,$$

in the classes $H^+_{p(\cdot),\rho} \times_m H^-_{p(\cdot),\rho}$ can be represented as follows:

$$F_0(z) = P_{m_0}(z) Z(z),$$

where $P_{m_0}(z)$ is an arbitrary polynomial of degree $m_0 \leq m$.So the following theorem is valid:

Theorem 7. *Let the* $\{\beta_k\}_1^r$*'s be defined by* (7) *and the inequalities* (9) *be true. If*

$$-\frac{1}{p(\tau_k)} < \alpha_k < \frac{1}{q(\tau_k)}, \quad k = \overline{1,m},$$

then the general solution of the homogeneous Riemann problem (5) *in the classes* $H^{+}_{p(\cdot),\,\rho} \times_m H^{-}_{p(\cdot),\,\rho}$ *can be represented as*

$$F(z) = P_{m_0}(z)\, Z(z)\,,$$

where $Z(\cdot)$ *is a canonical solution of homogeneous problem, and* $P_{m_0}(\cdot)$ *is a polynomial of degree* $m_0 \leq m$.

This theorem has the following direct:

Corollary 1. *Let all the conditions of Theorem 3.1 be satisfied. Then the homogeneous Riemann problem* (5) *is trivially solvable in the Hardy classes* $H^{+}_{p(\cdot),\rho} \times {}_{-1}H^{-}_{p(\cdot),\rho}$.

4. Reducing The Completeness of Exponential System with Complex Coefficients to Boundary Value Problems

Consider the following exponential system:

$$\left\{A(t)\, e^{int};\, B(t)\, e^{-i(n+1)t}\right\}_{n\in Z_+}, \tag{10}$$

where $A(t) \equiv |A(t)|\, e^{i\alpha(t)}$; $B(t) \equiv |B(t)|\, e^{i\beta(t)}$ are complex-valued functions on $[-\pi,\pi]$. We will consider the completeness of the system (10) in the space $L_{p(\cdot);\rho}$. It is known [6] that the conjugate space of $L_{p(\cdot);\rho}$ is isometrically isomorphic to the space $L_{q(\cdot);\rho}$: $\frac{1}{p(t)} + \frac{1}{q(t)} \equiv 1$. Therefore, the completeness of the system (10) in $L_{p(\cdot);\rho}$ is equivalent to the equality to zero of any function $f(t)$ from the space $L_{q(\cdot);\rho}$ which satisfies the relations

$$\int_{-\pi}^{\pi} A(t)\, e^{int}\overline{f(t)}dt = 0; \quad \int_{-\pi}^{\pi} B(t)\, e^{-i(n+1)t}\overline{f(t)}dt = 0,\, \forall n \in Z_+. \tag{11}$$

Assume that the following main condition is satisfied:

$$\underset{[-\pi,\pi]}{ess\sup}\left\{\, |A(t)|^{\pm 1};\, |B(t)|^{\pm 1}\right\} < +\infty. \tag{12}$$

From the first of equalities (11) we have

$$\int_{-\pi}^{\pi} A(t)\, e^{int}\overline{f}(t)\, dt = \frac{1}{i}\int_{\partial\omega} f^{+}(\tau)\,\tau^{n} d\tau = 0, \forall n \in Z_+, \tag{13}$$

where $f^{+}(\tau) \equiv A(\arg\tau)\ \overline{f}(\arg\tau)\ \bar{\tau}\ ,\ \ \tau \in \partial\omega$.

It is absolutely clear that $f^{+}(\tau) \in L_1(\partial\omega)$. Then it is well known (see [5], p.205) that the conditions (13) are equivalent to the existence of a function $F^{+}(z)$ from H_1^{+} whose nontangential boundary values on $\partial\omega$ coincide with $f^{+}(\tau) : F^{+}(\tau) = f^{+}(\tau)$ a.e. on $\partial\omega$.

Similarly, from the second of equalities (11) we have

$$\int_{-\pi}^{\pi} \overline{B(t)}e^{i(n+1)t} f(t)\, dt = \frac{1}{i}\int_{\partial\omega} f^{-}(\tau)\,\tau^{n} d\tau = 0, \forall n \in Z_+, \tag{14}$$

where $f^-(\tau) = \overline{B(\arg\tau)}\, f(\arg\tau)$, $\tau \in \partial\omega$. For the reason stated above, the equalities (14) are equivalent to the existence of a function $\Phi^+(z) \in H_1^+$ whose nontangential boundary values $\Phi^+(\tau)$ on $\partial\omega$ coincide with $f^-(\tau) : \Phi^+(\tau) = f^-(\tau)$ a.e. on$\partial\omega$.

It is absolutely clear that $F^+(\tau)\,;\,\Phi^+(\tau) \in L_{q(\cdot);\rho}(\partial\omega)$. Consequently, if we additionally require that $p(t) \in WL$, then from theorem in [7] we obtain the inclusion $F^+(z)\,;\,\Phi^+(z) \in H^+_{q(\cdot);\rho}$. Representing $f(t)$ in terms of $F^+(\tau)$ and $\Phi^+(\tau)$, we obtain the following conjugation problem:

$$F^+(\tau) - \frac{A(\arg\tau)}{B(\arg\tau)}\overline{\tau\,\Phi^+(\tau)} = 0, \tau \in \partial\omega.$$

Define the function $F^-(z)$ analytic outside the unit circle:

$$F^-(z) = \frac{1}{z}\overline{\Phi^+\left(\frac{1}{\bar z}\right)}, |z| > 1.$$

It is absolutely clear that $F^-(\infty) = 0$. Moreover, $F^-(\tau) = \bar\tau\,\overline{\Phi^+(\tau)}$, $\tau \in \partial\omega$. Then we arrive at the following Riemann problem:

$$\begin{cases} F^+(\tau) - G(\tau)\,F^-(\tau) = 0\ , & \tau \in \partial\omega, \\ F^-(\infty) = 0\,, \end{cases} \tag{15}$$

where

$$G(\tau) \equiv \frac{A(\arg\tau)}{B(\arg\tau)}, \tau \in \partial\omega.$$

By definition, we have $F^-(z) \in_{-1} H^-_{q(\cdot);\rho}$. Consequently, if the system (10) is incomplete in $L_{p(\cdot);\rho}$, then the Riemann problem (15) is non-trivially solvable in the classes $\left(H^+_{q(\cdot);\rho};\ {}_{-1}H^-_{q(\cdot);\rho}\right)$.

Now let's assume that the problem (15) is non-trivially solvable in the classes $\left(H^+_{q(\cdot);\rho};\ {}_{-1}H^-_{q(\cdot);\rho}\right)$, i.e. $F^+(z) \in H^+_{q(\cdot);\rho}$, $F^-(z) \in_{-1} H^-_{q(\cdot);\rho}$. Define

$\Phi_1^+(z) \equiv \overline{F^-\left(\frac{1}{\bar z}\right)}$ for $|z| < 1$.

We have $F^-(\tau) = \overline{\Phi_1^+(\tau)}$, $\tau \in \partial\omega$ and $\Phi^+(0) = 0$. Then it is clear that the function $\Phi^+(z) = z^{-1}\Phi_1^+(z)$ will be analytic when $|z| < 1$, and moreover, $\Phi^+(z) \in H^+_{q(\cdot);\rho}$. Thus,

$$F^+(\tau) - G(\tau)\,\overline{\tau\,\Phi^+(\tau)} = 0\ , \quad \tau \in \partial\omega,$$

or

$$\frac{F^+(\tau)}{A(\arg\tau)\,\bar\tau} = \frac{\overline{\Phi^+(\tau)}}{B(\arg\tau)}, \tau \in \omega.$$

Denote $f(t) = \frac{\overline{F^+(e^{it})}}{\overline{A(t)}e^{it}} = \frac{\Phi^+(e^{it})}{\overline{B(t)}}$.

It is absolutely clear that $f(t) \in L_{q(\cdot);\rho}$. From $F^+(z)$, $\Phi^+(z) \in H_1^+$ we obtain the equalities

$$\int_{\partial\omega} F^+(\tau)\,\tau^n d\tau = 0; \int_{\partial\omega} \Phi^+(\tau)\,\tau^n d\tau = 0, \forall n \in Z_+.$$

Expressing $F^+(\tau)$ and $\Phi^+(\tau)$ in terms of $f(\arg\tau)$ as $\tau \in \partial\omega$, we have

$$\int_{\partial\omega} A(t)\,e^{-it}\overline{f(t)}e^{int}de^{it} = i\int_{-\pi}^{\pi} A(t)\,e^{int}\overline{f}(t)\,dt = 0, \forall n \in Z_+;$$

$$\int_{\partial\omega} \overline{B(t)}f(t)\,e^{int}de^{it} = i\int_{-\pi}^{\pi} \overline{B(t)}e^{i(n+1)t}f(t)\,dt = 0, \forall n \in Z_+.$$

Obviously, $f(t) \neq 0$ on $[-\pi,\pi]$. Then these relations imply that the system (10) is incomplete in $L_{p(\cdot);\rho}$. So we have the following:

Theorem 8. *Let $p:\ 1 < p^- \leq p^+ < +\infty$, $p(t) \in WL$, and complex-valued coefficients $A(t)$; $B(t)$ satisfy the condition* (12). *Then the exponential system* (10) *is complete in $L_{p(\cdot);\rho}$ only if the Riemann problem* (15) *is only trivially solvable in the classes $\left(H^+_{q(\cdot);\rho};\ {}_{-1}H^-_{q(\cdot);\rho}\right)$.*

5. Completeness of Exponential System with Complex Coefficients in $L_{p(\cdot);\rho}$

In this section, we apply the results of previous sections to obtain the sufficient conditions for the completeness of exponential system with complex coefficients in $L_{p(\cdot);\rho}$. So let's consider the system

$$\left\{A(t)\,e^{int};\ B(t)\,e^{-i(n+1)t}\right\}_{n\in Z_+}, \tag{16}$$

where $A(t) \equiv |A(t)|\,e^{i\alpha(t)}$; $B(t) \equiv |B(t)|\,e^{i\beta(t)}$ are complex-valued functions on $[-\pi,\pi]$. Assume that the following conditions are satisfied:

$$1)\sup_{[-\pi,\pi]} vrai\left\{\left(|A|^{\pm 1};\ |B|^{\pm 1}\right)\right\} < +\infty,$$

2) The function $\theta(t) \equiv \alpha(t) - \beta(t)$ is piecewise continuous on $[-\pi,\pi]$ with points of discontinuity $\{s_i\}_1^r:\ -\pi < s_1 < \ldots < s_r < \pi$. Let $\{h_k\}_1^r = \theta(s_k+0) - \theta(s_k-0)$, $k=\overline{1,r}$, be the jumps of the function $\theta(t)$ at these points and $h_0 = \theta(-\pi) - \theta(\pi)$.

3) $\frac{h_k}{2\pi} + \frac{1}{p(s_k)} \notin Z$ (Z is a set of all integers), where$h_k$is a jump of the function $\theta(t) \equiv \alpha(t) - \beta(t)$ at the discontinuity point s_k, $k=\overline{0,r}$; $s_0 = \pi$.

Define the integers n_i, $i=\overline{1,r}$, from the following inequalities:

$$\begin{cases} -\frac{1}{p(s_k)} < \frac{h_k}{2\pi} + n_k - n_{k-1} < \frac{1}{q(s_k)}\,,\ k=\overline{1,r}\,, \\ \\ n_0 = 0\,. \end{cases} \tag{17}$$

Let
$$\Delta_r = \frac{1}{2\pi}\left[\alpha\left(-\pi\right) - \alpha\left(\pi\right) + \beta\left(\pi\right) - \beta\left(-\pi\right)\right] - n_r.$$
The following theorem is true:

Theorem 9. *Let the coefficients $A(t)$ and $B(t)$ of the system* (16) *satisfy the conditions 1)-3), where $G\left(e^{it}\right) \equiv \frac{A(t)}{B(t)}$, the integer n_r is defined by* (17)*, $p(t) \in WL$, $1 < p^- \leq p^+ < +\infty$. Then, if $\Delta_r \notin Z$ and $\Delta_r > -\frac{1}{p(\pi)}$, then the system (5.1) is complete in the space $L_{p(\cdot);\rho}$.*

Proof. Let the conditions 1)-3) be satisfied. We first assume that the inequalities (17) hold for $n_k = 0\,, k = \overline{1,r}$. Let $G(t) \equiv \frac{A(t)}{B(t)}$, $t \in [-\pi, \pi]$. As established above, the completeness of the system (16) in $L_{p(\cdot);\rho}$ is only equivalent to the trivial solvability of the Riemann problem
$$\begin{cases} F^+\left(e^{it}\right) - G(t)\, F^-\left(e^{it}\right) = 0\ , \ \ t \in [-\pi, \pi]\,, \\ \\ F^+ \in H^+_{q(\cdot);\rho}\,;\ F^- \in_{-1} H^{-1}_{q(\cdot);\rho}\,. \end{cases} \tag{18}$$
It follows from (17) that $n_r = 0$, and hence $h_0 = 2\pi\Delta_r$. Suppose $-\frac{1}{p(\pi)} < \Delta_r < \frac{1}{q(\pi)}$. Then, by Corollary 3.1, the problem (18) has only the trivial solution. Consequently, by Theorem 4.1, the system (10) is complete in $L_{p(\cdot);\rho}$ in the considered case. Now let $\Delta_r \notin Z$ and $\Delta_r > \frac{1}{q(\pi)}$, for example, $\Delta_r \in \left[\frac{1}{q(\pi)}, \frac{1}{q(\pi)} + 1\right)$. Consider the system
$$\left\{A(t)\, e^{int};\, B(t)\, e^{-int}\right\}_{n\in N}. \tag{19}$$
Reduce it to the following form:
$$\left\{\tilde{A}(t)\, e^{int};\, B(t)\, e^{-i(n+1)t}\right\}_{n\in Z_+}, \tag{20}$$
where $\tilde{A}(t) \equiv A(t)\, e^{it} \equiv |A(t)|\, e^{i\tilde{\alpha}(t)}$ and $\tilde{\alpha}(t) \equiv t + \alpha(t)$. Calculate $\tilde{\Delta}_r$, which corresponds to the system (20). We have
$$\tilde{\Delta}_r = \frac{1}{2\pi}\left[\tilde{\alpha}\left(-\pi\right) - \tilde{\alpha}\left(\pi\right) + \beta\left(\pi\right) - \beta\left(-\pi\right)\right] = \Delta_r - 1.$$
Thus, $\tilde{\Delta}_r \in \left(-\frac{1}{p(\pi)}, \frac{1}{q(\pi)}\right)$. Then it follows from the previous discussion that the system (20), and hence the system (16), is complete in $L_{p(\cdot);\rho}$. Continuing this process, we obtain the completeness of the system (16) in $L_{p(\cdot);\rho}$ for $\Delta_r > -\frac{1}{p(\pi)}$.

Now let's consider the general case. Let all the conditions of the theorem be satisfied. Express the unit function $e(t)$ on $[-\pi, \pi]$ in the form $e(t) \equiv e^{i \arg e(t)}$:
$$\arg\, e(t) \equiv \begin{cases} 0, & -\pi < t \leq s_1\,, \\ 2\pi n\,, & s_1 < t \leq s_2\,, \\ \quad\vdots & \\ 2\pi n_r\,, & s_r < t \leq \pi\,, \end{cases}$$

where n_k , $k = \overline{1, r}$ are the integers defined by (17). Replace the coefficient $A(t)$ with the function $A_0(t)$ which is equal to it: $A_0(t) = A(t) \cdot e(t)$. So, $\alpha_0(t) \equiv \arg A_0(t) = \alpha(t) + \arg e(t)$. Consider the system

$$\left\{A_0(t)\, e^{int};\ B(t)\, e^{-i(n+1)t}\right\}_{n \in Z_+} . \tag{21}$$

It is absolutely clear that the basis properties of the systems (16) and (21) in $L_{p(\cdot);\rho}$ are the same. We have

$$\theta_0(t) = \arg A_0(t) - \arg B(t) = \alpha(t) + \arg e(t) - \beta(t) = \theta(t) + \arg e(t) .$$

It is clear that the points of discontinuity of the functions $\theta_0(t)$ and $\theta(t)$ are the same. We have

$$h_k^0 = \theta_0(s_k + 0) - \theta_0(s_k - 0) = h_k + \arg e\,(s_k + 0) - \arg e(s_k - 0) =$$

$$= h_k + 2\pi(n_k - n_{k-1}) ,\ k = \overline{1, r} ,$$

where $n_0 = 0$. Thus

$$\frac{h_0^0}{2\pi} = \frac{h_k}{2\pi} + n_k - n_{k-1} ,\ k = \overline{1, r}.$$

On the other hand

$$h_0^0 = \theta_0(-\pi) - \theta_0(\pi) = \theta(-\pi) - \theta(\pi) +$$

$$+ \arg e(-\pi) - \arg e(\pi) = h_0 - 2\pi n_r \Rightarrow \tfrac{h_0^0}{2\pi} = \Delta_r .$$

Then it follows from the previous discussion that for $\Delta_r > -\frac{1}{p(\pi)}$ the system (21), and hence the system (16), is complete in $L_{p(\cdot);\rho}$.

Theorem is proved.

Let's apply the obtained theorem to the special case

$$\left\{e^{i[n + \alpha\, sign\, n]t}\right\}_{n \in Z} , \tag{22}$$

where $\alpha \in C$ is a complex parameter. Basis properties of the system (22) in the spaces $L_{p(\cdot);\rho}$ have been well studied. From Theorem 5.1 we have

Corollary 5.1. *Let* $p(t) \in WL$, $1 < p^- \leq p^+ < +\infty$ *and* $Re\alpha \in Z$. *If* $Re\alpha < \frac{1}{2p(\pi)}$, *then the system* (22) *is complete in* $L_{p(\cdot);\rho}$.

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Received 03 June 2021
Accepted 01 September 2021