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On the Generalised Hardy Inequality and the Best Constant

Sayali Mammadli

Abstract. Let Ω be a ball $B(0,R)$ in $\mathbb{R}^n$ and $p>1$. Using a different approach we prove the generalized Hardy inequality involving the distance to boundary

$$\left\|\mathrm{dist}(x,\partial\Omega)^{\alpha-1}u(x)\right\|_{L^p(\Omega)} \leq C_p\|\mathrm{dist}(x,\partial\Omega)^{\alpha}\nabla u(x)\|_{L^p(0,l)},$$

where $\alpha<(p-1)/p$ and $u\in \dot{W}^1_p(\Omega)$.

Key Words and Phrases: variable exponent, eigenvalue problem, Hardy's operator, compact embedding.

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1. Introduction

In this paper, we show an elementary approach to prove the weighted Sobolev inequality

$$\|\mathrm{dist}(x,\partial\Omega)^{\alpha-1}f(x)\|_{L^p(\Omega)} \leq C\|\mathrm{dist}(x,\partial\Omega)^{\alpha}\nabla f(.)\|_{L^p(\Omega)}, \tag{1}$$

for any $f\in \dot{W}^1_p(\Omega)$, where $-\infty<\alpha<1/p', 1<p<\infty$, the domain $\Omega\subset\mathbb{R}^n$ is bounded and satisfies some conditions, $\mathrm{dist}(x,\partial\Omega)$ is a distance from the point $x\in\Omega$ to the boundary $\partial\Omega$ of domain Ω. The space $\dot{W}^1_p(\Omega)$ is a closure of $C^1_0(\Omega)$, continuously differentiable functions vanishing on the $\partial\Omega$, in the norm

$$\|f\| = \|f\|_{L^p(\Omega)} + \|\nabla f\|_{L^p(\Omega)}.$$

Such inequalities take its beginning from the work by J. Necas [22], who proposed the inequality

$$\int_\Omega \frac{|f(x)|^p}{\mathrm{dist}(x,\partial\Omega)^p}\,dx \leq C_{n,p}(\Omega)\int_\Omega |\nabla f|^p\,dx \tag{2}$$

for any n dimensional bounded Lipschitz domain Ω for $p > 1$, where ∇f denotes the gradient of function f.

In general, inequality (2) does not hold in the non-regular domains. The equality is also not achieved (see, [23] about this and historical background). The inequality (2) gives room for various improvements. See, [4, 5, 18, 20] for the inequality (2) and the best constant $C_p = p/(p-1)$ for that on convex domains . Note that, the Hardy inequalities are derived for the different function spaces. For that we refer to [6, 9, 10, 11, 12, 14, 15] on variable exponent Lebesgue space settings (see also e.g. [1, 2, 3, 13, 16, 17, 19] for applications to pde's).

On the background of this inequalities, the following observation is interesting (see [9]). The Hardy inequality on finite interval (0,l) reads

$$\int_0^l \left(\frac{1}{x}\int_0^x f(t)dt\right)^p dx \le \left(\frac{p}{p-1}\right)^p \int_0^l f(x)^p\, dx, \tag{3}$$

where f is a positive measurable function on $(0,l)$ and $p > 1$. The constant $(\frac{p}{p-1})^p$ is sharp. After the change of variables $y = l - x$ the left hand side of (3) equals

$$\int_0^l \left(\frac{1}{l-y}\int_y^l g(s)\, ds\right)^p dy = \int_0^l \left(\frac{1}{l-y}\int_0^{l-y} f(t)\, dt\right)^p dy$$

$$= \int_0^l \left(\frac{1}{x}\int_0^x f(t)\, dt\right)^p dx,$$

where $g(t) = f(l-t)$. Changing again the variable $t = l - s$ on the right hand side (3), it is equal to

$$\left(\frac{p}{p-1}\right)^p \int_0^l f(t)^p\, dt = \left(\frac{p}{p-1}\right)^p \int_0^l f(l-s)^p\, ds = \left(\frac{p}{p-1}\right)^p \int_0^l g(s)^p\, ds$$

Therefore (3) yields that

$$\int_0^l \left(\frac{1}{l-y}\int_y^l g(s)\, ds\right)^p dy = \left(\frac{p}{p-1}\right)^p \int_0^l g(s)^p\, ds. \tag{4}$$

Since f is arbitrary, we get the inequality (4) for any measurable function g on interval $(0,l)$.

Let u be an absolutely continuous function on $(0,l)$ satisfying $u(0) = u(l) = 0$. Then

$$\frac{u(x)}{d(x)} = \frac{u(x)}{x},\ x \in (0, l/2) \quad \text{and} \quad \frac{u(x)}{d(x)} = \frac{u(x)}{l-x},\ x \in (l/2, l),$$

where $d(x) = min(x, l-x)$, $x \in (0,l)$. Now,

$$\frac{u(x)}{d(x)} = \frac{u(x)}{x}\chi_{0,l/2}(x) + \frac{u(x)}{l-x}\chi_{l/2,l}(x),$$

where $\chi_{(a,b)}$ denote a characteristic function of interval (a,b). Therefore, and applying (3), (4) we get the inequality

$$\|u(x)/d(x)\|^p_{L^p(0,l)} = \|u(x)/x\|^p_{L^p(0,l/2)} + \|u(x)/(l-x)\|^p_{L^p(l/2,l)}$$

$$\leq (p/(p-1))^p \left(\|u'(x)\|^p_{L^p(0,l/2)} + \|u'(x)\|^p_{L^p(l/2,l)}\right) = (p/(p-1))^p \|u'(x)\|^p_{L^p(0,l)}.$$

We get the inequality

$$\|u(x)/d(x)\|_{L^p(0,l)} \leq p/(p-1)\|u'(x)\|_{L^p(0,l)} \tag{5}$$

for all absolutely continuous function on interval $(0,l)$, with $u(0) = u(l) = 0$.

Lemma 1. *Let $f(t)$ be a positive measurable function on $(0,l)$. Let $p > 1, \alpha < \frac{1}{p'}$, $n > 0$. Then it holds the inequality*

$$\int_0^l \left(\int_x^l f(t)\,dt\right)^p (l-x)^{(\alpha-1)p} x^{n-1}\,dx$$
$$\leq \left(\frac{p}{p-1-\alpha p}\right)^p \int_0^l (l-x)^{\alpha p} x^{n-1} f(x)\,dx \tag{6}$$

Proof. Using the change of variables $l - x = z$ we get

$$\int_0^l \left(\int_x^l f(t)\,dt\right)^p (l-x)^{(\alpha-1)p} x^{n-1}\,dx$$

$$= \int_0^l \left(\int_{l-z}^l f(t)\,dt\right)^p z^{(\alpha-1)p}(l-z)^{n-1}\,dz,$$

applying again the change of variable $t = l - y$ this equal

$$\int_0^l \left(\int_0^z f(l-y)\,dy\right)^p z^{(\alpha-1)p}(l-z)^{n-1}\,dz$$

since $l - z \leq l - y$ as $u \in (0,z)$ the last integral is exceeded

$$\int_0^l \left(\int_0^z (l-y)^{\frac{n-1}{p}} f(l-y)\,du\right)^p z^{(\alpha-1)p}\,dz$$

On basis of the Hardy inequality (see, [8, p.23]) this is exceeded

$$\leq \left(\frac{p}{p-1-\alpha p}\right)^p \int_0^l y^{\alpha p}(l-y)^{n-1} f(l-y)^p\,du.$$

Using change of variable $l-y=x$ the last term equals

$$\left(\frac{p}{p-1-\alpha p}\right)^p \int_0^l (l-x)^{\alpha p} x^{n-1} f(x)^p\, dx.$$

This proves Lemma 1.

The inequality (6) gives the next assertion for the absolutely continuous functions.

Theorem 2. *Let $1<p<\infty$, $-\infty<\alpha<\frac{1}{p'}$, $n>1$. Let $u(x)$ be an absolutely continuos function on $(0,l)$ such that $\lim\limits_{x\to 0} u(x) = \lim\limits_{x\to l} u(x) = 0$. Then it holds the inequality*

$$\left\| x^{(n-1)/p} \frac{u(x)}{d(x)^{1-\alpha}} \right\|_{L^p(0,l)} \le \frac{p}{p-1-\alpha p} \left\| d(x)^\alpha x^{(n-1)/p} \frac{du}{dx} \right\|_{L^p(0,l)}, \quad (7)$$

where $d(x)=\min(x,l-x)$.

Note, the constant $\frac{p}{p-1-\alpha p}$ is exact but is not achievable (see, [6]).

Proof. Now,

$$\frac{u(x)}{d(x)} = \frac{u(x)}{x^{1-\alpha}} \chi_{0,l/2}(x) + \frac{u(x)}{(l-x)^{1-\alpha}} \chi_{l/2,l}(x),$$

where $\chi_{(a,b)}$ denote a characteristic function of interval (a,b). Therefore, and applying (6), (4) we get the inequality

$$\left\| \frac{u(x)}{d(x)^{1-\alpha}} x^{(n-1)/p} \right\|^p_{L^p(0,l)} = \int_0^{l/2} \left| \frac{u(x)}{x^{1-\alpha}} \right|^p x^{n-1}\, dx + \int_{l/2}^l \left| \frac{u(x)}{(l-x)^{1-\alpha}} \right|^p x^{n-1}\, dx. \quad (8)$$

Set in the inequality (6) $f(x)\chi_{(l/1,l)}$ in place of $f(x)$. Then

$$\frac{1}{n}\left(\int_{l/2}^l f(t)\, dt\right)^p (l/2)^{(\alpha-1)p+n} + \int_{l/2}^l \left(\int_x^l f(t)\, dt\right)^p (l-x)^{(\alpha-1)p} x^{n-1}\, dx$$
$$\le \left(\frac{p}{p-1-\alpha p}\right)^p \int_{l/2}^l (l-x)^{\alpha p} x^{n-1} f(x)\, dx \quad (9)$$

Therefore,

$$\int_{l/2}^l \left(\int_x^l f(t)\, dt\right)^p (l-x)^{(\alpha-1)p} x^{n-1}\, dx \le \left(\frac{p}{p-1-\alpha p}\right)^p \int_{l/2}^l (l-x)^{\alpha p} x^{n-1} f(x)\, dx \quad (10)$$

From (8) using inequality (10) and the Hardy inequality for power weights we get

$$\left\| \frac{u(x)}{d(x)^{1-\alpha}} x^{(n-1)/p} \right\|^p_{L^p(0,l)} \le \left(\frac{p}{p-n-\alpha p} \right)^p \left(\int_0^{l/2} \left| u'(x) \right|^p x^{\alpha p + n - 1}\, dx \right)$$

$$+ \left(\frac{p}{p-1-\alpha p} \right)^p \left(\int_{l/2}^{l} \left| u'(x) \right|^p (l-x)^{\alpha p} x^{n-1}\, dx \right).$$

Since $\left(\frac{p}{p-n-\alpha p}\right)^p < \left(\frac{p}{p-1-\alpha p}\right)^p$ we get the inequality

$$\left\| \frac{u(x)}{d(x)^{1-\alpha}} x^{(n-1)/p} \right\|^p_{L^p(0,l)} \le \left(\frac{p}{p-1-\alpha p} \right)^p \left(\int_0^{l} \left| \frac{du}{dx} \right|^p d(x)^{\alpha p} x^{n-1}\, dx \right).$$

This completes the proof of Theorem 2.

Usually, the constant $\left(\frac{p}{p-1-\alpha p}\right)^p$ in such type inequality is the best but it is not realized (see, e.g. [6] for $\alpha = 0$).

Theorem 3 (Main result)**.** *Let $1 < p < \infty$, $-\infty < \alpha < \frac{p-1}{p}$ and the ball $B_R = B(0,R)$ in $\mathbb{R}^n$. Then it holds the inequality*

$$\int_{B_R} (R-|x|)^{\alpha p} \left(\frac{|u(x)|}{R-|x|} \right)^p dx \le \left(\frac{p}{p-1-\alpha p} \right)^p \int_{B_R} (R-|x|)^{\alpha p} \left| \frac{\partial u(x)}{\partial |x|} \right|^p dx$$

for any $u \in \dot{W}^1_p(B_R)$; $\frac{\partial}{\partial |x|} = \sum_{j=1}^n \frac{x_j}{|x|} \cdot \frac{\partial}{\partial x_j}$

Proof. Let $u(x)$ be a $C_0^1(B_R)$ function. Using the duality in the weighted Lebesgue spaces, one can find the positive function $\phi \in L^p(B_R)$ with $\|\phi\|_{L^p(B_R)} = 1$ and such that

$$\left(\int_{B_R} \operatorname{dist}(x, \partial B_R)^{\alpha-1} \left| u(x) \right| dx \right)^{1/p} = \int_{B_R} \operatorname{dist}(x, \partial B_R)^{\alpha-1} \left| u(x) \right| \phi(x)\, dx$$

Using spherical system of coordinates, the right hand side may be written as

$$\int_{S_{n-1}} \left(\int_0^R (R-s)^{\alpha-1} |u(s,\theta)| \phi(s,\theta) s^{n-1}\, ds \right) |J(\theta)| d\theta, \tag{11}$$

where $|\frac{D(x)}{D(s,\theta)}| = s^{n-1}|J(\theta)|$ is a Jacobean of transformation from cartesian coordinate system x to the spherical (s,θ) (see, e.g. [21]), S_{n-1} is the unit sphere in $\mathbb{R}^n$. Applying the Newtonian-Leibniz formula, we have

$$u(s,\theta) = -\int_s^R \frac{\partial u(t,\theta)}{\partial t}\, dt, \quad s \in (0,R)$$

since $u(R,\theta)=0$. Inserting it from (11) we find the expression for that

$$\int_{S_{n-1}}\left(\int_0^R (R-s)^{\alpha-1}\left|\int_s^R \frac{\partial u(t,\theta)}{\partial t}\,dt\right|\phi(s,\theta)s^{n-1}\,ds\right)|J(\theta)|d\theta, \tag{12}$$

on base of the Holder inequality,

$$\leq \int_{S_{n-1}}\left\{\left[\int_0^R\left((R-s)^{\alpha-1}\int_s^R\left|\frac{\partial u(t,\theta)}{\partial t}\right|\,dt\right)^p s^{n-1}\,ds\right]^{1/p}\times\right.$$
$$\left.\times\left[\int_0^R \phi^p(s,\theta)\,s^{n-1}\,ds\right]^{1/p'}\right\}|J(\theta)|d\theta.$$

Using Lemma 1 this is exceeded (set $f=\frac{\partial u}{\partial s}$)

$$\leq \frac{p}{p-1-\alpha p}\cdot\int_{S_{n-1}}\left\{\left[\int_0^R (R-s)^{\alpha p}\left|\frac{\partial u(s,\theta)}{\partial s}\right|^p s^{n-1}\,ds\right]^{1/p}\times\right.$$
$$\left.\times\left[\int_0^R \phi^p(s,\theta)\,s^{n-1}\,ds\right]^{1/p'}\right\}|J(\theta)|d\theta. \tag{13}$$

Applying the Holder inequality in the interior integral and using that $\|\phi\|_{L^p(\Omega)}=1$ we get the final estimate for the right hand side of (13) it is estimated as

$$\leq \frac{p}{p-1-\alpha p}\cdot\left\{\int_{S_{n-1}}\left(\int_0^R (R-s)^{\alpha p}\left|\frac{\partial u(s,\theta)}{\partial s}\right|^p s^{n-1}\,ds\right)d\theta\right\}^{1/p}\times$$
$$\times\left\{\int_{S_{n-1}}\left(\int_0^R \phi^p(s,\theta)\,s^{n-1}\,ds\right)|J(\theta)|d\theta\right\}^{1/p'}$$
$$=\frac{p}{p-1-\alpha p}\cdot\left(\int_{B_R}(R-|x|)^{\alpha p}\left|\frac{\partial}{\partial|x|}u(x)\right|^p\,dx\right)^{1/p}\left(\int_{B_R}\phi^p(x)\,dx\right)^{1/p'}.$$
$$=\frac{p}{p-1-\alpha p}\cdot\left(\int_{B_R}(R-|x|)^{\alpha p}\left|\frac{\partial}{\partial|x|}u(x)\right|^p\,dx\right)^{1/p}.$$

Therefore,

$$\int_{B_R}(R-|x|)^{(\alpha-1)p}\,|u(x)|^p\,dx\leq\left(\frac{p}{p-1-\alpha p}\right)^p\int_{B_R}(R-|x|)^{\alpha p}\left|\frac{\partial}{\partial|x|}u(x)\right|^p\,dx \tag{14}$$

Theorem 3 has been proved.

In terms of $\mathrm{dist}(x,\partial\Omega)$, for domain $\Omega=B_R$, inequality (14) is written as

$$\int_\Omega|\mathrm{dist}(x,\partial\Omega)^{\alpha-1}u(x)|^p\,dx\leq\left(\frac{p}{p-1-\alpha p}\right)^p\int_\Omega\mathrm{dist}(x,\partial\Omega)^{\alpha p}\left|\frac{\partial u(x)}{\partial|x|}\right|^p\,dx$$

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Sayali Mammadli
Mathematics and Mechanics Institute of National Academy of Sciences
B.Vahabzade str., 9, AZ 1141, Baku, Azerbaijan E-mail: sayka-426@mail.ru

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