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Conditions for the boundedness of the G-fractional integral and G-maximal function in modified G-Morrey spaces

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Abstract. In this paper we find conditions for the strong and weak boundedness of the G-fractional integral and G-maximal operator.

Key Words and Phrases: G-fractional integral, G-maximal function, modified G-Morrey spaces, Hardy-Littlewood-Sobolev inequality.

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1. Introduction

The study of boundedness of the fractional integral operator, singular integrals, maximal function were studied by lots of researchers in the last decodes. Morrey estimates of such kind of operators is a more recent problem and is still very popular. Just as an example we recall the study made in [1,3,7,8,10,11].

In this paper we introduce modified Gegenbauer Morrey space (G- Morrey space) and prove Adams type theorem on the boundedness of the G- fractional interal. The result obtained is an analog of the corresponding theorem obtained for Riesz potential in [4].

Let $1 \leq p < \infty$ and $0 \leq \lambda \leq n$. The classical Morrey spaces is defined by

$$M_{p,\lambda}(R^n) = \left\{ f \in L^p_{loc}(R^n) : \|f\|_{L^{p,\lambda}} < \infty \right\}, \tag{1}$$

where

$$\|f\|_{L^{p,\lambda}} := \sup_{\theta} \left(\frac{1}{|\theta|^{\lambda/n}} \int_{\theta} |f(x)|^p \, dx \right)^{1/p},$$

the supremum, is taken over all cubes $Q \subset R^n$. It is well known that if $1 \leq p < \infty$ then $M_{p,0}(R^n) = L^p(R^n)$ and $M_{p,n}(R^n) = L^{\infty}(R^n)$.

Morrey spaces were originally introduced by Morrey in [15] to study the local behavior of solutions to second-order

Morrey spaces were originally introduced by Morrey in [15] to study the local behavior of solutions to secon-order elliptic partial differential equations.

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In [1], Adams for the Riesz potential

$$J_\alpha f(x) = \int_{R^n} \frac{f(y)\,dy}{|x-y|^{n-\alpha}},\ 0<\alpha<n$$

on the Morrey space proved the following theorem.

Theorem A. [1] Let $0<\alpha<n$ and let $0\le\lambda\le n$, $1\le p<(n-\lambda)/\alpha$.

1. If $1<p<(n-\lambda)/\alpha$ then the condition $1/p-1/q=\alpha/(n-\lambda)$ is necessary and sufficient for the boundedness of J_α from $M_{p,\lambda}(R^n)$ to $M_{q,\lambda}(R^n)$.
2. If $p=1$, then the condition $1-1/q=\alpha/(n-\lambda)$ is necessary and sufficient for the boundedness of J_α from $M_{1,\lambda}(R^n)$ to $M_{q,\lambda}(R^n)$.

In the work [12] is proved analog of this theorem for the Gegenbauer fractional integral on G- Morrey space.

The structure of the paper is as follows.

Section 1 is for informational purposes. In Section 2 are given some definition, notation and auxiliary results. In Section 4 is proved the theorem of strong and weak boundedness for maximal operator and also the Hardy-Littlewood-Sobolev type inequality for the G-fractional integral in modified G-Morrey spaces.

2. Definition, notation and auxiliary results

The generalized shift operator associated with the Gegenbauer differential operator G

$$G=G_\lambda=\left(x^2-1\right)^{\frac{1}{2}-\lambda}\frac{d}{dx}\left(x^2-1\right)^{\lambda+\frac{1}{2}}\frac{d}{dx},\ x\in(1,\infty),\ \lambda\in\left(0,\frac{1}{2}\right),$$

introduced in [7] has the form

$$A^\lambda_{ch\,y} f(ch\,x) = \frac{\Gamma\left(\lambda+\frac{1}{2}\right)}{\Gamma(\lambda)\Gamma\left(\frac{1}{2}\right)}\int_0^\pi f(ch\,x ch\,y - sh\,x sh\,y\cos\varphi)(\sin\varphi)^{2\lambda-1}d\varphi.$$

Let $L_p(R_+,G)\equiv L_{p,\lambda}(R_+)$, $1\le p\le\infty$, denote the space of $\mu_\lambda(x)=sh^{2\lambda}x$ measurable functions on $R_+=(0,\infty)$ with finite norm

$$\|f\|_{L_{p,\lambda}(R_+)} = \left(\int_0^\infty |f(ch\,x)|^p sh^{2\lambda}x dx\right)^{\frac{1}{p}},\ 1\le p<\infty,$$

$$\|f\|_{L_{\infty,\lambda}(R_+)} = \|f\|_{L_\infty} = \operatorname*{es\,sup}_{x\in R_+}|f(ch\,x)|.$$

For all measurable sets $E\subset R_+$ put $\mu E=|E|_\lambda=\int_E sh^{2\lambda}xdx$.

Also but $WL_{p,\lambda}(R_+)$, $1\le p<\infty$, denote the weak space $L_{p,\lambda}(R_+)$ of locally integrable functions $f(ch\,x)$, $x\in R_+$ with finite norm

$$\|f\|_{WL_{p,\lambda}(R_+)}=\sup_{r\in R_+} r\left|\{x\in R_+ : |f(chx)|>r\}\right|_\lambda^{\frac{1}{p}}$$

$$=\sup_{r\in R_+} r\left(\int_{\{x\in R_+:|f(chx)|>r\}} sh^{2\lambda}xdx\right)_\lambda^{\frac{1}{p}}.$$

In what follows, the expression $A\lesssim B$ will mean that there exist a constant C such that $0<A\le CB$, where C may depend on some inessential parameters. If $A\lesssim B$ and $B\lesssim A$, the we write $A\approx B$ and say that A and B are equivalent.

Denote $H_r=(0,r)\subset R_+$. Further, we need the following relation (see [14] Lemma 2.3 by $x=0$,$\gamma=2\lambda$)

$$|H_r|_\lambda=\int_0^r sh^{2\lambda}tdt\approx\left(sh\frac{r}{2}\right)^\gamma, \tag{2}$$

where

$$\gamma=\gamma_\lambda(r)=\begin{cases}2\lambda+1, & 0<r<2,\\ 4\lambda, & 2\le r<\infty,\end{cases}$$

and $0<\lambda<1/2$.

In [10] the Gegenbauer maximal function (G-maximal function) is defined as follows:

$$M_Gf(ch\,x)=\sup_{r>0}\frac{1}{|H_r|_\lambda}\int_{H_r}A_{ch\,y}^\lambda|f(ch\,x)|\,sh^{2\lambda}ydy.$$

In what follows we need the following Fefferman-Stein type inequality.

Theorem B ([9, Theorem 1.4]). For every $1\le p<\infty$ and every $0<t<\infty$ the inequality

$$\int_0^r A_{ch\,y}^\lambda(M_Gf(ch\,x))^p\,sh^{2\lambda}ydy\le\int_0^r A_{ch\,y}^\lambda|f(ch\,x)|^p\,sh^{2\lambda}ydy$$

is true.

Theorem C ([10, Theorem 1.5]). The Chebyshev-type inequality

$$\left|x\in(0,r):A_{ch\,y}^\lambda M_Gf(ch\,x)>\alpha\right|_\lambda\le\frac{1}{\alpha}\int_0^r A_{ch\,y}^\lambda M_Gf(ch\,x)\,sh^{2\lambda}ydy$$

is true for all $\alpha>0$ and $t>0$.

Theorem D [10] a) If $f\in L_{1,\lambda}(R)$, then for all $\alpha>0$ the inequality

$$|\{x\in R_+:M_Gf(ch\,x)>\alpha\}|_\lambda\lesssim\frac{1}{\alpha}\|f\|_{L_{1,\lambda}(R_+)}.$$

b) If $f\in L_{p,\lambda}(R_+)$, $1<p\le\infty$, then $M_Gf(ch\,x)\in L_{p,\lambda}(R_+)$ and

$$\|M_Gf\|_{L_{p,\lambda}(R_+)}\lesssim\|f\|_{L_{p,\lambda}(R_+)}.$$

Corollary E. If $f \in L_{p,\lambda}(R_+)$, $1 \le p \le \infty$, then

$$\lim_{r\to 0} \frac{1}{|H_r|_\lambda} \int_0^r A^\lambda_{ch\, y} f(ch\, x)\, sh^{2\lambda} y dy = f(ch\, x)$$

for a.e. $x \in R_+$.

3. Some embeddings into the G- Morrey and modified G-Morrey spaces.

We introduce the following nonation analogously in [8].

Definition 1. *Let $1 \le p < \infty$, $0 < \lambda < 1/2$, $0 \le \nu \le \gamma$, $\left[sh\frac{r}{2}\right]_1 = \min\left\{1,\, sh\frac{r}{2}\right\}$. We denote by $L_{p,\lambda,\nu}(R_+)$ the $G-$Morrey space, and by $\tilde{L}_{p,\lambda,\nu}(R_+)$ the modified $G-$Morrey space, as the set of locally integrable functions $f(ch\, x)$, $x \in R_+$ with the finite norms*

$$\|f\|_{L_{p,\lambda,\nu}(R_+)} = \sup_{x,r\in R_+} \left(\left(sh\frac{r}{2}\right)^{-\nu} \int_0^r A^\lambda_{ch\, y} |f(ch\, x)|^p\, sh^{2\lambda} y dy \right)^{\frac{1}{p}},$$

$$\|f\|_{\tilde{L}_{p,\lambda,\nu}(R_+)} = \sup_{x,r\in R_+} \left(\left[sh\frac{r}{2}\right]_1^{-\nu} \int_0^r A^\lambda_{ch\, y} |f(ch\, x)|^p\, sh^{2\lambda} y dy \right)^{\frac{1}{p}},$$

respectively.

Note that

$$\tilde{L}_{p,\lambda,0}(R_+) = L_{p,\lambda,0}(R_+) = L_{p,\lambda}(R_+).$$

$$\tilde{L}_{p,\lambda,0}(R_+) \subset L_{p,\lambda,\nu}(R_+) \bigcap L_{p,\lambda}(R_+)$$

and

$$\max\left\{ \|f\|_{L_{p,\lambda,\nu}},\, \|f\|_{L_{p,\lambda}} \right\} \le \|f\|_{\tilde{L}_{p,\lambda,\nu}}.$$

Definition 2. *Let $1 \le p < \infty$, $0 < \lambda < 1/2$, $0 \le \nu \le \gamma$. We denote by $WL_{p,\lambda,\nu}(R_+)$ the weak $G-$Morrey space and by $W\tilde{L}_{p,\lambda,\nu}(R_+)$ the modified weak $G-$Morrey space as the set of locally integrable functions $f(ch\, x)$, $x \in R_+$ with finite norms*

$$\|f\|_{WL_{p,\lambda,\nu}(R_+)} = \sup_{r\in R_+} r \sup_{x,t\in R_+} \left(\left(sh\frac{t}{2}\right)^{-\nu} \left| \left\{ y \in (0,t) : A^\lambda_{ch\, y} |f(ch\, x)| > r \right\} \right|_\lambda \right)^{\frac{1}{p}}$$

$$= \sup_{r\in R_+} r \sup_{x,t\in R_+} \left(\left(sh\frac{t}{2}\right)^{-\nu} \int_{\left\{y\in(0,t) : A^\lambda_{ch\, y}|f(ch\, x)|>r\right\}} sh^{2\lambda} y dy \right)^{\frac{1}{p}},$$

$$\|f\|_{W\tilde{L}_{p,\lambda,\nu}(R_+)} = \sup_{r\in R_+} r \sup_{x,t\in R_+} \left(\left[sh\frac{t}{2}\right]_1^{-\nu} \left| \left\{ y \in (0,t) : A^\lambda_{ch\, y} |f(ch\, x)| > r \right\} \right|_\lambda \right)^{\frac{1}{p}}$$

$$= \sup_{r\in R_+} r \sup_{x,t\in R_+} \left(\left[sh\frac{t}{2}\right]_1^{-\nu} \int_{\left\{y\in(0,t) : A^\lambda_{ch\, y}|f(ch\, x)|>r\right\}} sh^{2\lambda} y dy \right)^{\frac{1}{p}}$$

respectively.

Note that $WL_{p,\lambda,0}(R_+) = WL_{p,\lambda}(R_+) = W\tilde{L}_{p,\lambda,0}(R_+)$, $L_{p,\lambda,\nu}(R_+) \subset W\tilde{L}_{p,\lambda,\nu}(R_+)$ and $\|f\|_{WL_{p,\lambda,\nu}} \le \|f\|_{L_{p,\lambda,\nu}}$, $\tilde{L}_{p,\lambda,\nu}(R_+) \subset W\tilde{L}_{p,\lambda,\nu}(R_+)$ and $\|f\|_{W\tilde{L}_{p,\lambda,\nu}} \le \|f\|_{\tilde{L}_{p,\lambda,\nu}}$.

Lemma 1. *Let* $1 \le p < \infty$, $0 < \lambda < 1/2$, $0 \le \nu \le \gamma$. *Then*

$$L_{p,\lambda,\nu}(R_+) = L_{p,\lambda,\nu}(R_+) \bigcap L_{p,\lambda}(R_+)$$

and

$$\|f\|_{\tilde{L}_{p,\lambda,\nu}} = \max\left\{\|f\|_{L_{p,\lambda,\nu}}, \|f\|_{L_{p,\lambda}}\right\}.$$

Proof. Let $f \in \tilde{L}_{p,\lambda,\nu}(R_+)$. Then

$$\|f\|_{L_{p,\lambda}(R_+)} = \sup_{x,r\in R_+}\left(\int_0^r A^\lambda_{ch\,y}|f(ch\,x)|^p sh^{2\lambda}y dy\right)^{\frac{1}{p}}$$

$$\le \sup_{x,r\in R_+}\left(\left[sh\frac{r}{2}\right]_1^{-\nu}\int_0^r A^\lambda_{ch\,y}|f(ch\,x)|^p sh^{2\lambda}y dy\right)^{\frac{1}{p}}$$

$$= \|f\|_{\tilde{L}_{p,\lambda,\nu}},$$

and

$$\|f\|_{L_{p,\lambda,\nu}(R_+)} = \sup_{x,r\in R_+}\left(\left(sh\frac{r}{2}\right)^{-\nu}\int_0^r A^\lambda_{ch\,y}|f(ch\,x)|^p sh^{2\lambda}y dy\right)^{\frac{1}{p}}$$

$$\le \sup_{x,r\in R_+}\left(\left[sh\frac{r}{2}\right]_1^{-\nu}\int_0^r A^\lambda_{ch\,y}|f(ch\,x)|^p sh^{2\lambda}y dy\right)^{\frac{1}{p}}$$

$$= \|f\|_{\tilde{L}_{p,\lambda,\nu}}.$$

Therefore, $f \in L_{p,\lambda,\nu}(R_+) \bigcap L_{p,\lambda}(R_+)$ and the embedding

$$\tilde{L}_{p,\lambda,\nu}(R_+) \subset L_{p,\lambda,\nu}(R_+) \bigcap L_{p,\lambda}(R_+)$$

is valid.

Let $f \in L_{p,\lambda,\nu}(R_+) \bigcap L_{p,\lambda}(R_+)$. Then

$$\|f\|_{\tilde{L}_{p,\lambda,\nu}(R_+)} = \sup_{x,r\in R_+}\left(\left[sh\frac{r}{2}\right]_1^{-\nu}\int_0^r A^\lambda_{ch\,y}|f(ch\,x)|^p sh^{2\lambda}y dy\right)^{\frac{1}{p}}$$

$$= \max\left\{\sup_{x\in R_+, r\in(0,1]}\left(\left[sh\frac{r}{2}\right]^{-\nu}\int_0^r A^\lambda_{ch\,y}|f(ch\,x)|^p sh^{2\lambda}y dy\right)^{\frac{1}{p}},\right.$$

$$\left.\sup_{x\in R_+, r\in(1,\infty)}\left(\int_0^r A^\lambda_{ch\,y}|f(ch\,x)|^p sh^{2\lambda}y dy\right)^{\frac{1}{p}}\right\}$$

$$\leq \max\left\{\|f\|_{L_{p,\lambda,\nu}}, \|f\|_{L_{p,\lambda}}\right\}.$$

Therefore, $f \in \tilde{L}_{p,\lambda,\nu}(R_+)$ and the embedding $L_{p,\lambda,\nu}(R_+) \bigcap L_{p,\lambda}(R_+) \subset \tilde{L}_{p,\lambda,\nu}$ is valid. Thus $\tilde{L}_{p,\lambda,\nu}(R_+) = L_{p,\lambda,\nu}(R_+) \bigcap L_{p,\lambda}(R_+)$.

Let now $f \in \tilde{L}_{p,\lambda,\nu}(R_+)$. Then

$$\|f\|_{L_{p,\lambda,\nu}(R_+)} = \sup_{x,r\in R_+}\left(\left(sh\frac{r}{2}\right)^{-\nu}\int_0^r A_{ch\,y}^{\lambda}|f(ch\,x)|^p sh^{2\lambda}ydy\right)^{\frac{1}{p}}$$

$$= \sup_{x,r\in R_+}\left(\frac{\left[sh\frac{r}{2}\right]_1}{sh\frac{r}{2}}\right)^{\frac{\nu}{p}}\left(\left[sh\frac{r}{2}\right]_1^{-\nu}\int_0^r A_{ch\,y}|f(ch\,x)|^p sh^{2\lambda}ydy\right)^{\frac{1}{p}}$$

$$= \|f\|_{\tilde{L}_{p,\lambda,\nu}},$$

since by $0 < r < 2arcsh\,1$, $sh\frac{r}{2} < 1$ and $\left[sh\frac{r}{2}\right]_1 = sh\frac{r}{2}$. If $r \geq 2arcsh\,1$, then $sh\frac{r}{2} \geq 1$ and we have

$$\frac{\left[sh\frac{r}{2}\right]_1}{sh\frac{r}{2}} = \frac{1}{sh\frac{r}{2}} \leq 1.$$

4. Hardy-Littlewood-Sobolev inequality in modified $G-$Morrey spaces

In this section we study the $\tilde{L}_{p,\lambda,\nu}$- boundedness of the $G-$maximal operator M_G.

Theorem 1. *1) If $f \in \tilde{L}_{1,\lambda,\nu}(R_+)$, $0 \leq \nu < \gamma$, then $M_G f \in W\tilde{L}_{1,\lambda,\nu}(R_+)$ and*

$$\|M_G f\|_{W\tilde{L}_{1,\lambda,\nu}} \lesssim \|f\|_{\tilde{L}_{1,\lambda,\nu}}.$$

2) If $f \in \tilde{L}_{p,\lambda,\nu}(R_+)$, $1 < p < \infty$, then $M_G f \in W\tilde{L}_{p,\lambda,\nu}(R_+)$ and

$$\|M_G f\|_{\tilde{L}_{p,\lambda,\nu}} \lesssim \|f\|_{\tilde{L}_{p,\lambda,\nu}}.$$

Proof. 1) From the definition of weak modified $G-$Morrey spaces

$$\|M_G f\|_{W\tilde{L}_{1,\lambda,\nu}(R_+)} = \sup_{r\in R_+} r \sup_{x,t\in R_+}\left(\left[sh\frac{t}{2}\right]_1^{-\nu}\left|\left\{y \in (0,t) : A_{ch\,y}^{\lambda} M_G f(ch\,x) > r\right\}\right|_{\lambda}\right)^{\frac{1}{p}}.$$

Applying the Theorem B and also Theorem A we get

$$\|M_G f\|_{W\tilde{L}_{1,\lambda,\nu}} \lesssim \sup_{x,t\in R_+}\left(\left[sh\frac{t}{2}\right]_1^{-\nu}\int_0^r A_{ch\,y}^{\lambda}|f(ch\,x)| sh^{2\lambda}ydy\right)$$

$$= \|f\|_{\tilde{L}_{1,\lambda,\nu}}.$$

Assertion 2) follows from Theorem A.

We consider $G-$fractional integral introduced in [14].

$$J_G^{\alpha} d\left(ch\, x\right) = \int_0^{\infty} \left|H_y\right|_{\lambda}^{\frac{\alpha}{\gamma}-1} A_{ch\, y}^{\lambda} f\left(ch\, x\right) sh^{2\lambda} y dy.$$

The following Hardy-Littlewood-Sobolev inequality in modified $G-$Morrey spaces is valid.

Theorem 2. *Let $0 \leq \alpha < \gamma$, $0 \leq \nu < \gamma - \alpha p$ and $1 \leq p < \frac{\gamma-\nu}{\alpha}$.*

1) If $1 < p < \frac{\gamma-\nu}{\alpha}$, then the condition

$$\frac{\alpha}{\gamma} \leq \frac{1}{p} - \frac{1}{q} \leq \frac{\alpha}{\gamma-\nu}$$

is necessary and sufficient for the boundedness of the operator J_G^{α} from $\tilde{L}_{p,\lambda,\nu}\left(R_+\right)$ to $\tilde{L}_{q,\lambda,\nu}\left(R_+\right)$.

2) If $p = 1 < \frac{\gamma-\nu}{\alpha}$, then the condition

$$\frac{\alpha}{\gamma} \leq 1 - \frac{1}{q} \leq \frac{\alpha}{\gamma-\nu}$$

is necessary and sufficient for the boundedness of the operator J_G^{α} from $\tilde{L}_{1,\lambda,\nu}\left(R_+\right)$ to$\tilde{L}_{q,\lambda,\nu}\left(R_+\right)$.

Proof.

1) Sufficiency. Let $0 \leq \alpha < \gamma$, $0 \leq \nu < \gamma - \alpha p$, $1 < p < \frac{\gamma-\nu}{\alpha}$ and $f \in \tilde{L}_{p,\lambda,\nu}\left(R_+\right)$. From (2), we have

$$\left|J_G^{\alpha} f\left(ch\, x\right)\right| \lesssim \left(\int_0^r + \int_r^{\infty}\right) \frac{A_{ch\, y}^{\lambda} \left|f\left(ch\, x\right)\right|}{\left(sh\frac{y}{2}\right)^{\gamma-\alpha}} sh^{2\lambda} t dt$$

$$= A_1\left(x,r\right) + A_2\left(x,r\right). \tag{3}$$

We estimate $A_1\left(x,r\right)$. Let $0 < r < 2$, then by (2) we obtain

$$\left|A_1\left(x,r\right)\right| \lesssim \int_0^r \frac{A_{ch\, y}^{\lambda} \left|f\left(ch\, x\right)\right|}{\left(sh\,\frac{y}{2}\right)^{2\lambda+1-\alpha}} sh^{2\lambda} y dy \lesssim \sum_{j=0}^{\infty} \int_{r/2^{j+1}}^{r/2^j} \frac{A_{ch\, y}^{\lambda} \left|f\left(ch\, x\right)\right|}{\left(sh\frac{y}{2}\right)^{2\lambda+1-\alpha}} sh^{2\lambda} y dy$$

$$< \sum_{j=0}^{\infty} \left(sh\frac{r}{2^{j+1}}\right)^{\alpha} \left(sh\frac{r}{2^{j+2}}\right)^{-2\lambda-1} \int_0^{r/2^j} A_{ch\, y}^{\lambda} \left|f\left(ch\, x\right)\right| sh^{2\lambda} y dy.$$

Using the inequality (see [3], Lemma 2.2)

$$t \leq sh\, t \leq e^A t, \;\; A > 0 \tag{4}$$

and also $sh\, at \leq a\, sh\, t$ at $0 \leq a \leq 1$, we have

$$|A_1(x,r)| \lesssim \left(sh\frac{r}{2}\right)^{\alpha} \sum_{j=0}^{\infty} \left(2^{-j\alpha}\right) \left(sh\frac{r}{2^{j+1}}\right)^{-2\lambda-1} \int_0^{r/2^j} A^{\lambda}_{ch\, y} |f(ch\, x)|\, sh^{2\lambda} y dy$$

$$\lesssim \left(sh\frac{r}{2}\right)^{\alpha} M_G f(ch\, x) \sum_{j=0}^{\infty} 2^{-j\alpha}$$

$$\lesssim \left(sh\frac{r}{2}\right)^{\alpha} M_G f(ch\, x). \tag{5}$$

Let $2 \leq r < \infty$. Then

$$A_1(x,r) \lesssim \int_0^r \frac{A^{\lambda}_{ch\, y} |f(ch\, x)|}{\left(sh\frac{y}{2}\right)^{4\lambda-\alpha}} sh^{2\lambda} y dy$$

$$\lesssim \sum_{j=0}^{\infty} \int_{r/2^{j+1}}^{r/2^j} \frac{A^{\lambda}_{ch\, y} |f(ch\, x)|}{\left(sh\frac{y}{2}\right)^{4\lambda-\alpha}} sh^{2\lambda} y dy$$

$$\lesssim \sum_{j=0}^{\infty} \left(sh\frac{r}{2^{j+1}}\right)^{\alpha} \left(sh\frac{r}{2^{j+1}}\right)^{-4\lambda} \int_0^{r/2^j} A^{\lambda}_{ch\, y} |f(ch\, x)|\, sh^{2\lambda} y dy$$

$$\lesssim \left(sh\frac{r}{2}\right)^{\alpha} M_G f(ch\, x) \sum_{j=0}^{\infty} 2^{-j\alpha}$$

$$\lesssim \left(sh\frac{r}{2}\right)^{\alpha} M_G f(ch\, x). \tag{6}$$

Combining (5) and (6) we obtain

$$A_1(x,r) \lesssim \left(sh\frac{r}{2}\right)^{\alpha} M_G f(ch\, x),\ 0 < r < \infty. \tag{7}$$

Now consider $A_2(x,r)$. By Holders inequality we get

$$A_2(x,r) \lesssim \left(\int_r^{\infty} A^{\lambda}_{ch\, y} |f(ch\, x)|^p (sh\, y)^{-\beta} sh^{2\lambda} y dy\right)^{\frac{1}{p}}$$

$$\times \left(\int_r^{\infty} (sh\, y)^{(\beta/p+\alpha-\gamma)p'} sh^{2\lambda} y dy\right)^{\frac{1}{p'}}$$

$$= A_{2\cdot 1} \cdot A_{2\cdot 2} \tag{8}$$

Let $\nu < \beta < \gamma - \alpha p$. Using the inequality [7]

$$\left\| A^{\lambda}_{ch\, y} f \right\|_{\tilde{L}_{p,\lambda,\nu}} \leq \|f\|_{\tilde{L}_{p,\lambda,\nu}},$$

we obtain

$$A_{2\cdot 1} \lesssim \left(\sum_{j=0}^{\infty} \int_{2^j r}^{2^{j+1} r} A_{ch\, y}^{\lambda} \left| f\left(ch\, x \right) \right|^p \left(sh\, y \right)^{-\beta} sh^{2\lambda} y dy \right)^{\frac{1}{p}}$$

$$\lesssim \left\| A_{ch\, y}^{\lambda} f \right\|_{\tilde{L}_{p,\lambda.\nu}} \left(\sum_{j=0}^{\infty} \frac{\left[2^{j+1} sh \frac{r}{2} \right]_1^{\nu}}{\left(sh\, 2^j r \right)^{\beta}} \right)^{\frac{1}{p}}$$

$$\lesssim \left[2sh \frac{r}{2} \right]_1^{\nu/p} \left(sh\, r \right)^{-\beta/p} \left(\sum_{j=0}^{\infty} 2^{j(\nu-\beta)} \right)^{\frac{1}{p}} \left\| f \right\|_{\tilde{L}_{p,\lambda,\nu}}$$

$$\lesssim \left[sh \frac{r}{2} \right]_1^{\frac{\nu}{p}} \left(sh \frac{r}{2} \right)^{-\frac{\beta}{p}} \left\| f \right\|_{\tilde{L}_{p,\lambda,\nu}} \tag{9}$$

since $sh\, ax \geq ashx$ at $a \geq 1$.

For $A_{2\cdot 2}$ we have

$$A_{2\cdot 2} = \left(\int_r^{\infty} \left(sh\, y \right)^{(\beta/p+\alpha-\gamma)p'} sh^{2\lambda} y dy \right)^{\frac{1}{p'}}$$

$$\lesssim \left(sh\, r \right)^{\beta/p+\alpha-\gamma+\gamma/p'} \lesssim \left(sh\, r \right)^{\beta/p+\alpha-\gamma+\gamma(1-1/p)}$$

$$\lesssim \left(sh\, r \right)^{\beta/p+\alpha-\gamma/p} \lesssim \left(sh\, \frac{r}{2} \right)^{\beta/p+\alpha-\gamma/p}. \tag{10}$$

Taking into account (9) and (10) in (8), we obtain

$$A_2\left(x, r \right) \lesssim \left[sh \frac{r}{2} \right]^{\nu/p} \left(sh \frac{r}{2} \right)^{\alpha-\gamma/p} \left\| f \right\|_{\tilde{L}_{p,\lambda,\nu}}. \tag{11}$$

Thus from (5) and (11), we get

$$\left| J_G^{\alpha} f\left(ch\, x \right) \right| \lesssim \left(\left[sh \frac{r}{2} \right]_1^{\nu/p} \left(sh \frac{r}{2} \right)^{\alpha-\gamma/p} \left\| f \right\|_{\tilde{L}_{p,\lambda,\nu}} + \left(sh \frac{r}{2} \right)^{\alpha} M_G f\left(chx \right) \right)$$

$$\lesssim \min \left\{ \left(sh \frac{r}{2} \right)^{\alpha+(\nu-\gamma)/p} \left\| f \right\|_{\tilde{L}_{p,\lambda,\nu}} + \left(sh \frac{r}{2} \right)^{\alpha} M_G f\left(ch\, x \right), \right.$$

$$\left. \left(sh \frac{r}{2} \right)^{\alpha-\gamma/p} \left\| f \right\|_{\tilde{L}_{p,\lambda,\nu}} + \left(sh \frac{r}{2} \right)^{\alpha} M_G f\left(ch\, x \right) \right\}, \tag{12}$$

for all $r > 0$.
The right-hand side attains its minimum at

$$sh \frac{r}{2} = \left(\frac{\gamma - \alpha p}{\alpha p} \frac{\left\| f \right\|_{\tilde{L}_{p,\lambda,\nu}}}{M_G f\left(ch\, x \right)} \right)^{p/\gamma}, \tag{13}$$

and

$$sh\frac{r}{2}=\left(\frac{\gamma-\nu-\alpha p}{\alpha p}\frac{\|f\|_{\tilde{L}_{p,\lambda,\nu}}}{M_G f\left(ch\,x\right)}\right)^{\frac{p}{\gamma-\nu}}. \tag{14}$$

Applying (13) and (14) in (12), we get

$$\left|J_G^{\alpha}f\left(ch\,x\right)\right|\lesssim\min\left\{\left(\frac{M_G f\left(chx\right)}{\|f\|_{\tilde{L}_{p,\lambda,\nu}}}\right)^{1-\frac{\alpha p}{\gamma}},\left(\frac{M_G f\left(chx\right)}{\|f\|_{\tilde{L}_{p,\lambda,\nu}}}\right)^{1-\frac{\alpha p}{\gamma-\nu}}\right\}\|f\|_{\tilde{L}_{p,\lambda,\nu}}.$$

Then

$$\left|J_G^{\alpha}f\left(ch\,x\right)\right|\lesssim\left(M_G f\left(chx\right)\right)^{\frac{p}{q}}\|f\|_{\tilde{L}_{p,\lambda,\nu}}^{1-\frac{p}{q}}.$$

Hence, by Theorem 1, we have

$$\int_0^r\left|J_G^{\alpha}f\left(ch\,x\right)\right|^q sh^{2\lambda}xdx\lesssim\|f\|_{\tilde{L}_{p,\lambda,\nu}}^{q-p}\int_0^r\left|M_G f\left(ch\,x\right)\right|^p sh^{2\lambda}xdx$$

$$\lesssim\left[sh\frac{r}{2}\right]_1^{\nu}\|f\|_{\tilde{L}_{p,\lambda,\nu}}^{q}.$$

From this it follows that

$$\left\|J_G^{\alpha}f\left(ch\,x\right)\right\|_{L_{q,\lambda,\nu}}\lesssim\|f\|_{\tilde{L}_{p,\lambda,\nu}}^{q},$$

i.e., J_G^{α} in bounded from $\tilde{L}_{p,\lambda,\nu}\left(R_+\right)$ to $\tilde{L}_{q,\lambda,\nu}\left(R_+\right)$.

Necessity. Let $1<p<\left(\gamma-\nu\right)/\alpha$, $f\in\tilde{L}_{p,\lambda,\nu}\left(R_+\right)$ and J_G^{α} is bounded from $\tilde{L}_{p,\lambda,\nu}\left(R_+\right)$ to $\tilde{L}_{q,\lambda,\nu}\left(R_+\right)$. Let the function $f\left(chx\right)$ be non-negative and monotonically on R_+. The delates function $f_t(chx)$ is defined as follows [6]:

$$f\left(ch\left(th\frac{t}{2}\right)x\right)\le f_t\left(chx\right)\le f\left(ch\left(cth\frac{t}{2}\right)x\right),\ 0<t<2$$

$$f\left(ch\left(th\frac{t}{2}\right)x\right)\le f_t\left(chx\right)\le f\left(ch\left(sh\frac{t}{2}\right)x\right),\ 2\le t<\infty \tag{15}$$

We suppose $\left[sh\frac{t}{2}\right]_{1,+}=\max\left\{1,sh\frac{t}{2}\right\}$. Let $0<t<2$. Using the symmetry of the operator A_{chy}^{λ} (see [7]) $A_{chx}^{\lambda}f\left(chy\right)=A_{chy}^{\lambda}f\left(chx\right)$ we will have

$$\|f_t\|_{\tilde{L}_{p,\lambda,\nu}}=\sup_{x,r\in R_+}\left(\left[sh\frac{r}{2}\right]_1^{-\nu}\int_0^r A_{chy}^{\lambda}\left|f_t\left(chy\right)\right|^p sh^{2\lambda}ydy\right)^{\frac{1}{p}}$$

$$\le\sup_{x,r\in R_+}\left(\left[sh\frac{r}{2}\right]_1^{-\nu}\int_0^r A_{chx}^{\lambda}\left|f_t\left(ch\left(cth\frac{t}{2}\right)y\right)\right|^p sh^{2\lambda}ydy\right)^{\frac{1}{p}}$$

$$\left[\left(cth\frac{t}{2}\right)y=u,\ dy=\left(th\frac{t}{2}\right)du\right]$$

$$= \left(sh\frac{t}{2}\right)^{\frac{1}{p}} \sup_{x,r\in R_+} \left(\left[sh\frac{r}{2}\right]_1^{-\nu} \int_0^{rcth\frac{t}{2}} A^{\lambda}_{chy} \left|f\left(chu\right)\right|^p sh^{2\lambda}\left(th\frac{t}{2}\right) udu \right)^{\frac{1}{p}}$$

$$= \left(th\frac{t}{2}\right)^{\frac{2\lambda+1}{p}} \sup_{r\in R_+} \left(\frac{\left[\left(sh\frac{r}{2}\right)cth\frac{t}{2}\right]_1}{\left[sh\frac{r}{2}\right]_1} \right)^{\frac{\nu}{p}}$$

$$\times \sup_{x,r\in R_+} \left(\left[\left(sh\frac{r}{2}\right)cth\frac{t}{2}\right]_1^{-\nu} \int_0^{rcth\frac{t}{2}} A^{\lambda}_{chx} \left|f\left(chu\right)\right|^p sh^{2\lambda} udu \right)^{\frac{1}{p}}.$$

$$\leq \left(sh\frac{t}{2}\right)^{\frac{2\lambda+1}{p}} \left[cth\frac{t}{2}\right]_{1,+}^{\frac{\nu}{p}} \|f\|_{\tilde{L}_{p,\lambda,\nu}} \leq \left(th\frac{t}{2}\right)^{\frac{2\lambda+1-\nu}{p}} \|f\|_{\tilde{L}_{p,\lambda,\nu}}$$

$$= \left(\frac{sh\frac{t}{2}}{ch\frac{t}{2}}\right)^{\frac{2\lambda+1-\nu}{p}} \|f\|_{\tilde{L}_{p,\lambda,\nu}} \lesssim \frac{1}{\left(ch\frac{t}{2}\right)^{\frac{2\lambda+1-\nu}{p}-\alpha}} \|f\|_{\tilde{L}_{p,\lambda,\nu}}$$

$$\lesssim \left(sh\frac{t}{2}\right)^{\alpha+\frac{\nu-2\lambda-1}{p}} \|f\|_{\tilde{L}_{p,\lambda,\nu}}$$

$$\lesssim \left(sh\frac{t}{2}\right)^{\alpha+\frac{\nu-\gamma}{p}} \|f\|_{\tilde{L}_{p,\lambda,\nu}}, 0 < t < 2. \tag{16}$$

On the other hand, by $0 < t < 2$, we get

$$\|f_t\|_{\tilde{L}_{p,\lambda,\nu}} = \sup_{x,r\in R_+} \left(\left[sh\frac{r}{2}\right]_1^{-\nu} \int_0^{r} A^{\lambda}_{chy} \left|f_t\left(chx\right)\right|^p sh^{2\lambda} ydy \right)^{\frac{1}{p}}$$

$$\geq \sup_{x,r\in R_+} \left(\left[sh\frac{r}{2}\right]_1^{-\nu} \int_0^{r} A^{\lambda}_{chx} \left|f_t\left(ch\left(th\frac{t}{2}\right)y\right)\right|^p sh^{2\lambda} ydy \right)^{\frac{1}{p}}$$

$$\left[\left(th\frac{t}{2}\right)y = u,\ y = \left(cth\frac{t}{2}\right)u,\ dy = \left(cth\frac{t}{2}\right)du\right]$$

$$= \left(cth\frac{t}{2}\right)^{\frac{1}{p}} \sup_{x,r\in R_+} \left(\left[sh\frac{r}{2}\right]_1^{-\nu} \int_0^{rth\frac{t}{2}} A^{\lambda}_{chx} \left|f\left(chu\right)\right|^p sh^{2\lambda}\left(cth\frac{t}{2}\right) udu \right)^{\frac{1}{p}}$$

$$\geq \left(cth\frac{t}{2}\right)^{\frac{2\lambda+1}{p}} \left(\sup_{r\in R_+} \frac{\left[\left(sh\frac{r}{2}\right)th\frac{t}{2}\right]_1}{\left[sh\frac{r}{2}\right]_1} \right)^{\frac{\nu}{p}} \|f\|_{\tilde{L}_{p,\lambda,\nu}}$$

$$= \left(cth\frac{t}{2}\right)^{\frac{2\lambda+1}{p}} \left[th\frac{t}{2}\right]_{1,+}^{\frac{\nu}{p}} \|f\|_{\tilde{L}_{p,\lambda,\nu}}$$

$$\geq \left(cth\frac{t}{2}\right)^{\frac{2\lambda+1}{p}-\frac{\nu}{p}-\alpha} \|f\|_{\tilde{L}_{p,\lambda,\nu}}$$

$$= \left(cth\frac{t}{2} \right)^{\frac{2\lambda+1-\nu}{p}-\alpha} \|f\|_{\tilde{L}_{p,\lambda,\nu}}$$

$$\geq \left(sh\frac{t}{2} \right)^{\alpha+\frac{\nu-\gamma}{p}} \|f\|_{\tilde{L}_{p,\lambda,\nu}} \tag{17}$$

Combing (16) and (17), we obtain

$$\|f_t\|_{\tilde{L}_{p,\lambda,\nu}} \approx \left(sh\frac{t}{2} \right)^{\alpha+\frac{\nu-\gamma}{p}} \|f\|_{\tilde{L}_{p,\lambda,\nu}}, \ 0 < t < 2. \tag{18}$$

Now, let $2 \leq t < \infty$, then from (15) we have

$$\|f_t\|_{\tilde{L}_{p,\lambda,\nu}} = \sup_{x,r\in R_+} \left(\left[sh\frac{r}{2} \right]_1^{-\nu} \int_0^r A_{ch\, y}^{\lambda} \left| f_t\left(ch\, x \right) \right|^p sh^{2\lambda} y dy \right)^{\frac{1}{p}}$$

$$\geq \sup_{x,r\in R_+} \left(\left[sh\frac{r}{2} \right]_1^{-\nu} \int_0^r A_{ch\, x}^{\lambda} \left| f\left(ch\, \left(th\frac{t}{2} \right) y \right) \right|^p sh^{2\lambda} y dy \right)^{\frac{1}{p}}$$

$$\left[\left(th\frac{t}{2} \right) y = u,\ y = \left(cth\frac{t}{2} \right) u,\ dy = \left(cth\frac{t}{2} \right) du \right]$$

$$= \left(cth\frac{t}{2} \right)^{\frac{1}{p}} \sup_{x,r\in R_+} \left(\left[sh\frac{r}{2} \right]_1^{-\nu} \int_0^{r\, th\frac{t}{2}} A_{ch\, y}^{\lambda} \left| f\left(ch\, u \right) \right|^p sh^{2\lambda} \left(cth\frac{t}{2} \right) udu \right)^{\frac{1}{p}}$$

$$\geq \left(cth\frac{t}{2} \right)^{\frac{2\lambda+1}{p}} \sup_{x,r\in R_+} \left(\left[sh\frac{r}{2} \right]_1^{-\nu} \int_0^{r\, th\frac{t}{2}} A_{ch\, x}^{\lambda} \left| f\left(ch\, u \right) \right|^p sh^{2\lambda} udu \right)^{\frac{1}{p}}$$

$$= \left(cth\frac{t}{2} \right)^{\frac{2\lambda+1}{p}} \sup_{x,r\in R_+} \left(\frac{\left[\left(sh\frac{r}{2} \right) th\frac{t}{2} \right]_1}{\left[sh\frac{r}{2} \right]_1} \right)^{\frac{\nu}{p}} \|f\|_{\tilde{L}_{p,\lambda,\nu}}$$

$$= \left(cth\frac{t}{2} \right)^{\frac{2\lambda+1}{p}} \left[th\frac{t}{2} \right]_1^{\frac{\nu}{p}} \|f\|_{\tilde{L}_{p,\lambda,\nu}} \geq \left(cth\frac{t}{2} \right)^{\frac{4\lambda-\nu}{p}-\alpha} \|f\|_{\tilde{L}_{p,\lambda,\nu}}$$

$$\geq \left(sh\frac{t}{2} \right)^{\alpha+\frac{\nu-4\lambda}{p}} \|f\|_{\tilde{L}_{p,\lambda,\nu}} = \left(sh\frac{t}{2} \right)^{\alpha+\frac{\nu-\gamma}{p}} \|f\|_{\tilde{L}_{p,\lambda,\nu}}, \ 2 \leq t < \infty. \tag{19}$$

On the other hand, at $2 \leq t < \infty$, we get

$$\|f_t\|_{\tilde{L}_{p,\lambda,\nu}} = \sup_{x,r\in R_+} \left(\left[sh\frac{r}{2} \right]_1^{-\nu} \int_0^r A_{ch\, y}^{\lambda} \left| f_t\left(ch\, x \right) \right|^p sh^{2\lambda} y dy \right)^{\frac{1}{p}}$$

$$\leq \sup_{x,r\in R_+} \left(\left[sh\frac{r}{2} \right]_1^{-\nu} \int_0^r A_{ch\, y}^{\lambda} \left| f\left(ch\, \left(sh\frac{t}{2} \right) y \right) \right|^p sh^{2\lambda} y dy \right)^{\frac{1}{p}}$$

$$\left[\left(sh\frac{t}{2}\right)y=0,\ \ dy=\frac{du}{sh\frac{t}{2}}\right]$$

$$=\left(sh\frac{t}{2}\right)^{-\frac{1}{p}}\sup_{x,r\in R_+}\left(\left[sh\frac{r}{2}\right]_1^{-\nu}\int_0^{rsh\frac{t}{2}}A_{ch\,x}^{\lambda}\left|f\left(ch\,u\right)\right|^p sh^{2\lambda}\frac{u}{sht}du\right)^{\frac{1}{p}}$$

$$\leq\left(sh\frac{t}{2}\right)^{-\frac{2\lambda+1}{p}}\sup_{x,r\in R_+}\left(\left[sh\frac{r}{2}\right]_1^{-\nu}\int_0^{rsh\frac{t}{2}}A_{ch\,x}^{\lambda}\left|f\left(ch\,u\right)\right|^p sh^{2\lambda}udu\right)^{\frac{1}{p}}$$

$$\leq\left(sh\frac{t}{2}\right)^{-\frac{4\lambda}{p}}\left(\sup_{r\in R_+}\frac{\left[\left(sh\frac{r}{2}\right)sh\frac{t}{2}\right]_1}{\left[sh\frac{r}{2}\right]_1}\right)^{\frac{\nu}{p}}\|f\|_{\tilde{L}_{p,\lambda,\nu}}$$

$$=\left(sh\frac{t}{2}\right)^{-\frac{4\lambda}{p}}\left[sh\frac{t}{2}\right]_1^{\frac{\nu}{p}}\|f\|_{\tilde{L}_{p,\lambda,\nu}}$$

$$\leq\left(sh\frac{t}{2}\right)^{\alpha+\frac{\nu-4\lambda}{p}}\|f\|_{\tilde{L}_{p,\lambda,\nu}}$$

$$=\left(sh\frac{t}{2}\right)^{\alpha+\frac{\nu-\gamma}{p}}\|f\|_{\tilde{L}_{p,\lambda,\nu}},2\leq t<\infty. \tag{20}$$

Combing (19) and (20), we obtain

$$\|f\|_{\tilde{L}_{p,\lambda,\nu}}\approx\left(sh\frac{t}{2}\right)^{\alpha+\frac{\nu-\gamma}{p}}\|f\|_{\tilde{L}_{p,\lambda,\nu}}, \tag{21}$$

Thus from (18) and (21), we have

$$\|f_t\|_{\tilde{L}_{p,\lambda,\nu}}\approx\left(sh\frac{t}{2}\right)^{\alpha+\frac{\nu-\gamma}{p}}\|f\|_{\tilde{L}_{p,\lambda,\nu}},\ 0<t<\infty. \tag{22}$$

From (2) $0<t<2$, we have

$$\|J_G^{\alpha}f_t\|_{\tilde{L}_{q,\lambda,\nu}}=\sup_{x,r\in R_+}\left(\left[sh\frac{r}{2}\right]_1^{-\nu}\int_0^{r}\left|J_G^{\alpha}f_t\left(ch\,y\right)\right|^q sh^{2\lambda}ydy\right)^{\frac{1}{q}}$$

$$\leq\sup_{x,r\in R_+}\left(\left[sh\frac{r}{2}\right]_1^{-\nu}\int_0^{r}\left|J_G^{\alpha}f\left(ch\left(cth\frac{t}{2}\right)y\right)\right|^q sh^{2\lambda}ydy\right)^{\frac{1}{q}}$$

$$\left[\left(cth\frac{t}{2}\right)y=z,\qquad dy=\left(th\frac{t}{2}\right)dz\right]$$

$$=\left(th\frac{t}{2}\right)^{\frac{1}{q}}\sup_{x,r\in R_+}\left(\left[sh\frac{r}{2}\right]_1^{-\nu}\int_0^{rcth\frac{t}{2}}\left|J_G^{\alpha}f\left(ch\,z\right)\right|^q sh^{2\lambda}\left(th\frac{t}{2}\right)zdz\right)^{\frac{1}{q}}$$

$$\le \left(th\frac{t}{2}\right)^{\frac{2\lambda+1}{q}} \sup_{x,r\in R_+}\left(\left[sh\frac{r}{2}\right]_1^{-\nu}\int_0^{rcth\frac{t}{2}} |J_G^\alpha f\,(ch\,z)|^q\, sh^{2\lambda}zdz\right)^{\frac{1}{q}}$$

$$= \left(th\frac{t}{2}\right)^{\frac{2\lambda+1}{q}} \left(\sup_{r\in R_+}\frac{\left[\left(sh\frac{r}{2}\right)cth\frac{t}{2}\right]_1}{\left[sh\frac{r}{2}\right]_1}\right)^{\frac{\nu}{q}} \|J_G^\alpha f\|_{\tilde{L}_{q,\lambda,\nu}}$$

$$= \left(th\frac{t}{2}\right)^{\frac{2\lambda+1}{q}} \left[cth\frac{t}{2}\right]_1^{\frac{\nu}{q}} \|J_G^\alpha f\|_{\tilde{L}_{q,\lambda,\nu}}$$

$$\le \left(cth\frac{t}{2}\right)^{-\frac{2\lambda+1}{q}} \left[cth\frac{t}{2}\right]_{1,+}^{\frac{\nu}{q}} \|J_G^\alpha f\|_{\tilde{L}_{q,\lambda,\nu}}$$

$$\le \left(cth\frac{t}{2}\right)^{\frac{\nu-2\lambda-1}{q}} \|J_G^\alpha f\|_{\tilde{L}_{q,\lambda,\nu}}$$

$$\le \left(sh\frac{t}{2}\right)^{\frac{\nu-2\lambda-1}{q}} \|J_G^\alpha\|_{\tilde{L}_{q,\lambda,\nu}}$$

$$= \left(sh\frac{t}{2}\right)^{\frac{\nu-\gamma}{q}} \|J_G^\alpha\|_{\tilde{L}_{q,\lambda,\nu}}\,, \quad 0 < t < 2. \tag{23}$$

On the other hand by $0 < t < 2$, we get

$$\|J_G^\alpha f_t\|_{\tilde{L}_{q,\lambda,\nu}} = \sup_{x,r\in R_+}\left(\left[sh\frac{r}{2}\right]_1^{-\nu}\int_0^{r} |J_G^\alpha f_t\,(ch\,y)|^q\, sh^{2\lambda}ydy\right)^{\frac{1}{q}}$$

$$\ge \sup_{x,r\in R_+}\left(\left[sh\frac{r}{2}\right]_1^{-\nu}\int_0^{r} \left|J_G^\alpha f\left(ch\left(th\frac{t}{2}\right)y\right)\right|^q sh^{2\lambda}ydy\right)^{\frac{1}{q}}$$

$$\left[\left(th\frac{t}{2}\right)y = z, \qquad dy = \left(cth\frac{t}{2}\right)dz\right]$$

$$= \left(cth\frac{t}{2}\right)^{\frac{1}{q}} \sup_{x,r\in R_+}\left(\left[sh\frac{r}{2}\right]_1^{-\nu}\int_0^{rth\frac{t}{2}} |J_G^\alpha f\,(ch\,z)|^q\, sh^{2\lambda}\left(cth\frac{t}{2}\right)dz\right)^{\frac{1}{q}}$$

$$\ge \left(cth\frac{t}{2}\right)^{\frac{2\lambda+1}{q}} \left(\sup_{r\in R_+}\frac{\left[\left(sh\frac{r}{2}\right)th\frac{t}{2}\right]_1}{\left[sh\frac{r}{2}\right]_1}\right)^{\frac{\nu}{q}} \|J_G^\alpha f\|_{\tilde{L}_{q,\lambda,\nu}}$$

$$= \left(cth\frac{t}{2}\right)^{\frac{2\lambda+1}{q}} \left(th\frac{t}{2}\right)_1^{\frac{\nu}{q}} \|J_G^\alpha f\|_{\tilde{L}_{q,\lambda,\nu}}$$

$$\ge \left(\frac{ch\frac{t}{2}}{sh\frac{t}{2}}\right)^{\frac{2\lambda+1-\nu}{q}} \|J_G^\alpha f\|_{\tilde{L}_{q,\lambda,\nu}} \ge \left(sh\frac{t}{2}\right)^{\frac{\nu-2\lambda-1}{q}} \|J_G^\alpha f\|_{\tilde{L}_{q,\lambda,\nu}}$$

$$= \left(sh\frac{t}{2}\right)^{\frac{\nu-\gamma}{q}} \|J_G^\alpha f\|_{\tilde{L}_{q,\lambda,\nu}}, \ 0 < t < 2. \tag{24}$$

Thus from (23) and (24), we obtain

$$\|J_G^\alpha f\|_{\tilde{L}_{q,\lambda,\nu}} \approx \left(sh\frac{t}{2}\right)^{\frac{\nu-\gamma}{q}} \|J_G^\alpha f\|_{\tilde{L}_{q,\lambda,\nu}}, \ \ 0 < t < 2. \tag{25}$$

Now we consider the case, then $2 \le t < \infty$. From (15), we have

$$\|J_G^\alpha f_t\|_{\tilde{L}_{q,\lambda,\nu}} = \sup_{x,r\in R_+} \left(\left[sh\frac{r}{2}\right]_1^{-\nu} \int_0^r |J_G^\alpha f_t\,(ch\,y)|^q\, sh^{2\lambda} y dy \right)^{\frac{1}{q}}$$

$$\ge \sup_{x,r\in R_+} \left(\left[sh\frac{r}{2}\right]_1^{-\nu} \int_0^r \left|J_G^\alpha f\left(ch\left(th\frac{t}{2}\right)\, y\right)\right|^q sh^{2\lambda} y dy \right)^{\frac{1}{q}}$$

$$\left[\left(th\frac{t}{2}\right) y = z, \qquad dy = \left(cth\frac{t}{2}\right) dz\right]$$

$$= \left(cth\frac{t}{2}\right)^{\frac{1}{q}} \sup_{x,r\in R_+} \left(\left[sh\frac{r}{2}\right]_1^{-\nu} \int_0^{rth\frac{t}{2}} |J_G^\alpha f\,(ch\,z)|^q\, sh^{2\lambda} \left(cth\frac{t}{2}\right) zdz \right)^{\frac{1}{q}}$$

$$\ge \left(cth\frac{t}{2}\right)^{\frac{2\lambda+1}{q}} \left(\sup_{r\in R_+} \frac{\left[\left(sh\frac{r}{2}\right) th\frac{t}{2}\right]_1}{\left[sh\frac{r}{2}\right]_1} \right)^{\frac{\nu}{q}} \|J_G^\alpha f\|_{\tilde{L}_{q,\lambda,\nu}}$$

$$= \left(cth\frac{t}{2}\right)^{\frac{4\lambda}{q}} \left(th\frac{t}{2}\right)_1^{\frac{\nu}{q}} \|J_G^\alpha f\|_{\tilde{L}_{q,\lambda,\nu}}$$

$$\ge \left(ch\frac{t}{2}\right)^{\frac{4\lambda-\nu}{q}} \|J_G^\alpha f\|_{\tilde{L}_{q,\lambda,\nu}} \ge \left(sh\frac{t}{2}\right)^{\frac{\nu-\gamma}{q}} \|J_G^\alpha f\|_{\tilde{L}_{q,\lambda,\nu}}, \, 2 \le t < \infty. \tag{26}$$

On the other and by $2 \le t < \infty$, we obtain

$$\|J_G^\alpha f_t\|_{\tilde{L}_{q,\lambda,\nu}} \le \sup_{x,r\in R_+} \left(\left[sh\frac{r}{2}\right]_1^{-\nu} \int_0^r \left|J_G^\alpha f_t\left(ch\,\left(sh\frac{t}{2}\right) y\right)\right|^q sh^{2\lambda} y dy \right)^{\frac{1}{q}}$$

$$\left[\left(sh\frac{t}{2}\right) y = z, \qquad dz = \frac{dz}{sh\frac{t}{2}}\right]$$

$$= \left(sh\frac{t}{2}\right)^{-\frac{1}{q}} \sup_{x,r\in R_+} \left(\left[sh\frac{r}{2}\right]_1^{-\nu} \int_0^r |J_G^\alpha f\,(cht)|^q\, sh^{2\lambda} \left(\frac{z}{sh\frac{t}{2}}\right) dz \right)^{\frac{1}{q}}$$

$$= \left(sh\frac{t}{2}\right)^{-\frac{2\lambda+1}{q}} \sup_{x,r\in R_+} \left(\left[sh\frac{r}{2}\right]_1^{-\nu} \int_0^{rsh\frac{t}{2}} |J_G^\alpha f_t\,(chz)|^q\, sh^{2\lambda} zdz \right)^{\frac{1}{q}}$$

$$= \left(sh\frac{t}{2}\right)^{\frac{-2\lambda+1}{q}} \left(\sup_{r\in R_+} \frac{\left[sh\frac{r}{2}\left(sh\frac{t}{2}\right)\right]_1}{\left[sh\frac{r}{2}\right]_1}\right)^{\frac{\nu}{q}} \|J_G^\alpha f\|_{\tilde{L}_{q,\lambda,\nu}}$$

$$\le \left(sh\frac{t}{2}\right)^{-\frac{4\lambda}{q}} \left[sh\frac{t}{2}\right]^{\frac{\nu}{q}} \|J_G^\alpha f\|_{\tilde{L}_{q,\lambda,\nu}}$$

$$\le \left(sh\frac{t}{2}\right)^{\frac{\nu-4\lambda}{q}} \|J_G^\alpha f\|_{\tilde{L}_{q,\lambda,\nu}}$$

$$= \left(sh\frac{t}{2}\right)^{\frac{\nu-\gamma}{q}} \|J_G^\alpha f\|_{\tilde{L}_{q,\lambda,\nu}}, \ 2 \le t < \infty. \tag{27}$$

Combing (26) and (27), we have

$$\|J_G^\alpha f\|_{\tilde{L}_{q,\lambda,\nu}} \approx \left(sh\frac{t}{2}\right)^{\frac{\nu-\gamma}{q}} \|f\|_{\tilde{L}_{q,\lambda,\nu}}, \ 2 < t < \infty. \tag{28}$$

Thus (23) and (28), we obtain

$$\|J_G^\alpha f_t\|_{\tilde{L}_{q,\lambda,\nu}} \approx \left(sh\frac{t}{2}\right)^{\frac{\nu-\gamma}{q}} \|J_G^\alpha f\|_{\tilde{L}_{q,\lambda,\nu}}, \ 0 < t < \infty. \tag{29}$$

Since J_G^α is bounded from $\tilde{L}_{p,\lambda,\nu}(R_+)$ to $\tilde{L}_{q,\lambda,\nu}(R_+)$, i.e.

$$\|J_G^\alpha f\|_{\tilde{L}_{q,\lambda,\nu}} \lesssim \|f\|_{\tilde{L}_{p,\lambda,\nu}},$$

then taking into account (18) and (29), we obtain

$$\|J_G^\alpha f_t\|_{\tilde{L}_{q,\lambda,\nu}} \approx \left(sh\frac{t}{2}\right)^{\frac{\nu-\gamma}{q}} \|J_G^\alpha f\|_{\tilde{L}_{q,\lambda,\nu}} \lesssim \left(sh\frac{t}{2}\right)^{\frac{\nu-\gamma}{q}} \|f_t\|_{\tilde{L}_{p,\lambda,\nu}}$$

$$\lesssim \left(sh\frac{t}{2}\right)^{\alpha+(\nu-\gamma)\left(\frac{1}{p}-\frac{1}{q}\right)} \|f\|_{\tilde{L}_{p,\lambda,\nu}}$$

$$\lesssim \|f\|_{\tilde{L}_{p,\lambda,\nu}} \begin{cases} \left(sh\frac{t}{2}\right)^{\alpha-\gamma\left(\frac{1}{p}-\frac{1}{q}\right)}, \ 0 < t < 2 \\ \left(sh\frac{t}{2}\right)^{\alpha+(\nu-\gamma)\left(\frac{1}{p}-\frac{1}{q}\right)}, \ 2 \le t < \infty \end{cases}.$$

If $\frac{1}{p} - \frac{1}{q} < \frac{\alpha}{\gamma}$, then at $t \to 0$ we have $\|J_G^\alpha f_t\|_{\tilde{L}_{q,\lambda,\nu}} = 0$, for all $f \in \tilde{L}_{p,\lambda,\nu}(R_+)$
As well as is $\frac{1}{p} - \frac{1}{q} > \frac{\alpha}{\nu-\gamma}$, then at $t \to \infty$ we get $\|J_G^\alpha f_t\|_{\tilde{L}_{q,\lambda,\nu}} = 0$, for all $f \in \tilde{L}_{p,\lambda,\nu}(R_+)$.
Therefore $\frac{\alpha}{\gamma} \le \frac{1}{p} - \frac{1}{q} \le \frac{\alpha}{\gamma-\nu}$.

Sufficiency. Let $f \in \tilde{L}_{1,\lambda,\nu}(R_+)$, then

$$\left|\{x \in (0,r) : |J_G^\alpha f(ch\,x)| > 2\beta\}\right|_\lambda$$

$$\leq |\{x \in (0,r) : A_1(x,r) > \beta\}|_\lambda + |\{x \in (0,r) : A_2(x,r) > \beta\}|_\lambda .$$

Also

$$A_2(x,r) = \int_r^\infty A^\lambda_{ch\,y} \frac{f(ch\,x)\, sh^{2\lambda} y dy}{\left(sh\frac{y}{2}\right)^{\gamma-\alpha}}$$

$$\leq \sum_{j=0}^\infty \int_{2^j r}^{2^{j+1} r} \frac{A^\lambda_{ch\,y} |f(ch\,x)|\, sh^{2\lambda} y dy}{\left(sh\frac{y}{2}\right)^{\gamma-\alpha}}$$

$$\leq \left\| A^\lambda_{ch\,y} f \right\|_{\tilde{L}_{1,\lambda,\gamma}} \sum_{j=0}^\infty \frac{\left[2^{j+1} sh\frac{r}{2}\right]_1^\nu}{\left(2^j sh\frac{r}{2}\right)^{\gamma-\alpha}} \lesssim \left(sh\frac{r}{2}\right)^{\alpha-\gamma} \left[sh\frac{r}{2}\right]_1^\nu \|f\|_{\tilde{L}_{1,\lambda,\gamma}}$$

$$\lesssim \left(sh\frac{r}{2}\right)^{\alpha-\gamma} \left[sh\frac{r}{2}\right]_1^\nu \|f\|_{\tilde{L}_{1,\lambda,\gamma}} . \tag{30}$$

$$\lesssim \|f\|_{\tilde{L}_{1,\lambda,\gamma}} \begin{cases} \left(sh\frac{r}{2}\right)^{\alpha+\nu-\gamma}, & if\ 0 < r < 2arsh\,1, \\ \left(sh\frac{r}{2}\right)^{\alpha-\gamma}, & if\ 2arcsh\,1 \leq r < \infty. \end{cases} \tag{31}$$

According the inequality (7) and Theorem C, we obtain

$$|\{x \in (0,r) : A_1(x,r) > \beta\}|_\lambda$$

$$\lesssim \left| \left\{ x \in (0,r) : M_G f(chx) > \frac{\beta}{C sh^\alpha \frac{r}{2}} \right\} \right|_\lambda \lesssim$$

$$\lesssim \frac{1}{\beta} \left(sh^\alpha \frac{r}{2}\right) \left[sh\frac{r}{2}\right]_1^\nu \|f\|_{\tilde{L}_{1,\lambda,\gamma}}, \quad 0 < r < \infty. \tag{32}$$

If $\left(sh\frac{r}{2}\right)^{\alpha-\gamma} \left[sh\frac{r}{2}\right]_1^\nu \|f\|_{\tilde{L}_{1,\lambda,\gamma}} = \beta$, then from (30) we obtain $|A_2(x,r)| \lesssim \beta$ and consequently, $|\{x \in (0,r) : A_2(x,r) > \beta\}|_\lambda = 0$. Then by $0 < r < 2arcsh\,1$ $\left(sh\frac{r}{2}\right)^{\alpha+\nu-\gamma} \|f\|_{\tilde{L}_{1,\lambda,\gamma}} = \beta$ and from (31), we have

$$|\{x \in (0,r) : |J^\alpha_G f(ch\,x)| > 2\beta\}|_\lambda \lesssim \frac{1}{\beta} \left(sh^\alpha \frac{r}{2}\right) \left[sh\frac{r}{2}\right]_1^\nu \|f\|_{\tilde{L}_{1,\lambda,\gamma}}$$

$$= \left(sh\frac{r}{2}\right)^{\alpha-\gamma} \left[sh\frac{r}{2}\right]_1^\nu = \left(\beta^{-1} \|f\|_{\tilde{L}_{1,\lambda,\gamma}}\right)^{\frac{\gamma-\nu}{\gamma-\nu-\alpha}} \left[sh\frac{r}{2}\right]_1^\nu . \tag{33}$$

And for $2arcsh\,1 < r < \infty$, $\beta = \left(sh\frac{r}{2}\right)^{\alpha-\gamma} \|f\|_{\tilde{L}_{1,\lambda,\gamma}}$ and from (32), we have

$$|\{x \in (0,r) : |J^\alpha_G f(ch\,x)| > 2\beta\}|_\lambda \lesssim \frac{1}{\beta} \left(sh^\alpha \frac{r}{2}\right) \left[sh\frac{r}{2}\right]_1^\nu \|f\|_{\tilde{L}_{1,\lambda,\gamma}}$$

$$= \left(sh\frac{r}{2}\right)^{\alpha} \left[sh\frac{r}{2}\right]_1^\nu = \left(\beta^{-1} \|f\|_{\tilde{L}_{1,\lambda,\gamma}}\right)^{\frac{\gamma}{\gamma-\alpha}} \left[sh\frac{r}{2}\right]_1^\nu . \tag{34}$$

Finally from (33) and (34), we obtain

$$|\{x \in (0,r) : |J^\alpha_G f(ch\,x)| > 2\beta\}|_\lambda$$

$$\lesssim \left[sh\frac{r}{2}\right]_1^{\nu} \min\left\{\left(\beta^{-1}\left\|f\right\|_{\tilde{L}_{1,\lambda,\gamma}}\right)^{\frac{\gamma}{\gamma-\alpha}}, \left(\beta^{-1}\left\|f\right\|_{\tilde{L}_{1,\lambda,\gamma}}\right)^{\frac{\gamma-\nu}{\gamma-\nu-\alpha}}\right\}$$

$$\lesssim \left[sh\frac{r}{2}\right]_1^{\nu} \left(\beta^{-1}\left\|f\right\|_{\tilde{L}_{1,\lambda,\gamma}}\right)^{q},$$

where by condition of the theorem

$$\frac{\gamma}{\gamma-\alpha} \le q \le \frac{\gamma-\nu}{\gamma-\nu-\alpha} \Leftrightarrow \frac{\alpha}{\gamma} \le 1-\frac{1}{q} \le \frac{\alpha}{\gamma-\nu}.$$

Necessity. Preliminarily we established the estimates for $\|J_G^{\alpha} f_t\|_{W\tilde{L}_{q,\lambda,\nu}}$. From (15) for $0<t<2$, we have

$$\|J_G^{\alpha} f_t\|_{W\tilde{L}_{q,\lambda,\nu}} = \sup_{r\in R_+}\sup_{x,u\in R_+}\left(\left[sh\frac{u}{2}\right]_1^{-\nu}\int_{\left\{y\in(0,u):A_{ch\,y}^{\lambda}\left|J_G^{\alpha} f_t(ch\,x)\right|>r\right\}} sh^{2\lambda} y dy\right)^{\frac{1}{q}}$$

$$\ge \sup_{r\in R_+} r \sup_{x,u\in R_+}\left(\left[sh\frac{u}{2}\right]_1^{-\nu}\int_{\left\{y\in(0,u):A_{ch\,x}^{\lambda}\left|J_G^{\alpha} f_t\left(ch\left(th\frac{t}{2}\right)y\right)\right|>r\right\}} sh^{2\lambda} y dy\right)^{\frac{1}{q}}$$

$$\left[\left(th\frac{t}{2}\right)y=z,\ dy=\left(cth\frac{t}{2}\right)dz\right]$$

$$=\left(cth\frac{t}{2}\right)^{1+\frac{1}{q}}\times$$

$$\times \sup_{r\in R_+}\left(rth\frac{t}{2}\right)\sup_{x,u\in R_+}\left(\left[sh\frac{u}{2}\right]_1^{-\nu}\int_{\left\{z\in\left(0,\,uth\frac{t}{2}\right):A_{ch\,x}^{\lambda}\left|J_G^{\alpha} f(ch)\right|>rth\frac{t}{2}\right\}} sh^{2\lambda}\left(\left(cth\frac{t}{2}\right)z\right)dz\right)^{\frac{1}{q}}$$

$$\ge \left(cth\frac{t}{2}\right)^{\frac{1}{q}}\sup_{r\in R_+}\left(rth\frac{t}{2}\right)\sup_{u\in R_+}\left(\frac{\left[\left(sh\frac{u}{2}\right)th\frac{t}{2}\right]_1}{\left[sh\frac{u}{2}\right]_1}\right)^{\frac{\nu}{q}}$$

$$\times\sup_{x,u\in R_+}\left(\left[\left(sh\frac{u}{2}\right)th\frac{t}{2}\right]_1^{-\nu}\int_{\left\{z\in\left(0,\,uth\frac{t}{2}\right):A_{ch\,x}^{\lambda}\left|J_G^{\alpha} f(czh)\right|>rth\frac{t}{2}\right\}} sh^{2\lambda}\left(cth\frac{t}{2}\right)zdz\right)^{\frac{1}{q}}$$

$$\ge\left(cth\frac{t}{2}\right)^{\frac{2\lambda+1}{q}}\left[th\frac{t}{2}\right]_1^{\frac{\nu}{q}}\times$$

$$\times\sup_{r\in R_+}\left(\left(sh\frac{r}{2}\right)th\frac{t}{2}\right)\sup_{x,u\in R_+}\left(\left[\left(sh\frac{u}{2}\right)th\frac{t}{2}\right]_1^{-\nu}\int_{\left\{z\in\left(0,\,uth\frac{t}{2}\right):A_{ch\,x}^{\lambda}\left|J_G^{\alpha} f(ch)\right|>rth\frac{t}{2}\right\}} sh^{2\lambda} zdz\right)^{\frac{1}{q}}$$

$$\ge\left(th\frac{t}{2}\right)^{-\frac{2\lambda-1}{q}}\left[th\frac{t}{2}\right]_1^{\frac{\nu}{q}}\|J_G^{\alpha} f_t\|_{W\tilde{L}_{q,\lambda,\nu}}.$$

$$\geq \left(th\frac{t}{2}\right)^{\frac{\nu-\gamma}{q}} \|J_G^\alpha f_t\|_{W\tilde{L}_{q,\lambda,\nu}}. \tag{35}$$

On the other hand at $0 < r < 2$, we have

$$\|J_G^\alpha f_t\|_{W\tilde{L}_{q,\lambda,\nu}} = \sup_{r\in R_+}\sup_{x,u\in R_+}\left(\left[sh\frac{u}{2}\right]_1^{-\nu}\int_{\left\{y\in(0,u):A^\lambda_{ch\,x}\left|J_G^\alpha f_t\left(ch\left(cth\,\frac{x}{t2}\right)y\right)\right|>r\right\}} sh^{2\lambda}y dy\right)^{\frac{1}{q}}$$

$$\left[\left(cth\frac{t}{2}\right)y = z,\, dy = \left(th\frac{t}{2}\right)dz\right]$$

$$= \left(th\frac{t}{2}\right)^{\frac{1}{q}}\sup_{r\in R_+}\sup_{x,u\in R_+}\left(\left[sh\frac{u}{2}\right]_1^{-\nu}\int_{\left\{z\in\left(0,\,ucth\frac{t}{2}\right):A^\lambda_{ch\,x}\left|J_G^\alpha f(ch\,z)\right|>r\right\}} sh^{2\lambda}\left(th\frac{t}{2}\right)zdz\right)^{\frac{1}{q}}$$

$$\leq \left(th\frac{t}{2}\right)^{\frac{2\lambda+1}{q}}\sup_{u\in R_+}\left(\frac{\left[\left(sh\frac{u}{2}\right)cth\frac{t}{2}\right]_1}{\left[sh\frac{u}{2}\right]_1}\right)^{\frac{\nu}{q}}\|J_G^\alpha f\|_{W\tilde{L}_{q,\lambda,\nu}}$$

$$\lesssim \left(th\frac{t}{2}\right)^{\frac{2\lambda+1-q}{\nu}}\|J_G^\alpha f\|_{W\tilde{L}_{q,\lambda,\nu}} \lesssim \left(sh\frac{t}{2}\right)^{\frac{\nu-2\lambda-1}{q}}\|J_G^\alpha f\|_{W\tilde{L}_{q,\lambda,\nu}}$$

$$= \left(sh\frac{t}{2}\right)^{\frac{\nu-\gamma}{q}}\|J_G^\alpha f\|_{W\tilde{L}_{q,\lambda,\nu}}. \tag{36}$$

From (35) and (36) it follows that

$$\|J_G^\alpha f_t\|_{W\tilde{L}_{q,\lambda,\nu}} \approx \left(sh\frac{t}{2}\right)^{\frac{\gamma}{q}}\|J_G^\alpha f_t\|_{W\tilde{L}_{q,\lambda,\nu}},\, 0<r<2. \tag{37}$$

Now we consider the case then $2 \leq t < \infty$. From (15), we get

$$\|J_G^\alpha f_t\|_{W\tilde{L}_{q,\lambda,\nu}} \geq \sup_{r\in R_+} r \sup_{x,u\in R_+}\left(\left[sh\frac{u}{2}\right]_1^{-\nu}\int_{\left\{y\in(0,u):\left|A^\lambda_{ch\,x}J_G^\alpha f_t\left(ch\left(th\frac{t}{2}\right)y\right)\right|>r\right\}} sh^{2\lambda}y dy\right)^{\frac{1}{q}}$$

$$\left[\left(th\frac{t}{2}\right)y = z,\, dy = \left(cth\frac{t}{2}\right)dz\right]$$

$$= \left(cth\frac{t}{2}\right)^{\frac{1}{q}}\sup_{r\in R_+}\sup_{x,u\in R_+}\left(\left[sh\frac{u}{2}\right]_1^{-\nu}\int_{\left\{z\in\left(0,\,uth\frac{t}{2}\right):\left|A^\lambda_{ch\,x}J_G^\alpha f(ch\,z)\right|>r\right\}} sh^{2\lambda}\left(cth\frac{t}{2}\right)zdz\right)^{\frac{1}{q}}$$

$$= \left(cth\frac{t}{2}\right)^{\frac{2\lambda+1}{q}+1}\sup_{r\in R_+} rth\frac{t}{2}\sup_{x,u\in R_+}\left(\left[sh\frac{u}{2}\right]_1^{-\nu}\int_{\left\{z\in\left(0,\,uth\frac{t}{2}\right):\left|A^\lambda_{ch\,x}J_G^\alpha f(ch\,z)\right|>rth\frac{t}{2}\right\}} sh^{2\lambda}zdz\right)^{\frac{1}{q}}$$

$$\geq \left(cth\frac{t}{2}\right)^{\frac{2\lambda+1}{q}} \sup_{r\in R_+} \left(\frac{\left[\left(sh\frac{u}{2}\right) th\frac{t}{2}\right]_1}{\left[sh\frac{u}{2}\right]_1}\right)^{\frac{\nu}{q}} \|J_G^\alpha f\|_{W\tilde{L}_{q,\lambda,\nu}}$$

$$= \left(cth\frac{t}{2}\right)^{\frac{4\lambda}{q}} \left[th\frac{t}{2}\right]_1^{\frac{\nu}{q}} \|J_G^\alpha f\|_{W\tilde{L}_{q,\lambda,\nu}}$$

$$\geq \left(th\frac{t}{2}\right)^{\frac{\nu-4\lambda}{q}} \|J_G^\alpha f\|_{W\tilde{L}_{q,\lambda,\nu}} \geq \left(sh\frac{t}{2}\right)^{\frac{\nu-4\lambda}{q}} \|J_G^\alpha f\|_{W\tilde{L}_{q,\lambda,\nu}}$$

$$= \left(sh\frac{t}{2}\right)^{\frac{\nu-\gamma}{q}} \|J_G^\alpha f\|_{W\tilde{L}_{q,\lambda,\nu}}, \ \ 2 \leq t < \infty. \tag{38}$$

On the other than, we have

$$\|J_G^\alpha f_t\|_{W\tilde{L}_{q,\lambda,\nu}} \leq \sup_{r\in R_+} r \sup_{x,u\in R_+} \left(\left[sh\frac{u}{2}\right]_1^{-\nu} \int_{\left\{y\in(0,u):\left|A_{ch\,x}^\lambda J_G^\alpha f_t\left(ch\left(sh\frac{t}{2}\right)y\right)\right|>r\right\}} sh^{2\lambda} y dy\right)^{\frac{1}{q}}$$

$$\left[\left(sh\frac{t}{2}\right) y = z, \, dy = \frac{dz}{sh\frac{t}{2}}\right]$$

$$= \left(sh\frac{t}{2}\right)^{-\frac{1}{q}} \sup_{r\in R_+} r \sup_{x,u\in R_+} \left(\left[sh\frac{u}{2}\right]_1^{-\nu} \int_{\left\{y\in\left(0,ush\frac{t}{2}\right):\left|A_{ch\,x}^\lambda J_G^\alpha f(chz)\right|>rsh\frac{t}{2}\right\}} sh^{2\lambda} y dy\right)^{\frac{1}{q}}$$

$$\leq \frac{\left(sh\frac{t}{2}\right)^{-\frac{2\lambda+1}{q}}}{sh\frac{t}{2}} \sup_{r\in R_+} \left(rsh\frac{t}{2}\right) \sup_{u\in R_+} \left(\frac{\left[\left(sh\frac{u}{2}\right) sh\frac{t}{2}\right]_1}{\left[sh\frac{u}{2}\right]_1}\right)^{\frac{\nu}{q}} \|J_G^\alpha f\|_{W\tilde{L}_{q,\lambda,\nu}}$$

$$\lesssim \left(sh\frac{t}{2}\right)^{\frac{\nu-2\lambda-1}{q}} \|J_G^\alpha f_t\|_{W\tilde{L}_{q,\lambda,\nu}} \lesssim \left(sh\frac{t}{2}\right)^{\frac{\nu-4\lambda}{q}} \|J_G^\alpha f\|_{W\tilde{L}_{q,\lambda,\nu}}$$

$$= \left(sh\frac{t}{2}\right)^{\frac{\nu-\lambda}{q}} \|J_G^\alpha f_t\|_{W\tilde{L}_{q,\lambda,\nu}}, \ \ 2 \leq t < \infty. \tag{39}$$

According to (38) and (39), we obtain

$$\|J_G^\alpha f_t\|_{W\tilde{L}_{q,\lambda,\nu}} \approx \left(sh\frac{t}{2}\right)^{\frac{\nu-\lambda}{q}} \|J_G^\alpha f_t\|_{W\tilde{L}_{q,\lambda,\nu}}, \ \ 2 \leq t < \infty. \tag{40}$$

Thus from (37) and (40), we have

$$\|J_G^\alpha f_t\|_{W\tilde{L}_{q,\lambda,\nu}} \approx \left(sh\frac{t}{2}\right)^{\frac{\nu-\gamma}{q}} \|J_G^\alpha f_t\|_{W\tilde{L}_{q,\lambda,\nu}}, \ \ 0 < t < \infty. \tag{41}$$

From the boundedness J_G^α from $\tilde{L}_{1,\lambda,\nu}(R_+)$ to $W\tilde{L}_{q,\lambda,\nu}(R_+)$ and from (22) and (41), we have

$$\|J_G^\alpha f_t\|_{W\tilde{L}_{q,\lambda,\nu}} \leq \left(sh\frac{t}{2}\right)^{\frac{\gamma-\nu}{q}} \|J_G^\alpha f_t\|_{W\tilde{L}_{q,\lambda,\nu}}$$

$$\lesssim \left(sh\frac{t}{2}\right)^{\frac{\gamma-\nu}{q}} \|f_t\|$$

$$\lesssim \left(sh\frac{t}{2}\right)^{\frac{\gamma-\nu}{q}} \left(sh\frac{t}{2}\right)^{\alpha+\nu-\gamma} \|f\|_{\tilde{L}_{1,\lambda,\nu}}$$

$$= \left(sh\frac{t}{2}\right)^{\alpha-(\gamma-\nu)\left(1-\frac{1}{q}\right)} \|f\|_{\tilde{L}_{1,\lambda,\nu}}$$

$$\lesssim \|f\|_{\tilde{L}_{1,\lambda,\gamma}} \begin{cases} \left(sh\frac{t}{2}\right)^{\alpha-\gamma\left(1-\frac{1}{q}\right)}, & if\ 0<t<2arcsh\,1, \\ \left(sh\frac{t}{2}\right)^{\alpha-(\gamma-\nu)\left(1-\frac{1}{q}\right)}, & if\ 2arcsh\,1<t<\infty. \end{cases}$$

If $1-\frac{1}{q}<\frac{\alpha}{\gamma}$, then at $t\to 0$, we have $\|J_G^\alpha f_t\|_{W\tilde{L}_{q,\lambda,\nu}}=0$ for all $f\in\tilde{L}_{1,\lambda,\nu}(R_+)$. Similarly, if $1-\frac{1}{q}>\frac{\alpha}{\gamma-\nu}$, then at $t\to\infty$ we obtain $\|J_G^\alpha f_t\|_{W\tilde{L}_{q,\lambda,\nu}}=0$ for all $f\in\tilde{L}_{1,\lambda,\nu}(R_+)$. Therefore, $\frac{\alpha}{\gamma}\leq 1-\frac{1}{q}\leq\frac{\alpha}{\gamma-\nu}$.

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