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On the Completeness and Minimality of Eigenfunctions of a Non-self-adjoint Spectral Problem With Spectral Parameter in the Boundary Condition

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Abstract. The article considers the following spectral problem:

$$-y'' + q\,(x)\,y = \lambda y, \quad x \in (0,1)\,,$$

$$\left.\begin{aligned} y(0) &= 0, \\ y'(0) &= (a\lambda + b)y(1), \end{aligned}\right\}$$

where $q(x)$ is a complex-valued summable function, λ is a spectral parameter, a and b are arbitrary complex numbers ($a \neq 0$.) The theorems on completeness and minimality of eigenunctions of a spectral problem in $L_p(0,1) \oplus C$ and $L_p(0,1)$ are proved.

Key Words and Phrases: eigenvalues, eigenunctions, complete and minimal system.

2010 Mathematics Subject Classifications: 34B05, 34B24, 34L10, 34L20

1. Introduction

Consider the following spectral problem:

$$-y'' + q\,(x)\,y = \lambda y, \quad x \in (0,1)\,, \tag{1}$$

$$\left.\begin{aligned} y(0) &= 0, \\ y'(0) &= (a\lambda + b)y(1), \end{aligned}\right\} \tag{2}$$

where $q(x)$ is a complex-valued function, λ is a spectral parameter, a and b are arbitrary complex numbers ($a \neq 0$.) The purpose of this article is to prove the corresponding theorems on the completeness and minimality of a system of eigenfunctions of the spectral problem (1), (2) in the spaces $L_p(0,1) \oplus C$ and $L_p(0,1)$. There are numerous articles and monographs on the study of the spectral properties of problems posed for ordinary differential operators and including spectral parameters in the boundary conditions (see, e.g., [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14]). One can cite articles from recent works [15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26]. Special mention should be made of the works [8, 9, 14, 25, 26] directly related to our paper. So, the case $q(x) \equiv 0$, $b = 0$ is considered in [8, 9], and in [14] under the additional condition $q(x) = q(1-x)$, it is considered the case $b = 0$. Other generalizations of the boundary conditions (2) are also found in [25, 26], where questions of the uniform convergence of spectral expansions and also, under an additional condition $q(x) = q(1-x)$, the basis properties of eigenfunctions in the spaces $L_p(0,1)$ are studied.

2. Auxiliary facts and initial results

In order to obtain the main results, we need some abstract results on complete and minimal systems in the direct sum of a Banach space with a finite-dimensional one. A system $\{u_n\}_{n\in N}$ of a Banach space X is called complete in X if the closure of the linear span of this system coincides with the entire space X, and is called minimal if no element of this system is included in the closed linear span of other elements of this system. Recall also that a system is complete in X if and only if there is no nonzero linear continuous functional that annihilates all elements of this system. A system is minimal in X if and only if it has a biorthogonal system.

Let $X_1 = X \oplus C^m$ and $\{\hat{u}_n\}_{n\in N} \subset X_1$ be some minimal system, and $\left\{\widehat{\vartheta}_n\right\}_{n\in N} \subset X_1^* = X^* \oplus C^m$ is its biorthogonal system :

$$\hat{u}_n = (u_n; \alpha_{n1}, ..., \alpha_{nm}); \qquad \widehat{\vartheta}_n = (\vartheta_n; \beta_{n1}, ..., \beta_{nm}).$$

Let $J = \{n_1, ..., n_m\}$ be some set of m distinct natural numbers and $N_J = N \backslash J$. Assume

$$\delta = det \left\| \beta_{n_i j} \right\|_{i,j=\overline{1,m}}.$$

The following theorem is true.

Theorem 1. *[27, 28] Let $\{\hat{u}_n\}_{n\in N}$ be minimal in X_1 with conjugated system $\left\{\widehat{\vartheta}_n\right\}_{n\in N} \subset X_1^*$. If $\delta \neq 0$, then the system $\{u_n\}_{n\in N_J}$ is minimal in X. In this case, the orthogonally conjugate system has the form*

$$\vartheta_n^* = \frac{1}{\delta} \begin{vmatrix} \vartheta_n & \vartheta_{n1} & ... & \vartheta_{nm} \\ \beta_{n1} & \beta_{n_1 1} & ... & \beta_{n_m 1} \\ ...... & ...... & ...... & ...... \\ \beta_{nm} & \beta_{n_1 m} & \cdots & \beta_{n_m m} \end{vmatrix}.$$

If $\{\hat{u}_n\}_{n\in N}$ is complete and minimal in X_1 and $\delta \neq 0$, then $\{u_n\}_{n\in N_0}$ is complete and minimal in X. If the system $\{\hat{u}_n\}_{n\in N}$ is complete and minimal in X_1 and $\delta = 0$, then the system $\{u_n\}_{n\in N_0}$ is not complete in X.

Accept $\lambda = -\rho^2$. Let us denote the forms included in the boundary conditions (2) as follows:

$$\left.\begin{aligned} U_1(y) &= y(0), \\ U_2(y) &= y'(0) - \left(a\rho^2 + b\right) y(1). \end{aligned}\right\} \tag{3}$$

After these notation, the problem (1), (2) can be written as follows:

$$-y'' + q(x)\,y + \rho^2 y = 0, x \in (0,1), \tag{4}$$

$$\left.\begin{aligned} U_1(y) &= 0, \\ U_2(y) &= 0. \end{aligned}\right\} \tag{5}$$

It is known that there is a system of fundamental solutions $y_1(x)$ and $y_2(x)$ of the equation (4) in the interval $(0,1)$ and these solutions are regular functions of ρ and at large values of $|\rho|$ with respect to the variable $x \in [0,1]$ uniformly satisfy the following asymptotic relationships:

$$\left.\begin{aligned} y_1^{(j)}(x) &= (\rho\omega_1)^j e^{\rho\omega_1 x}\left[1 + O\left(\tfrac{1}{\rho}\right)\right], \\ y_2^{(j)}(x) &= (\rho\omega_2)^j e^{\rho\omega_2 x}\left[1 + O\left(\tfrac{1}{\rho}\right)\right], \end{aligned}\right\} \tag{6}$$

here $j = 0, 1;$ ρ belongs to one of the four S -sectors [29, p. 62], and $\omega_1,\ \omega_2$ are different square roots of -1, numbered so that for $\rho \in S$ the inequality $Re(\rho\omega_1) \le Re(\rho\omega_2)$ holds. For example, for the sector $S_0 = \left\{\rho : 0 \le arg\ \rho \le \frac{\pi}{2}\right\}$ we have $\omega_1 = i,\ \omega_2 = -i$.

The solution of the equation (1) (or (4)) should be in the form of

$$y(x) = c_1 y_1(x) + c_2 y_2(x).$$

Let us choose the constants c_1 and c_2 so that the function $y(x)$ satisfies the boundary conditions (5). Then to find the constants c_1, c_2 we get the following system of algebraic equations:

$$\left.\begin{aligned} c_1 U_1(y_1) + c_2 U_1(y_2) &= 0, \\ c_1 U_2(y_1) + c_2 U_2(y_2) &= 0. \end{aligned}\right\}$$

It is known that there is a non-trivial solution of this system of algebraic equations when its main determinant (characteristic determinant) $\Delta(\rho)$ equals zero. Thus, the number $\lambda = \rho^2$ is a eigen value of the spectral problem (1) - (2) if and only if it is a solution of the following equation:

$$\Delta(\rho) = \begin{vmatrix} U_1(y_1) & U_1(y_2) \\ U_2(y_1) & U_2(y_2) \end{vmatrix} = U_1(y_1)\,U_2(y_2) - U_2(y_1)\,U_1(y_2) = 0. \tag{7}$$

Considering the asymptotic formulas (6) in the expressions of U_1 and U_2 in (3), we obtain the following asymptotic relations:

$$U_1(y_1) = 1 + O(\frac{1}{\rho}), \qquad U_1(y_2) = 1 + O(\frac{1}{\rho}),$$

$$U_2(y_1) = i\rho\left[1 + O(\frac{1}{\rho})\right] - (a\rho^2 + b)e^{i\rho}\left[1 + O(\frac{1}{\rho})\right],$$

$$U_2(y_2) = -i\rho\left[1 + O(\frac{1}{\rho})\right] - (a\rho^2 + b)e^{-i\rho}\left[1 + O(\frac{1}{\rho})\right].$$

By substituting these asymptotic relations in the expression of $\Delta(\rho)$ in (7) and using Birkhof's sign $[A] = A + O\left(\frac{1}{\rho}\right)$ we obtain:

$$\Delta(\rho) = \begin{vmatrix} [1] & [1] \\ i\rho[1] - (a\rho^2 + b)e^{i\rho}[1] & -i\rho[1] - (a\rho^2 + b)e^{-i\rho}[1] \end{vmatrix}$$

Calculating the determinant $\Delta(\rho)$ and consider that when the complex number ρ enters the sector $Re\rho \geq 0$, $Im\rho \geq 0$, the inequality $Re(i\rho) \leq 0 \leq Re(-i\rho)$ satisfies, then we get the following asymptotic relation:

$$\Delta(\rho) = \left(a\rho^2 + b\right)e^{-i\rho}\left[e^{2i\rho} - 1 + O\left(\frac{1}{\rho}\right)\right] - 2i\rho[1] =$$

$$= \left(a\rho^2 + b\right)e^{-i\rho}\left[e^{2i\rho} - 1 - \frac{2i\rho}{a\rho^2 + b}e^{i\rho} + O(\frac{1}{\rho})\right]. \tag{8}$$

So, the eigen values of the problem (1),(2) are the root of the equation

$$\Delta_0(\rho) = e^{2i\rho} - 1 - \frac{2i\rho}{a\rho^2 + b}e^{i\rho} + O(\frac{1}{\rho}) = 0. \tag{9}$$

Note that the number $\lambda = -\frac{b}{a}$ (i.e. $\rho = \pm\sqrt{\frac{a}{b}}i$) cannot be an eigen value, because in this case the function $y(x)$ satisfies the initial conditions $y(0) = 0$, $y'(0) = 0$, from which $y(x) \equiv 0$ is obtained . The roots of the equation $f(\rho) = e^{2i\rho} - 1 = 0$ are the numbers $\widetilde{\rho}_k = \pi k, k = 0, \pm 1, \ldots$.. Since $\Delta(\rho)$ is an even function, we will consider only the roots of this function in the right hemisphere. Draw a circle γ_k with the same radius δ $(0 < \delta < \frac{\pi}{2})$ around each point $\widetilde{\rho}_k$. If we denote the region outside these circles by Q_δ, then the function $f(\rho) = e^{2i\rho} - 1$ in this region is bounded by a definite positive constant from below. Indeed, since the function $f(\rho)$ is a periodic function with a period π, it suffices to investigate this function in a vertical stripe bounded by the straight lines $Rez = \pm\frac{\pi}{2}$. While in this stripe the following relations

$$\lim_{Im\rho \to -\infty} |f(\rho)| = +\infty,$$

$$\lim_{Im\rho \to +\infty} |f(\rho)| = 1,$$

are true. Since the function $f(\rho)$ does not vanish outside the circle γ_0 in this band, it is bounded from below by an absolute value positive number α outside the circle γ_0. At large values of $|\rho|$ the inequality $\left|O(\frac{1}{\rho})\right| < \alpha$ is also satisfied. Therefore, according to Rouché's theorem, at sufficiently large values of k, equation (9) has only one root inside the circle γ_k, and if we denote it by ρ_k, then from equation (9) we get the asymptotic formula

$$\rho_k = \pi k + O\left(\frac{1}{k}\right). \tag{10}$$

In addition, since the function $f(\rho) = e^{2i\rho} - 1$ is bounded below by a certain positive number in the domain Q_δ it follows that for sufficiently large $|\rho|$ the function $\Delta_0(\rho) = e^{2i\rho} - 1 - \frac{2i\rho}{a\rho^2+b}e^{i\rho} + O(\frac{1}{\rho})$ is also bounded below by a certain positive number in domain $S_0 \cap Q_\delta$.

Taken into account the asymptotic formula (6) for sufficiently large $|\rho|$ we get the inequality

$$|\Delta(\rho)| \geq M_\delta |\rho|^2 e^{\tau}, \tag{11}$$

where the constant M_δ independent of ρ, only depends on the number $\delta > 0$.

Thus, the following theorem is proved.

Theorem 2. *The characteristic determinant $\Delta(\rho)$ of the spectral problem* (1),(2) *has the following properties:*

1. *there exists a positive number M_δ such that, in domain $S_0 \cap Q_\delta$ for the sufficiently large $|\rho|$ the inequality $|\Delta(\rho)| \geq M_\delta |\rho|^2 e^{\tau}$ holds;*

2. *The zeros of the function $\Delta(\rho)$ are asymptotically simple and have asymptotics as follows:*

$$\rho_k = \pi k + O\left(\frac{1}{k}\right), k = 0, 1, 2, ...$$

3. Construction of the Green function of the spectral problem (1), (2)

To construct the Green function of the problem (1), (2), it is necessary to obtain an integral representation for the solution of the corresponding non-homogeneous equation. Let us write the non-homogeneous equation as follows

$$-y'' + q(x)y = \lambda y + f(x), x \in (0, 1). \tag{12}$$

When the number λ is not an eigenvalue, if we apply the method of variation of the constant to find the solution of equation (12) that satisfies the boundary conditions (2), we obtain the following formula for the solution $y(x)$ of this equation:

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \int_0^1 g(x, \xi) f(\xi) d\xi, \; x \in (0, 1), \tag{13}$$

where

$$g(x,\xi)=\begin{cases} \frac{1}{2}\frac{1}{W(\xi)}\left(y_1\left(\xi\right)y_2\left(x\right)-y_2\left(\xi\right)y_1\left(x\right)\right), & x>\xi, \\ -\frac{1}{2}\frac{1}{W(\xi)}\left(y_1\left(\xi\right)y_2\left(x\right)-y_2\left(\xi\right)y_1\left(x\right)\right), x<\xi, \end{cases}$$

$W(x)$ is the Wronskan of the functions $y_1(x), y_2(x)$, i.e.

$$W(\xi)=\begin{vmatrix} y_1(\xi) & y_2(\xi) \\ y_1'(\xi) & y_2'(\xi) \end{vmatrix}.$$

Let us claim that the general solution (13) of equation (12) satisfies the boundary conditions (2), i.e. is the solution of the boundary value problem (12), (2). This means that the constants c_1, c_2 must be solutions of the following non-homogeneous system of algebraic equations:

$$\begin{cases} c_1U_1(y_1)+c_2U_1(y_2)+\int_0^1 U_1(g)f(\xi)d\xi=0. \\ c_1U_2(y_1)+c_2U(y_2)+\int_0^1 U_2(g)f(\xi)d\xi=0. \end{cases}$$

Since λ is not an eigenvalue, the main determinant of this system is differ from zero, and therefore there exists only one solution. Solving this system and substituting the found values of the constants c_1 and c_2 in equation (13), we obtain the following formula:

$$y\left(x\right)=\int_0^1 G\left(x,\xi,\rho\right)f\left(\xi\right)d\xi. \tag{14}$$

In formula (14) $G(x,\xi,\rho)$ is a Green's function and defined as follows:

$$G\left(x,\xi,\rho\right)=\frac{1}{\Delta\left(\rho\right)}\begin{vmatrix} y_1\left(x\right) & y_2\left(x\right) & g\left(x,\xi\right) \\ U_1\left(y_1\right) & U_1\left(y_2\right) & U_1\left(g\right) \\ U_2\left(y_1\right) & U\left(y_2\right) & U_2\left(g\right) \end{vmatrix}, x,\xi\in[0,1], \tag{15}$$

where

$$g(x,\xi)=\begin{cases} \frac{1}{2}\left(z_1\left(\xi\right)y_2\left(x\right)+z_2\left(\xi\right)y_1\left(x\right)\right), & x\geq\xi, \\ -\frac{1}{2}(z_1(\xi)y_2(x)+z_2(\xi)y_1(x), & x<\xi, \end{cases}$$

$$z_1\left(\xi\right)=\frac{y_2(\xi)}{W(\xi)},\ z_2\left(\xi\right)=-\frac{y_1(\xi)}{W(\xi)},$$

$$U_1\left(g\right)=-\frac{1}{2}\left(U_1\left(y_2\right)z_1\left(\xi\right)+U_1\left(y_1\right)z_2\left(\xi\right)\right),$$

$$U_2\left(g\right)=-\frac{1}{2}\left(z_1\left(\xi\right)y_2'\left(0\right)+z_2\left(\xi\right)y_1'\left(0\right)\right)-\left(a\rho^2+b\right)\frac{1}{2}\left(z_1\left(\xi\right)y_2\left(1\right)+z_2\left(\xi\right)y_1\left(1\right)\right).$$

So, the following lemma is proved.

Lemma 1. *The Green function of the spectral problem* (1),(2) *is defined by the formula* (15).

4. Evaluation of the linearized operator's resolvent. Theorems on the completeness

Let us now reduce the study of the spectral problem (1), (2) to the study of the spectral problem $L\hat{y} = \lambda\hat{y}$ for an operator L acting in the space $L_p(0,1) \oplus C$. The operator L is defined as follows:

$$D(L) = \left\{\hat{y} \in L_p \oplus C : \hat{y} = (y(x), ay(1)), y \in W_p^2(0,1), l(y) \in L_p(0,1), y(0) = 0\right\},$$

for $\hat{y} \in D(L)$ it is true $L\hat{y} = (l(y); y^{'}(0) - by(1))$.

Lemma 2. *Operator L is a closed operator with a compact resolvent and is dense everywhere in the domain $L_p(0,1) \oplus C$. The eigenvalues of the operator L coincide with the eigenvalues of problem* (1), (2). *Each eigen or associated function $y(x)$ of the problem* (1), (2) *corresponds to an eigen or associated vector $\hat{y} = (y(x), ay(1))$ of the operator L.*

Proof. Let's define the function $F\hat{y} = y(0)$ for the vector $\hat{y} = (y\,(x)\,, ay\,(1))\,,\ y\,(x) \in W_p^2\,(0,1)$. It can be easily checked that the functional F is bounded in space $W_p^2(0,1) \oplus C$ and unbounded in space $L_p(0,1) \oplus C$. Then the considered operator L is a finite-dimensional contraction of the maximum operator $\tilde{L}$, defined as follows:

$$\tilde{L} : L_p \oplus C \to L_p \oplus C,$$

$$D\left(\tilde{L}\right) = \left\{\hat{y} \in L_p \oplus C : \hat{y} = (y\,(x)\,, ay\,(1))\,, y \in W_p^2\,(0,1)\,, l\,(y) \in L_p\,(0,1)\right\},$$

$$\tilde{L}\hat{y} = \left(l\,(y)\,, y^{'}(0) - by\,(1)\right), \quad \forall\ \hat{y} \in D(\tilde{L}).$$

Then (see [30, 31]) we obtain that the operator L is a closed operator with a compact resolvent and its domain is dense everywhere. The second part of the lemma is examined directly.

Note that since the operator L is closed and dense defined everywhere, it has an adjoint, and the adjoint operator L^* will be the linear operator generated by the spectral problem

$$-z^{''} + \overline{q(x)}z = \lambda z, \tag{16}$$

$$\left.\begin{aligned} z(1) &= 0, \\ z^{'}(1) &= -(\overline{a}\lambda + \overline{b})z(0), \end{aligned}\right\} \tag{17}$$

in the space $L_q(0,1) \oplus C$, where $q = \frac{p}{p-1}$.

To construct the resolvent operator $R\,(\lambda) = (L - \lambda I)^{-1}$ take an arbitrary element $\tilde{f} = (f(x), \beta) \in L_p(0,1) \oplus C$ and consider the operator equation $(L - \lambda I)\hat{y} = \hat{f}$. To solve this equation, it is necessary to find a solution to equation (12) that satisfies condition

$$\left.\begin{aligned} y(0) &= 0, \\ y'(0) - (a\lambda + b)y(1) &= \beta. \end{aligned}\right\}. \tag{18}$$

It is obvious that, for each regular number λ the element $\hat{y} = (y(x,\lambda), ay(1,\lambda)) \in D(L)$ will be the solution of the equation $L\hat{y} - \lambda\hat{y} = \tilde{f}$ if and only if the function $y(x)$ will be a solution of the non-homogeneous equation (12), (18). We can present the solution $y(x,\lambda)$ of equations (12), (18) in the form of the sum of two functions:

$$y(x,\lambda) = \phi(x,\lambda) + h(x,\lambda)$$

thus, $\phi(x,\lambda)$ is the solution of the problem (12), (18), and $h(x,\lambda)$ is the solution of the problem (1), (18). The representation (14) for the function $\phi(x,\lambda)$ has already been obtained. Now let's take a representation for the function $h(x,\lambda)$ Let's denote it briefly by $h(x)$. Then let us seek it in the form

$$h(x) = a_1 y_1(x) + a_2 y_2(x), x \in (0,1), \tag{19}$$

where the constants a_1, a_2 must be the solution of the following system of algebraic equations:

$$\begin{cases} U_1(h) = 0, \\ U_2(h) = \beta, \end{cases}$$

or

$$\begin{cases} a_1 U_1(y_1) + a_2 U_1(y_2) = 0, \\ a_1 U_2(y_1) + a_2 U_2(y_2) = \beta. \end{cases}$$

By solving this system of equations, we have

$$a_1 = -\frac{\beta}{\Delta(\rho)} U_1(y_2), \quad a_2 = \frac{\beta}{\Delta(\rho)} U_1(y_1).$$

If we substitute them in (18), we obtain

$$h(x) = \frac{\beta}{\Delta(\rho)}(-U_1(y_2)y_1(x) + U_1(y_1)y_2(x)). \tag{20}$$

Thus, if the number λ is a regular point of the operator L, then we obtain the following representation for the solution $y(x,\lambda)$ of the problem (12), (18):

$$y(x,\lambda) = \int_0^1 G(x,\xi,f)f(\xi)d\xi + \frac{\beta}{\Delta(\lambda)}(-U_1(y_2)y_1(x) + U_1(y_1)y_2(x)) \tag{21}$$

where $G(x,\xi,\lambda)$ is a Green function and is determined by equation (15) .

Now we can proceed to a direct estimate of the resolvent $R(\lambda) = (L - \lambda I)^{-1}$. Let Ω_δ be the image of the domain Q_δ in the complex λ -plane under the mapping $\lambda = \rho^2$.

Theorem 3. *For the resolvent of the operator L, which linearizes the spectral problem* (1),(2), *in the domain Ω_δ for large values of $|\lambda|$ the following estimate is valid*

$$\|R(\lambda)\| \leq \frac{M_\delta}{|\lambda|^{\frac{1}{2}}}. \tag{22}$$

Proof. Let $\hat{f} = (f(x), \beta \in L_p \oplus C$ be an arbitrary fixed element. To estimate the resolvent it is necessary to estimate the vector $(y\,(x)\,, ay\,(1)) \in L_p \oplus C$. Let us show that if $\rho \in Q_\delta$, $Im\rho \geq 0$, then for sufficiently large $|\rho|$, for the solution $y(x, \rho)$ of the problem (12), (18) uniformly with respect to the variable $x \in [0, 1]$ the following inequality

$$|(y(x, \rho)| \leq \frac{C}{|\rho|}$$

is true; where the constant C is independent of ρ, but depend only on element $\hat{f} \in L_p(0, 1) \oplus C$ and δ. Let us accept $\lambda = -\rho^2, \rho = s + i\tau, \tau \geq 0$. Then according to (21) the following representation

$$y(x, \rho) = \int_0^1 G\,(x, \xi, \rho)\, f\,(\xi)\, d\xi + \frac{\beta}{\Delta\,(\rho)}\,(-U_1(y_2)y_1(x) + U_1(y_1)y_2(x)) =$$

$$= \int_0^1 G(x, \xi, \rho) f(\xi) d\xi + h(x, \rho))$$

is true, where

$$h(x, \rho) = \beta \frac{-U_1(y_2)y_1(x) + U_2(y_1)y_2(x)}{\Delta(\rho)}. \tag{23}$$

Using asymptotic formulas (3), we can write the following:

$$U_1(y_2)y_1(x) = e^{i\rho x}[1] \cdot [1] = e^{i\rho x}[1] = O(1), (Rei\ \rho \leq 0),$$

$$U_2(y_1)y_2(x) = e^{-i\rho x}[1](i\rho\,[1] - \left(a\rho^2 + b\right) e^{i\rho[1]} = a\rho^2 e^{-i\rho(1-x)}\,[1] = O\left(e^{\tau}\right).$$

Note that these asymptotic formulas uniformly satisfy with respect to the variable $x \in [0, 1]$. Considering these asymptotic formulas in (23), we obtain that as $|\rho| \to \infty$, the increase in the numerator of the fraction in (23) is like $O(e^\tau)$. On the other hand, taking into account the inequality (11) and the above estimate of the increase in the numerator of the fraction in (23), for sufficiently large $|\rho|$ in the domain Q_δ the following estimate

$$|h(x, \rho)| \leq M_\delta^{'} e^{-\tau} \leq \frac{M_\delta^{'}}{|\rho|^2} \tag{24}$$

is obtained, here the constant $M_\delta^{'}$ is independent of ρ.

Now, let us estimate the function $\phi(x, \rho)$. Taking into account the asymptotic formulas (3) in the following expressions

$$z_1(\xi) = \frac{y_1(\xi)}{W(\xi)}, \quad z_2(\xi) = -\frac{y_2(\xi)}{W(\xi)},$$

we have:

$$z_1(\xi) = \frac{y_2(\xi)}{\begin{vmatrix} y_1(\xi) & y_2(\xi) \\ y_2^{'}(\xi) & y_2^{'}(\xi) \end{vmatrix}} = \frac{e^{-i\rho\xi}[1]}{\begin{vmatrix} e^{i\rho\xi}[1] & e^{-i\rho\xi}[1] \\ & \\ i\rho e^{i\rho\xi}[1] & -i\rho e^{-i\rho\xi}[1] \end{vmatrix}} = \frac{e^{-i\rho\xi}\,[1]}{i\rho \begin{vmatrix} [1] & [1] \\ [1] & [1] \end{vmatrix}} = \frac{e^{-i\rho\xi}}{-2i\rho}[1] = \frac{i}{2\rho} e^{-i\rho\xi}[1] \tag{25}$$

$$z_2(\xi) = -\frac{y_1(\xi)}{\begin{vmatrix} y_1(\xi) & y_2(\xi) \\ y_1'(\xi) & y_2'(\xi) \end{vmatrix}} = -\frac{e^{i\rho\xi}[1]}{\begin{vmatrix} e^{i\rho\xi}[1] & e^{-i\rho\xi}[1] \\ i\rho e^{i\rho\xi}[1] & -i\rho e^{-i\rho\xi}[1] \end{vmatrix}} = -\frac{e^{-i\rho\xi}[1]}{i\rho \begin{vmatrix} [1] & [1] \\ [1] & [1] \end{vmatrix}} = -\frac{i}{2\rho}e^{i\rho\xi}[1] \tag{26}$$

Consider the function $G(x,\xi,\rho)$ in the case of $x \geq \xi$ (the case of $x < \xi$ is considered similarly).

The determinant (12), which determines the function $G\ (x,\xi,\ \rho)$, can be transformed as follows: multiply the first column of the determinant by $\frac{1}{2}z_2(\xi)$, and the second column by $-\frac{1}{2}z_1(\xi)$ and add to last column. Using asymptotic formulas (3), (25), (26), we obtain the following formulas for the elements of the last column of the determinant in (15)

$$P_1 = g\,(x,\xi) + \frac{1}{2}y_1\,(x)\,z_2\,(\xi) - \frac{1}{2}y_2\,(x)\,z_1\,(\xi) =$$

$$= \frac{1}{2}z_1(\xi)y_2(x) + \frac{1}{2}z_2(\xi)y_1(x) + \frac{1}{2}y_1(x)z_2(\xi) - \frac{1}{2}y_2\,(x)\,z_1\,(\xi) =$$

$$= y_1\,(x)\,z_2\,(\xi) = e^{i\rho x}\,[1]\left(\frac{-i}{2\rho}e^{-i\rho\xi}\right)[1] = -\frac{i}{2\rho}e^{i\rho(x-\xi)}\,[1]\,, \tag{27}$$

$$P_2 = U_1\,(g) + \frac{1}{2}z_2\,(\xi)\,U_1\,(y_1) - \frac{1}{2}z_1\,(\xi)\,U_1\,(y_2) =$$

$$= -\frac{1}{2}z_1(\xi)U_1(y_2) - \frac{1}{2}z_2(\xi)U_1(y_1) + \frac{1}{2}z_2(\xi)U_1(y_1) -$$

$$-\frac{1}{2}z_1\,(\xi)\,U_1\,(y_2) = -z_1\,(\xi)\,U_1\,(y_2) = -\frac{i}{2\rho}e^{-i\rho\xi}\,[1]\cdot[1] = -\frac{i}{2\rho}e^{i\rho\xi}\,[1]\,, \tag{28}$$

$$P_3 = -\frac{1}{2}z_1\,(\xi)\,y_2'(0) - \frac{1}{2}z_2\,(\xi)\,y_1'(0) - \frac{1}{2}\left(a\rho^2+b\right)z_1\,(\xi)\,y_2\,(1)$$

$$-\frac{1}{2}(a\rho^2+b)z_2(\xi)y_1(1) + \frac{1}{2}z_2\,(\xi)\,y_1'\,(0)$$

$$-\frac{1}{2}z_2(\xi)(a\rho^2+b)y_1(1) - \frac{1}{2}z_1(\xi)y_2'(0) + \frac{1}{2}(a\rho^2+b)z_1(\xi)y_2(1) =$$

$$= -z_1\,(\xi)\,y_2'(0) - \left(a\rho^2+b\right)z_2\,(\xi)\,y_1\,(1) = i\rho\,[1]\,\frac{i}{2\rho}e^{i\rho\xi}\,[1] -$$

$$(a\rho^2+b)e^{i\rho}[1](-\frac{i}{2\rho}e^{-i\rho\xi}[1]) = -\frac{1}{2}e^{i\rho\xi}[1] + \frac{(a\rho^2+b)i}{2\rho}e^{i\rho(1-\xi)} \tag{29}$$

Substituting formulas (27), (28), (29) into the formula (15) of the Green function, we obtain:

$$G(x,\xi,\rho) = \frac{1}{\Delta(\rho)}\begin{vmatrix} y_1(x) & y_2(x) & P_1 \\ U_1(y_1) & U_1(y_2) & P_2 \\ U_2(y_1) & U_2(y_2) & P_3 \end{vmatrix} = \frac{e^{i\rho}}{(a\rho^2+b)\Delta_0(\rho)}\times$$

$$\times \begin{vmatrix} e^{i\rho x}[1] & e^{-i\rho x}[1] & -\frac{i}{2\rho}e^{i\rho(x-\xi)}[1] \\ i\rho[1]-(a\rho^2+b)e^{i\rho}[1] & -i\rho[1]-(a\rho^2+b)e^{-i\rho}[1] & \frac{a\rho^2+b}{2\rho}ie^{i\rho(1-\xi)}[1]-\frac{1}{2}e^{i\rho\xi}[1] \end{vmatrix}.$$

Since the last formula contains $0 \le x \le 1$, $0 \le \xi \le 1$, $x \ge \xi$ and $Re(\ i\rho) \le 0$ the powers of the exponents included in the determinant are complex numbers, the real part of which is not positive. We have shown that the function $\Delta_0(\rho)$ is bounded below by some positive number. Thus, the function $G(x,\xi,\rho)$ for large values of $\rho \in S_0 \cap Q_\delta, 0 \le \xi \le x \le 1$, and $|\rho|$ satisfies the following inequality

$$|G(x,\xi,\rho)| \le \frac{C}{|\rho|}; \tag{30}$$

this inequality is satisfied uniformly with respect to the variables x and ξ. Now, taking into account the inequalities (20) and (30), we obtain the following estimate for the solution $y(x,\rho)$ of equations (12), (18) for the fixed element $\hat{f} \in L_p(0,1) \oplus C$:

$$|y(x,\rho| = \left| \int_0^1 G(x,\xi,\rho) f(\xi)\, d\xi + h(x,\rho) \right| \le$$

$$\le \int_0^1 |G(x,\xi,\rho)|\, |f(\xi)|\, d\xi + |h(x,\rho| \le$$

$$\le \frac{C}{|\rho|} (\int_0^1 |f(\xi)|\, d\xi + |\beta|) \le$$

$$\le \frac{C}{|\rho|} \left\| \hat{f} \right\|_{L_p \oplus C}. \tag{31}$$

Hence, we have the inequality

$$\|y\|_{L_p} \le \frac{C}{|\rho|} \left\| \hat{f} \right\|_{L_p \oplus C}.$$

Since the estimate (31) is satisfied uniformly with respect to the variable $x \in [0,1]$, the estimate for $|y(1)|$ is obtained by writing $x = 1$ in (31). Thus, the inequality (22) is true for each $\lambda \in \Omega_\delta$.

Theorem is proved.

Using the Theorem 3, let's prove the following theorem, which is the main result of this section.

Theorem 4. *The system of eigen and associated elements of the operator L is a complete and minimal system in the space $L_p(0,1) \oplus C$, $1 < p < \infty$.*

Proof. The minimality of the system of eigenvectors and associated vectors of the operator L in the space $L_p(0,1) \oplus C$, $1 < p < \infty$, is a consequence of the fact that the resolvent of the operator L is a compact operator in this space [32]. Therefore, we prove the completeness of this system. According to Theorem 2, the resolvent of the operator

L satisfies estimate (22) . This estimate means that the resolvent $R(\lambda) = (L - \lambda I)^{-1}$ satisfies the inequality

$$\left\| R\left(\rho^2\right) \right\| \leq \frac{C_\delta}{|\rho|}, \ \rho \in Q_\delta, \ |\rho| \geq r_0. \tag{32}$$

Let us assume that the system of root vectors of the operator L is not complete in space $L_p(0,1) \oplus C$. Then there exists a vector $\hat{g} \in L_q(0,1) \oplus C$ orthogonal to all root subspaces of the operator L, i.e.

$$\left\langle Q_n \hat{f}, \hat{g} \right\rangle = 0, \forall \hat{f} \in L_p(0,1) \oplus C, \ n = 0, 1, 2, ...,$$

and hence $Q_n^* \hat{g} = 0, \ n = 0, 1, 2, ...$; here Q_n denotes the Riesz projectors of the operator L :

$$Q_n = \frac{1}{2\pi i} \oint_{|\lambda - \lambda_n| = r} R(\lambda) d\lambda.$$

In this case it is obvious that $Q_n^*, \ n \in N_0, \ (N_0 = N \cup \{0\})$, will be the Riesz projectors of the adjoint operator L^*. It follows that $R(\lambda, L^*)\, \hat{g}$ will be an entire function in the entire λ - plane. On the other hand, based on estimate (32), the inequality

$$\left\| R(\lambda, L^*) \right\| \leq \frac{C_\delta}{|\lambda|^{\frac{1}{2}}}, \ \lambda \in \Omega_\delta, \ |\lambda| \geq r_0^2, \tag{33}$$

is true. Then, by the maximum principle, inequality (33) is satisfied in the entire λ-plane and $R(\lambda, L^*)\, \hat{g} \to 0$ as $|\lambda| \to \infty$, and by Liouville's theorem this means that an entire function $R(\lambda, L^*)\, \hat{g}$ is a constant function. Then differentiating this function and taking into account that $\frac{d}{d\lambda} R(\lambda, L^*) = R^2(\lambda, L^*)$ we obtain that $R^2(\lambda, L^*)\, \hat{g} = 0$. Since for all $\lambda \in \rho(L^*)$ the operator $R(\lambda, L^*)$ is single-valued, we obtain that $\hat{g} = 0$, which means that the root vectors of the operator L form a complete system in the space $L_p(0,1) \oplus C$. Theorem is proved.

From Theorem 4 it also follows that the system of eigenfunctions and associated functions of the spectral problem (1),(2) is overflowing in space $L_p(0,1)$, and in this system one function is superfluous. Therefore, we clarify the question of which function can be excluded from this system while maintaining the completeness and minimality properties. Let the system $\{\hat{z}_n\}_{n=0}^{\infty}$ be biorthogonal system to $\{\hat{y}_n\}_{n=0}^{\infty}$. It is a system of root vectors of the adjoint operator L^* moreover $\hat{z}_n = (z_n(x), \overline{a} z_n(0))$, where $z_n(x)$ is an eigenfunction or an associated function of the adjoint spectral problem (16),(17).

The following theorem is true.

Theorem 5. *The system $\{y_n(x)\}_{n=0, n \neq n_0}^{\infty}$, obtained from the system of eigen and associated functions $\{y_n(x)\}_{n=0, n \neq n_0}^{\infty}$, of the spectral problem (1),(2) after removing any eigenfunction $y_{n_0}(x)$, corresponding to a simple eigenvalue, is complete and minimal in the space $L_p(0,1)$, $1 < p < \infty$. In this case, the biorthogonal system has the form $\{\vartheta_n(x)\}_{n=0, n \neq n_0}^{\infty}$, where*

$$\vartheta_n(x) = z_n(x) - \frac{z_n(0)}{z_{n_0}(0)} z_{n_0}(x).$$

Proof. As follows from Theorem 1, a sufficient condition for the completeness and minimality of the system $\{y_n(x)\}_{n=0,n\neq n_0}^{\infty}$ is the condition $z_{n_0}(0) \neq 0$. For any simple eigenvalue λ_{n_0} this condition is satisfied, because, otherwise, we get that the function $z_{n_0}(x)$ is a solution to equation (16), satisfying the initial conditions $z_{n_0}(1) = 0$, $z'_{n_0}(1) = 0$, so this solution is trivial, i.e. $z_{n_0}(x) \equiv 0$, which contradicts the fact that it is an eigenfunction. Thus, the assertion of the theorem follows from Theorem 1.

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