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Uniform Convergence of Spectral Expansions for a Boundary Value Problem with a Boundary Condition Depending on the Spectral Parameter

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Abstract. In this paper, we consider the spectral problem for ordinary differential equations of fourth order with a spectral parameter contained in one of the boundary conditions. The uniform convergence of spectral expansions in terms of the system of eigenfunctions of this problem is studied.

Key Words and Phrases: spectral problem, eigenvalue, eigenfunction, Riesz basis, Fourier series

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1. Introduction

We consider the following eigenvalue problem

$$\ell(y)(x) \equiv y^{(4)}(x) - (q(x)y'(x))' = \lambda y(x),\ 0 < x < l, \tag{1.1}$$

$$y'(0)\cos\alpha - y''(0)\sin\alpha = 0, \tag{1.2a}$$

$$y(0)\cos\beta + Ty(0)\sin\beta = 0, \tag{1.2b}$$

$$(a\lambda + b)\,y'(l) + (c\lambda + d)\,y''(l) = 0, \tag{1.2c}$$

$$y(l)\cos\delta - Ty(l)\sin\delta = 0, \tag{1.2d}$$

where $\lambda \in \mathbb{C}$ is a spectral parameter, $Ty \equiv y''' - qy'$, q is a positive absolutely continuous function on $[0, l]$, α, β, δ, a, b, c, d are real constants such that $0 \le \alpha, \beta \le \pi/2$, $\pi/2 \le \delta < \pi$ (with the exception of the case $\beta = \delta = \pi/2$), $\sigma = bc - ad > 0$.

Note that problem (1.1), (1.2) for $\alpha = \beta = 0$ arises when describing small bending vibrations of an elastic cantilever homogeneous beam, in cross sections of which a longitudinal force acts, the left end of which is fixed, and a load is attached to the right end by means of a weightless rod, which is held in equilibrium by means of an elastic spring (see, e.g., [6, 17]).

The uniform convergence Fourier series expansions in the systems of root functions of Sturm-Liouville problems were studied in [7-9, 11-13, 15].

Problem (1.1), (1.2) in the case $\alpha = \beta = 0$ was studied in [1], where, in particular, it was proved that the eigenvalues of this problem are real and simple and form an infinitely increasing sequence. Moreover, the location of the eigenvalues on the real axis is studied, the oscillatory properties of the eigenfunctions are investigated, and the basis property in the space $L_p(0, l)$, $1 < p < \infty$, of the system of eigenfunctions of this problem with one arbitrary remote function is established.

The purpose of this paper is to study the uniform convergence of spectral expansions in terms of eigenfunctions of problem (1.1), (1.2).

2. Preliminary

Consider the boundary condition

$$y'(0)\cos\gamma + y''(0)\sin\gamma = 0, \tag{1.2c'}$$

where $\gamma \in \left[0, \frac{\pi}{2}]\right]$.

By following the argument in Theorem 5.2 of [4] we can prove that for each fixed α, β the eigenvalues of problem (1.1), (1.2a) (1.2b), (1.2c′), (1.2d) are real, simple and form infinitely increasing sequence $\{\lambda_k(\gamma, \delta)\}_{k=1}^{\infty}$ such that $\lambda_k(\gamma, \delta) > 0$ for $k \geq 2$, and for each γ there exists $\delta_0(\gamma) \in \left[\frac{\pi}{2}, \pi\right)$ such that $\lambda_1(\gamma, \delta) > 0$ for $\delta \in [0, \delta_0(\gamma))$, $\lambda_1(\gamma, \delta) = 0$ for $\delta = \delta_0(\gamma)$, $\lambda_1(\gamma, \delta) < 0$ for $\delta \in (\delta_0(\gamma), \pi)$. Moreover, the eigenfunction $y_{k,\gamma,\delta}(x)$, corresponding to the eigenvalue $\lambda_k(\gamma, \delta)$, for $k \geq 2$ has exactly $k - 1$ simple zeros, for $k = 1$ has no zeros if $\delta \in [0, \delta_0(\gamma)]$, has an arbitrary number of simple zeros in the interval $(0, 1)$ if $\delta \in (\delta_0(\gamma), \pi)$.

For the study of spectral properties of problem (1.1), (1.2) we consider solutions of the initial-boundary problem (1.1), (1.2a) (1.2b), (1.2d).

Theorem 2.1. *For every fixed $\lambda \in \mathbb{C}$ there exists a unique non-trivial solution $y(x, \lambda)$ of problem (1.1), (1.2a) (1.2b), (1.2d) up to a constant multiplier.*

The proof of this lemma is similar to that of [1, Lemma 2.3] (see also [10, Theorem 2.1]).

Remark 2.1. Let $y(x, \lambda)$ be the solution of (1.1), (1.2a) (1.2b), (1.2d) normalized by the condition $|y(0)| + |Ty(0)| = 1$ for $\lambda > 0$, and $|y'(l)| + |y''(l)| = 1$ for $\lambda \leq 0$. Since Eq. (1.1) depends linearly of the parameter λ, it follows from the general theory of ordinary differential equations (see, e.g., [16, Ch. I]) that for every fixed $x \in [0, l]$ the function $y(x, \lambda)$ is an entire function of the parameter λ.

Let $\alpha, \beta \in [0, \pi/2]$ and $\delta \in [\pi/2, \pi)$ be arbitrary fixed, and let $\mathcal{B}_k = (\lambda_{k-1}(0, \delta), \lambda_k(0, \delta))$, $k = 1, 2, \ldots$, where $\lambda_0(0, \delta) = -\infty$.

It is obvious that the eigenvalues $\lambda_k(0, \delta)$ and $\lambda_k(\pi/2, \delta)$, $k \in \mathbb{N}$, of problem (1.1), (1.2a) (1.2b), (1.2c′), (1.2d) for $\gamma = 0$ and $\gamma = \pi/2$ are zeros of entire functions $y'(l, \lambda)$ and $y''(l, \lambda)$ respectively. Note that the function $F(\lambda) = y''(l, \lambda)/y'(l, \lambda)$ is defined in

$\mathcal{B} \equiv \left(\bigcup_{k=1}^{\infty} \mathcal{B}_k \right) \cup (\mathbb{C} \backslash \mathbb{R})$ and is a meromorphic function of finite order, and the eigenvalues $\lambda_k(\pi/2, \delta)$ and $\lambda_k(0, \delta)$, $k \in \mathbb{N}$, are zeros and poles of this function respectively.

Lemma 3.1. *The following formula holds:*

$$\frac{dF(\lambda)}{d\lambda} = -\frac{1}{y'^2(l, \lambda)} \int_0^l y^2(x, \lambda) dx, \ \lambda \in \mathcal{B}. \tag{2.1}$$

The proof of this lemma literally repeats the proof of [11, formula (30)].

Lemma 2.2 *The following limit relation holds:*

$$\lim_{\lambda \to -\infty} F(\lambda) = +\infty. \tag{2.2}$$

The proof of this lemma is similar to that of [1, Lemma 2.8].

In view of [5, Property 1], by (2.1) and (2.2) we get

$$\lambda_1(\pi/2, \delta) < \lambda_1(0, \delta) < \lambda_2(\pi/2, \delta) < \lambda_2(0, \delta) < \ \ldots \ . \tag{2.3}$$

Let $m(\lambda) = ay'(l, \lambda) + cy''(l, \lambda)$.

Remark 2.2. If follows from boundary condition (1.2c) that if λ is an eigenvalue of problem (1.1), (1.2), then $m(\lambda) \neq 0$.

We denote by $s(\lambda)$, $\lambda \in \mathbb{R}$, the number of zeros of the function $y(x, \lambda)$ contained in the interval $(0, l)$.

By following the arguments in Lemma 2.11 from [1] we verify that the following oscillation theorem is valid for the function $y(x, \lambda)$.

Lemma 2.3. *If $\lambda \in (\lambda_{k-1}(0, \delta), \lambda_k(\pi/2, \delta))$ for $k \geq 3$, then $k-2 \leq s(\lambda) \leq k-1$, and if $\lambda \in [\lambda_k(\pi/2, \delta), \lambda_k(0, \delta)]$ for $k \geq 3$, then $s(\lambda) = k-1$. Moreover, if $\delta \in [\pi/2, \delta_0(0)]$, then $s(\lambda) = 0$ for $\lambda \in [0, \lambda_1(0, \delta)]$, $0 \leq s(\lambda) \leq 1$ for $\lambda \in (\lambda_1(0, \delta), \lambda_2(\pi/2, \delta))$ and $s(\lambda) = 1$ for $\lambda \in [\lambda_2(\pi/2, \delta), \lambda_2(0, \delta)]$, if $\delta \in [\delta_0(0), \delta_0(\pi/2))$, then $0 \leq s(\lambda) \leq 1$ for $\lambda \in [0, \lambda_2(\pi/2, \delta)$, $s(\lambda) = 1$ for $\lambda \in [\lambda_2(\pi/2, \delta), \lambda_2(0, \delta)]$, and if $\delta \in [\delta_0(\pi/2, \pi)$, then $s(\lambda) = 1$ for $\lambda \in [0, \lambda_2(0, \delta)]$.*

3. The properties of eigenvalues and eigenfunctions of problem (1.1), (1.2).

We introduce the following boundary condition

$$ay'(l) + cy''(l) = 0. \tag{1.2c''}$$

Note that, boundary condition (1.2c$''$) in the case $a = 0$ ($c = 0$) coincides with condition (1.2c$'$) for $\gamma = \pi/2$ ($\gamma = 0$). By [2, p. 768] the eigenvalues of problem (1.1), (1.2a) (1.2b), (1.2c$''$), (1.2d) for each fixed α, β and for $ac \neq 0$ are real, simple and form infinitely increasing sequence $\{\tau_k(\delta)\}_{k=1}^{\infty}$ such that for any fixed α, β and δ the relations hold

$$\lambda_1(\pi/2,\delta) < \tau_1(\delta) < \lambda_1(0,\delta) < \lambda_1(\pi/2,\delta) < \tau_2(\delta) < \lambda_1(0,\delta) < \ldots \qquad (3.1a)$$

in the case $a/c > 0$, and

$$\tau_1(\delta) < \lambda_1(\pi/2,\delta) < \lambda_1(0,\delta) < \tau_2(\delta) < \lambda_2(\pi/2,\delta) < \lambda_2(0,\delta) < \ldots \qquad (3.1b)$$

in the case $a/c < 0$.

For $a \neq 0$ ($c \neq 0$) we define the number k_a (k_c) from the inequality

$$\lambda_{k_a-1} \leq -b/a < \lambda_{k_a} \, (\lambda_{k_c-1} < -d/c \leq \lambda_{k_c}).$$

Remark 3.1. If $ac \neq 0$, then $k_a \leq k_c + 1$ for $ac > 0$, and $k_a \geq k_c$ for $ac < 0$.

Theorem 3.1. *The eigenvalues of problem (1.1), (1.2) are real and simple and form an infinitely increasing sequence* $\{\lambda_k\}_{k=1}^{\infty}$ *such* $\lambda_k > 0$ *for* $k \geq 3 + \operatorname{sgn}|c|$. *Moreover, for* $k > k_1 = \max\{k_a, k_c\} + 2$, *the eigenvalues have the following arrangement on the real axis:*

$$\lambda_{k-1}(0,\delta) < \lambda_k < \lambda_k(\pi/2,\delta) < \lambda_k(0,\delta), \ \textit{if} \ \ c = 0, \qquad (3.2a)$$

$$\lambda_{k-2}(0,\delta) < \tau_{k-1}(\delta) = \lambda_{k-1}(\pi/2,\delta) < \lambda_k < \lambda_{k-1}(0,\delta), \ \textit{if} \ \ a = 0, \qquad (3.2b)$$

$$\lambda_{k-2}(0,\delta) < \lambda_{k-1}(\pi/2,\delta) < \tau_{k-1}(\delta) < \lambda_k < \lambda_{k-1}(0,\delta), \ \textit{if} \ \ ac > 0, \qquad (3.2c)$$

$$\lambda_{k-2}(0,\delta) < \tau_{k-1}(\delta) < \lambda_k < \lambda_{k-1}(\pi/2,\delta) < \lambda_{k-1}(0,\delta), \ \textit{if} \ \ ac < 0. \qquad (3.2d)$$

Remark 3.2. Using Lemma 2.3 from Theorem 3.1 one can obtain the oscillatory properties of eigenfunctions corresponding to all positive eigenvalues. For example, if $c = 0$, then the function $y_k(x)$ ($k \geq 1$ for $\delta \leq \delta_0(\pi/2)$ and $k_a \geq 2$; $k \geq 2$ for $\delta \leq \delta_0(\pi/2)$ and $k_a = 1$, for $\delta_0(\pi/2) < \delta \leq \delta_0(0)$ and for $\delta > \delta_0(0)$ and $k_a \geq 3$; $k \geq 3$ for $\delta > \delta_0(0)$ and $k_a \leq 2$) has exactly $k-1$ simple zeros for $k < k_a$, has either $k-2$ or $k-1$ simple zeros in the interval $(0,1)$ for $k \geq k_a$.

4. Asymptotic formulas for eigenvalues and eigenfunctions of problems (1.1), (1.2a) (1.2b), (1.2c$''$), (1.2d) with $q \equiv 0$ and (1.1), (1.2)

Lemma 4.1. *Let* $q \equiv 0$ *in Eq. (1.1). Then the following asymptotic formulas for the eigenvalues and eigenfunctions of problem (1.1), (1.2a) (1.2b), (1.2c$''$), (1.2d) with* $q \equiv 0$ *are valid:*

$$\sqrt[4]{\tau_k}(\delta) = \left(k - \frac{1+3\,\mathrm{sgn}\beta}{4}\right)\frac{\pi}{l} + O\left(\frac{1}{k^2}\right), \ \textit{if}\ \alpha = 0,\ c = 0, \qquad (4.1a)$$

$$\sqrt[4]{\tau_k}(\delta) = \left(k - \frac{2+3\,\mathrm{sgn}\beta}{4}\right)\frac{\pi}{l} + \frac{(1+\mathrm{sgn}\beta)\cot\alpha}{2k\pi} + O\left(\frac{1}{k^2}\right), \ \textit{if}\ \alpha \in (0,\pi/2],\ c = 0, \qquad (4.1b)$$

$$\sqrt[4]{\tau_k}(\delta) = \left(k - \frac{2+3\,\mathrm{sgn}\beta}{4}\right)\frac{\pi}{l} + \frac{a/c}{k\pi} + O\left(\frac{1}{k^2}\right),\ \ if\ \alpha = 0,\ c \neq 0, \tag{4.1c}$$

$$\sqrt[4]{\tau_k}(\delta) = \left(k - \tfrac{3(1+\mathrm{sgn}\beta)}{4}\right)\tfrac{\pi}{l} + \tfrac{2a/c+(1+\mathrm{sgn}\beta)\cot\alpha}{2k\pi} + O\left(\tfrac{1}{k^2}\right), \quad if\ \ \alpha \in (0, \pi/2]\,,\ c \neq 0, \tag{4.1d}$$

$$v_{k,\,\delta}(x) = \sqrt{\tfrac{1+\mathrm{sgn}\beta}{l}}\left\{(1-\mathrm{sgn}\,\beta)\sin\sqrt[4]{\tau_k}\,x - (-1)^{\mathrm{sgn}\,\beta}\cos\sqrt[4]{\tau_k}\,x + (1-\mathrm{sgn}\,\beta)\,e^{-\sqrt[4]{\tau_k}\,x} + O\left(\tfrac{1}{k^2}\right)\right\},\ \ if\ \alpha = 0,\ c = 0, \tag{4.2a}$$

$$v_{k,\,\delta}(x) = \sqrt{\tfrac{2-\,\mathrm{sgn}\beta}{l}}\left\{\sin\sqrt[4]{\tau_k}\,x - \mathrm{sgn}\,\beta\cdot\cos\sqrt[4]{\tau_k}\,x - \mathrm{sgn}\,\beta\ \cdot e^{-\sqrt[4]{\tau_k}\,x} + \mathrm{sgn}\beta\,\tfrac{\cot\alpha}{(2-\mathrm{sgn}\beta)\sqrt[4]{\tau_k}}\sin\sqrt[4]{\tau_k}\,x - (1+\mathrm{sgn}\beta)\tfrac{\cot\alpha}{2\sqrt[4]{\tau_k}}\cos\sqrt[4]{\tau_k}\,x + (1+\mathrm{sgn}\beta)\,\tfrac{\cot\alpha}{2\sqrt[4]{\tau_k}}\,e^{-\sqrt[4]{\tau_k}\,x} + O\left(\tfrac{1}{k^2}\right)\right\},\ \ if\ \alpha \in (0, \pi/2]\,,\ c = 0, \tag{4.2b}$$

$$v_{k,\,\delta}(x) = \sqrt{\tfrac{1+\mathrm{sgn}\beta}{l}}\left\{(1-\mathrm{sgn}\beta)\sin\sqrt[4]{\tau_k}\,x - (-1)^{\mathrm{sgn}\,\beta}\cos\sqrt[4]{\tau_k}\,x + (1-\mathrm{sgn}\,\beta)\,e^{-\sqrt[4]{\tau_k}\,x} + (-1)^{k+1}\left(\tfrac{\sqrt{2}}{2}\right)^{\mathrm{sgn}\beta} e^{\sqrt[4]{\tau_k}\,(x-l)} + (-1)^{k+1}\left(\tfrac{\sqrt{2}}{2}\right)^{\mathrm{sgn}\beta}\tfrac{a/c}{\rho_k}\,e^{\sqrt[4]{\tau_k}\,(x-l)} + O\left(\tfrac{1}{k^2}\right)\right\},\ \ if\ \alpha = 0,\ c \neq 0, \tag{4.2c}$$

$$v_{k,\,\delta}(x) = \sqrt{\tfrac{2-\,\mathrm{sgn}\beta}{l}}\left\{\sin\sqrt[4]{\tau_k}\,x - \mathrm{sgn}\,\beta\cdot\cos\sqrt[4]{\tau_k}\,x - \mathrm{sgn}\,\beta\cdot e^{-\sqrt[4]{\tau_k}\,x} + (-1)^{k+1-\,\mathrm{sgn}\,\beta}\left(\tfrac{\sqrt{2}}{2}\right)^{1-\,\mathrm{sgn}\beta} e^{\sqrt[4]{\tau_k}\,(x-l)} - \mathrm{sgn}\beta\cdot\tfrac{\cot\alpha}{\rho_k}\sin\sqrt[4]{\tau_k}\,x - \tfrac{\cot\alpha}{(2-\mathrm{sgn}\beta)\rho_k}\cos\sqrt[4]{\tau_k}\,x + \tfrac{\cot\alpha}{(2-\mathrm{sgn}\beta)\rho_k}\,e^{-\sqrt[4]{\tau_k}\,x} + (-1)^{k+\mathrm{sgn}\beta}\left(\tfrac{\sqrt{2}}{2}\right)^{1-\mathrm{sgn}\beta}\tfrac{a/c}{\rho_k}e^{\rho_k(x-l)} + O\left(\tfrac{1}{k^2}\right)\right\},\ \ if\ \alpha \in \left(0, \tfrac{\pi}{2}\right],\ c \neq 0, \tag{4.2d}$$

where relations (4.2a)-(4.2d) hold uniformly for $x \in [0, 1]$.

The proof of this lemma is similar to that of [3, Lemma 3.1].

By (4.1) from (4.2) by direct calculations we obtain

$$||v_{k,\,\delta}||_2^2 = 1 + O\left(k^{-2}\right), \tag{4.3}$$

where $||\cdot||_2$ is the norm in $L_2(0, l)$.

We denote by $\Psi_k(x)$, $k \in \mathbb{N}$ the normalized eigenfunction, corresponding to the eigenvalue τ_k of problem (1.1), (1.2a) (1.2b), (1.2c$'$), (1.2d) with $q \equiv 0$, i.e. $\Psi_k(x) = \frac{v_{k,\,\delta}(x)}{||v_{k,\,\delta}||_2}$. Then by (4.3) for $\Psi_k(x)$ the asymptotic formulas (4.2a)-(4.2d) are valid.

The function $q_0(x)$, $x \in [0, l]$, and the number q_0 we define as follows:

$$q_0(x) = \int_0^x q(t)dt, \; q_0 = \int_0^l q(t)dt.$$

Lemma 4.2. *For the eigenvalues and eigenfunctions of problem (1.1), (1.2) we have the following asymptotic formulas:*

$$\sqrt[4]{\lambda_k} = \left(k - \frac{5 + 3\,\mathrm{sgn}\beta}{4}\right)\frac{\pi}{l} + \frac{q_0}{4k\pi} + O\left(\frac{1}{k^2}\right), \; \textit{if } \alpha = 0, \; c = 0, \tag{4.4a}$$

$$\sqrt[4]{\lambda_k} = \left(k - \tfrac{6+3\,\mathrm{sgn}\beta}{4}\right)\tfrac{\pi}{l} + \tfrac{q_0 + 2(1+\mathrm{sgn}\beta)\cot\alpha}{4k\pi} + O\left(\tfrac{1}{k^2}\right), \quad \textit{if } \alpha \in (0, \pi/2], \; c = 0, \tag{4.4b}$$

$$\sqrt[4]{\lambda_k} = \left(k - \frac{6 + 3\,\mathrm{sgn}\beta}{4}\right)\frac{\pi}{l} + \frac{q_0 + 4a/c}{4k\pi} + O\left(\frac{1}{k^2}\right), \; \textit{if } \alpha = 0, \; c \neq 0, \tag{4.4c}$$

$$\sqrt[4]{\lambda_k} = \left(k - \tfrac{7+3\,\mathrm{sgn}\beta)}{4}\right)\tfrac{\pi}{l} + \tfrac{q_0 + 4a/c + 2(1+\mathrm{sgn}\beta)\cot\alpha}{4k\pi} + O\left(\tfrac{1}{k^2}\right), \quad \textit{if } \alpha \in (0, \pi/2], \; c \neq 0, \tag{4.4d}$$

$$y_k(x) = \sqrt{\tfrac{1+\mathrm{sgn}\,\beta}{l}}\left\{(1 - \mathrm{sgn}\,\beta)\sin\sqrt[4]{\lambda_k}x - (-1)^{\mathrm{sgn}\beta}\cos\sqrt[4]{\lambda_k}x + (1 - \mathrm{sgn}\,\beta)\,e^{-\sqrt[4]{\lambda_k}x} + (-1)^{\mathrm{sgn}\,\beta}\tfrac{(1-\mathrm{sgn}\,\beta)q_0 - q_0(x)}{4\varrho_k}\sin\sqrt[4]{\lambda_k}x - (-1)^{\mathrm{sgn}\,\beta}\tfrac{q_0 + (1-\mathrm{sgn}\,\beta)q_0(x)}{4\varrho_k}\cos\sqrt[4]{\lambda_k}x + (1 - \mathrm{sgn}\,\beta)\tfrac{q_0 - q_0(x)}{4\varrho_k}e^{\sqrt[4]{\lambda_k}x} + O\left(\tfrac{1}{k^2}\right)\right\}, \quad \textit{if } \alpha = 0, \; c = 0, \tag{4.5a}$$

$$y_k(x) = \sqrt{\tfrac{2-\mathrm{sgn}\beta}{l}}\left\{\sin\sqrt[4]{\lambda_k}x - \mathrm{sgn}\,\beta\cdot\cos\sqrt[4]{\lambda_k}x - \mathrm{sgn}\,\beta\cdot e^{-\sqrt[4]{\lambda_k}x} - \mathrm{sgn}\,\beta\,\tfrac{q_0(x)+4\cot\alpha}{4\rho_k}\sin\sqrt[4]{\lambda_k}x - \tfrac{q_0(x)+2(1+\mathrm{sgn}\,\beta)\cot\alpha}{4\rho_k}\cos\sqrt[4]{\lambda_k}x + \tfrac{\mathrm{sgn}\,\beta\cdot q_0(x)+2(1+\mathrm{sgn}\,\beta)\cot\alpha}{4\rho_k}e^{-\sqrt[4]{\lambda_k}x} + O\left(\tfrac{1}{k^2}\right)\right\}, \; \textit{if } \alpha \in (0, \pi/2], \; c = 0, \tag{4.5b}$$

$$y_k(x) = \sqrt{\tfrac{1+\text{sgn}\beta}{l}} \left\{ (1-\text{sgn}\,\beta)\sin\sqrt[4]{\lambda_k}x - (-1)^{\text{sgn}\,\beta}\cos\sqrt[4]{\lambda_k}x + \right.$$

$$(1-\text{sgn}\,\beta)\, e^{-\sqrt[4]{\lambda_k}x} + (-1)^{k+\text{sgn}\beta}\left(\tfrac{\sqrt{2}}{2}\right)^{\text{sgn}\beta} e^{\sqrt[4]{\lambda_k}(x-l)} +$$

$$(-1)^{\text{sgn}\,\beta}\frac{(1-\text{sgn}\,\beta)(q_0+4a/c)-q_0(x)}{4\varrho_k}\sin\sqrt[4]{\lambda_k}x - \qquad (4.5c)$$

$$(-1)^{\text{sgn}\,\beta}\frac{q_0+4a/c+(1-\text{sgn}\,\beta)\,q_0(x)}{4\varrho_k}\cos\sqrt[4]{\lambda_k}x + (1-\text{sgn}\,\beta)\frac{q_0+4a/c-q_0(x)}{4\varrho_k}e^{-\sqrt[4]{\lambda_k}x} +$$

$$\left.(-1)^{k+\text{sgn}\beta}\left(\tfrac{\sqrt{2}}{2}\right)^{\text{sgn}\beta}\frac{q_0(x)}{4\varrho_k}e^{\sqrt[4]{\lambda_k}(x-l)} + O\left(\tfrac{1}{k^2}\right)\right\}, \ \textit{if } \alpha = 0,\ c \neq 0,$$

$$y_k(x) = \sqrt{\tfrac{2-\text{sgn}\beta}{l}}\left\{\sin\sqrt[4]{\lambda_k}x - \text{sgn}\,\beta\cdot\cos\sqrt[4]{\lambda_k}x - \text{sgn}\,\beta\,\cdot\,e^{-\sqrt[4]{\lambda_k}x} + \right.$$

$$(-1)^{k+\text{sgn}\beta}\left(\tfrac{\sqrt{2}}{2}\right)^{\text{sgn}\beta} e^{\sqrt[4]{\lambda_k}(x-l)} - \text{sgn}\,\beta\,\frac{4\cot\alpha+q_0(x)}{4\varrho_k}\sin\sqrt[4]{\lambda_k}x -$$

$$\frac{q_0(x)+2(1+\text{sgn}\,\beta)\cot\alpha}{4\varrho_k}\cos\sqrt[4]{\lambda_k}x + \frac{\text{sgn}\beta\cdot q_0(x)+4\cot\alpha}{4\varrho_k}e^{-\sqrt[4]{\lambda_k}x} \qquad (4.5d)$$

$$+(-1)^{k+\text{sgn}\beta}\frac{q_0(x)-q_0+4a/c}{4\varrho_k}e^{\sqrt[4]{\lambda_k}(x-l)} + O\left(\tfrac{1}{k^2}\right), \ \textit{if } \alpha\in(0,\pi/2],\ c\neq 0,$$

where relations (4.5a)-(4.5d) hold uniformly for $x \in [0, 1]$.

The proof of this lemma is similar to that of [3, Lemma 3.2].

5. Uniform convergence of expansions in Fourier series of subsystems of eigenfunctions of problem (1.1), (1.2)

Let

$$\delta_k = ||y_k||_2^2 + \sigma^{-1}m_k^2. \qquad (5.1)$$

Since $\sigma > 0$ and $m_k \neq 0$ it follows from (5.1) that

$$\delta_k > 0,\ k\in\mathbb{N}. \qquad (5.2)$$

Theorem 5.1. *Let* r *be the any fixed positive integer. Then the system* $\{y_k(x)\}_{k=1,\,k\neq r}^{\infty}$ *of eigenfunctions of problem (1.1), (1.2) forms a basis in* $L_p(0,l)$, $1 < p < \infty$, *and for* $p = 2$ *this basis is a Riesz basis. The system* $\{u_k(x)\}_{k=1,\,k\neq r}^{\infty}$, *conjugate to the system* $\{y_k(x)\}_{k=1,\,k\neq r}^{\infty}$, *is defined by the equality:*

$$u_k(x) = \delta_k^{-1}\left\{y_k(x) - m_k m_r^{-1} y_r(x)\right\},\ k\in\mathbb{N},\ k\neq r, \qquad (5.3)$$

The proof of this theorem repeats the proof of Theorem 4.1 of [1].

If r is an arbitrary fixed natural number, then by Theorem 5.1 the Fourier series expansion

$$f(x)=\sum_{k=1,\,k\neq r}^{\infty}(f,u_k)\,y_k(x), \tag{5.4}$$

in the system $\{y_k(x)\}_{k=1,\,k\neq r}^{\infty}$ of any continuous function $f(x)$ on $[0,1]$ converges in $L_p\,(0,l)$, $1<p<\infty$, and converges unconditionally for $p=2$.

The main result of this paper is the following theorem.

Theorem 5.1. *Let r be the an arbitrary fixed positive integer, $f(x)$ is continuous function on the interval $[0,l]$ and has uniformly convergent on $[0,1]$ Fourier series in the system $\{\Psi_k(x)\}_{k=1}^{\infty}$. Then the series (5.4) converges uniformly on $[0,1]$.*

Proof. If $\alpha=\beta=0$ and $c=0$ in boundary conditions $(1.2a)$-$(1.2c)$ and $(1.2c'')$, then it follows from $(4.1a)$ and $(4.2a)$ that for eigenvalues and eigenfunctions of problem (1.1), (1.2a), (1.2b), (1.2c$''$), (1.2d) with $q\equiv 0$ the following asymptotic formulas hold:

$$\sqrt[4]{\tau_k(\delta)}=\left(k-\frac{1}{4}\right)\frac{\pi}{l}+O\left(\frac{1}{k^2}\right), \tag{5.5}$$

$$\Psi_k(x)=\sqrt{\tfrac{1}{l}}\left\{\sin\left(k-\tfrac{1}{4}\right)\tfrac{\pi}{l}x-\cos\left(k-\tfrac{1}{4}\right)\tfrac{\pi}{l}x+\,e^{-\left(k-\frac{1}{4}\right)\frac{\pi}{l}\,x}+O\left(\tfrac{1}{k^2}\right)\right\}, \tag{5.6}$$

where (5.6) holds uniformly for $x\in[0,l]$.

It follows from (4.4a) and (4.5a) that for eigenvalues and eigenfunctions of problem (1.1), (1.2) with $\alpha=\beta=0$ and $c=0$ the asymptotic formulas are valid:

$$\sqrt[4]{\lambda_k}=\left(k-\frac{5}{4}\right)\frac{\pi}{l}+\frac{q_0}{4k\pi}+O\left(\frac{1}{k^2}\right), \tag{5.7}$$

$$\begin{gathered}y_k(x)=\sqrt{\tfrac{1}{l}}\left\{\sin\left(k-\tfrac{5}{4}\right)\tfrac{\pi}{l}\,x-\cos\left(k-\tfrac{5}{4}\right)\tfrac{\pi}{l}\,x+\,e^{-\left(k-\frac{5}{4}\right)\frac{\pi}{l}\,x}+\right.\\ \tfrac{(q_0-\,q_0(x))l+q_0x}{4k\pi}\sin\left(k-\tfrac{5}{4}\right)\tfrac{\pi}{l}\,x-\tfrac{(q_0+q_0(x))l-q_0x}{4k\pi}\cos\left(k-\tfrac{5}{4}\right)\tfrac{\pi}{l}\,x+\\ \left.\tfrac{(q_0-q_0(x))l-q_0x}{4k\pi}\,e^{-\,\left(k-\frac{5}{4}\right)\frac{\pi}{l}\,x}+O\left(\tfrac{1}{k^2}\right)\right\},\end{gathered} \tag{5.8}$$

where (5.8) holds uniformly for $x\in[0,l]$.

By asymptotic formulas (5.6) and (5.8) we have

$$\begin{gathered}y_k(x)=\Psi_{k-1}(x)+\sqrt{\tfrac{1}{l}}\left\{\tfrac{(q_0-\,q_0(x))l+q_0x}{4k\pi}\sin\left(k-\tfrac{5}{4}\right)\tfrac{\pi}{l}\,x-\tfrac{(q_0+q_0(x))l-q_0x}{4k\pi}\times\right.\\ \left.\cos\left(k-\tfrac{5}{4}\right)\tfrac{\pi}{l}\,x+\tfrac{(q_0-q_0(x))l-q_0x}{4k\pi}\,e^{-\,\left(k-\frac{5}{4}\right)\frac{\pi}{l}\,x}+O\left(\tfrac{1}{k^2}\right)\right\}.\end{gathered} \tag{5.9}$$

In view of (5.8) we get

$$y_k'(l)=(-1)^k\sqrt{2}\sqrt{\frac{1}{l}}\left(\frac{q_0}{4}+O\left(\frac{1}{k}\right)\right),$$

$$y_k''(l) = (-1)^k \sqrt{2} \sqrt{\frac{1}{l}} \frac{k^2\pi^2}{l^2} \left(1 + \frac{q_0 l - 10\pi}{4k\pi} + O\left(\frac{1}{k^2}\right)\right),$$

which implies that

$$m_k = a y_k'(l) + c y_k''(l) = -\frac{b y_k'(l) + d y_k''(l)}{\lambda_k} = O\left(\frac{1}{k^2}\right). \tag{5.10}$$

Direct calculations show that

$$||y_k||_2^2 = 1 + O\left(\frac{1}{k^2}\right). \tag{5.11}$$

Then by (5.10) and (5.11) it follows from (5.1) that

$$\delta_k = ||y_k||_2^2 + \sigma^{-1} m_k^2 = 1 + O\left(\frac{1}{k^2}\right). \tag{5.12}$$

Let r be the any fixed positive integer. By (5.10)-(5.12), from (5.3) we get

$$u_k(x) = \delta_k^{-1} \left\{ y_k(x) - m_k \, m_r{}^{-1} \, y_r(x) \right\} = y_k(x) + O\left(\frac{1}{k^2}\right). \tag{5.13}$$

Note that for uniformly convergence of series (5.4) it is necessary and sufficient uniform convergence of the series

$$\sum_{k=r+1}^{\infty} (f, u_k) y_k(x). \tag{5.14}$$

By (5.13) we have

$$\sum_{k=r+1}^{\infty} (f, u_k) y_k(x) = \sum_{k=r+1}^{\infty} (f, y_k) y_k(x) + \sum_{k=r+1}^{\infty} O\left(\frac{1}{k^2}\right). \tag{5.15}$$

it follows from (5.9) that

$$y_k(x) = \Phi_{k-1}(x) + O\left(\frac{1}{k}\right). \tag{5.16}$$

According to (5.16) we have

$$\sum_{k=r+1}^{\infty} (f, y_k) y_k(x) = \sum_{k=r+1}^{\infty} (f, y_k) \Phi_{k-1}(x) + \sum_{k=r+1}^{\infty} (f, y_k) O\left(\tfrac{1}{k}\right). \tag{5.17}$$

Since $\{y_k(x)\}_{k=1,\, k\neq r}^{\infty}$ is a Riesz basis in $L_2(0,l)$ the following estimate holds

$$\sum_{k=l+1}^{\infty} \left|(f, y_k) O\left(\tfrac{1}{k}\right)\right| \leq \text{const} \left\{ \sum_{k=l+1}^{\infty} |(f, y_k)|^2 + \sum_{k=l+1}^{\infty} \tfrac{1}{k^2} \right\} < +\infty.$$

Hence to study the uniform convergence of the series (5.14), it suffices to study the uniform convergence of the series

$$\sum_{k=r+1}^{\infty} (f, y_k)\Psi_{k-1}(x) \tag{5.18}$$

Let

$$p_1(x) = \sqrt{\frac{1}{l}}\,\frac{(q_0 - q_0(x))l + q_0 x}{4\pi},\ p_2(x) = \sqrt{\frac{1}{l}}\,\frac{(q_0 + q_0(x))l - q_0 x}{4\pi},$$

$$p_3(x) = \sqrt{\frac{1}{l}}\,\frac{(q_0 - q_0(x))l - q_0 x}{4\pi}, e_{k,1}(x) = \sin\left(k - \frac{5}{4}\right)\frac{\pi}{l}\,x,\ e_{k,2}(x) = \cos\left(k - \frac{5}{4}\right)\frac{\pi}{l}\,x,$$

$$e_{k,3}(x) = e^{-\left(k - \frac{5}{4}\right)\frac{\pi}{l}x},\ x \in [0, l].$$

Then by (5.9) we get

$$y_k(x) = \Psi_{k-1}(x) + k^{-1}p_1(x)e_{k,1}(x) + k^{-1}p_2(x)\,e_{k,2}(x) + k^{-1}p_3(x)\,e_{k,3}(x) + O\left(\frac{1}{k^2}\right),$$

whence implies that

$$\sum_{k=r+1}^{\infty} (f, y_k)\Phi_{k-1}(x) = \sum_{k=r+1}^{\infty} (f, \Phi_{k-1})\,\Phi_{k-1}(x) +$$

$$\sum_{k=r+1}^{\infty} k^{-1}(fp_1, e_{k,1})\,\Phi_{k-1}(x) + \sum_{k=r+1}^{\infty} k^{-1}(fp_2, e_{k,2})\,\Phi_{k-1}(x) +$$

$$\sum_{k=r+1}^{\infty} k^{-1}(fp_3, e_{k,3})\,\Phi_{k-1}(x) + \sum_{k=r+1}^{\infty} O\left(k^{-2}\right)\Phi_{k-1}(x). \tag{5.19}$$

By virtue of [14, Lemma 5] each of the systems $\{e_{k,j}\}_{k=1}^{\infty}$, $j = 1,\ 2.\ 3$, is a Bessel system. Therefore, we have the following estimates

$$\sum_{k=l+1}^{\infty} \left|\frac{(fp_j, e_{k,\,j})}{k}\right| \leq \text{const}\left(\sum_{k=l+1}^{\infty}\frac{1}{k^2} + \sum_{k=l+1}^{\infty} |(fp_j, e_{k,\,j})|^2\right) \leq \text{const}\,(1+||f||_2^2),\ j = 1,\ 2,\ 3.$$

By virtue of the condition of this theorem the series $\sum\limits_{k=r+1}^{\infty} (f, \Phi_{k-1})\,\Phi_{k-1}(x)$ converges uniformly on the interval $[0, 1]$. Then, as seen from (5.19), the series (5.18) converges uniformly on $[0, 1]$.

The rest cases are treated in a similar way. The proof of this theorem is complete.

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