Caspian Journal of Applied Mathematics, Ecology and Economics
V. 10, No 1, 2022, July
ISSN 1560-4055

On the Basicity of Eigenfunctions of a Non-self-adjoint Spectral Problem with a Spectral Parameter in the Boundary Condition in Lebesgue Spaces

T.B. Gasymov*, U.G. Hashimova, R.J. Taghiyeva

Abstract. In this work we consider the following spectral problem

$$-y'' + q\,(x)\,y = \lambda y, \quad x \in (0,1)\,,$$

$$\left.\begin{array}{c} y(0) = 0 \\ y'(0) = (a\lambda + b)y(1) \end{array}\right\},$$

where $q(x)$ is a complex-valued summable function, λ is a spectral parameter, a and b are arbitrary complex numbers ($a \neq 0$). We prove theorems on the basicity of eigenfunctions and associated functions of the spectral problem in the Lebesgue spaces $L_p(0,1) \oplus C$ and $L_p\,(0,1)\,, 1 < p < \infty$, as well as in their weighted analogs with a general weight function satisfying the Mackenhaupt condition.

Key Words and Phrases: eigenvalues, eigenunctions, complete and minimal system, basicity.

2010 Mathematics Subject Classifications: 34B05, 34B24, 34L10, 34L20

---

1. Introduction

Consider the following spectral problem:

$$-y^{''} + q\,(x)\,y = \lambda y, \quad x \in (0,1)\,, \tag{1}$$

$$\left.\begin{array}{c} y(0) = 0, \\ y'(0) = (a\lambda + b)y(1), \end{array}\right\} \tag{2}$$

where $q(x)$ is a complex-valued summable function, λ is a spectral parameter, a and b are arbitrary complex numbers ($a \neq 0$). In this work it is proved the theorems on the basicity of eigenfunctions and associated functions of the spectral problem in the Lebesgue spaces $L_p(0,1) \oplus C$ and $L_p\,(0,1)\,, 1 < p < \infty$, as well as in their weighted analogs with a general weight function satisfying the Mackenhaupt condition. Numerous works are devoted to

*Corresponding author.

spectral problems for ordinary differential operators with a spectral parameter in boundary conditions (see, e.g., [1-16]). Of the latter, let us note the works [17-26]. The works [8,9,14,25,26,27,28] are directly related to our work. The case $q(x) \equiv 0,\ b = 0$ is considered in [8,9]. The case $b = 0$, was considered in [14], and other generalizations of boundary conditions (2) were considered in [25,26]. Note that the theorems on the basicity in $L_p(0,1)$ under the additional assumption $q(x) = q(1-x)$, and the theorems on the uniform convergence of spectral expansions for the potential $q(x)$ from class $L_2(0,1)$ were proved in [14,25,26]. In [27], asymptotic formulas for the eigenvalues (1),(2) and eigenfunctions were found, and in [28], theorems on the completeness and minimality of the root functions of problem (1),(2) in the Lebesgue spaces $L_p(0,1) \oplus C$ and $L_p(0,1)$, $1 < p < \infty$ were proved.

2. Needed information and preliminary results

In obtaining the main results, we need some concepts and facts from the theory of bases in a Banach space.

Definition 1. *A basis $\{u_n\}_{n\in N}$ of a space X is called a p-basis, if for any $x \in X$ the condition*

$$\left(\sum_{n=1}^{\infty} |\langle x, \vartheta_n\rangle|^p\right)^{\frac{1}{p}} \leq M\,\|x\|,$$

is fulfilled, where $\{\vartheta_n\}_{n\in N}$ is a biorthogonal system to $\{u_n\}_{n\in N}$.

Definition 2. *Sequences $\{u_n\}_{n\in N}$ and $\{\phi_n\}_{n\in N}$ of a Banach space X are said to be p close if the condition*

$$\sum_{n=1}^{\infty} \|u_n - \phi_n\|^p < \infty,$$

is fulfilled.

Let us recall that two systems in a Banach space are said to be isomorphic (or equivalent) if there exists a bounded linear operator in this space with a bounded inverse that maps one of these systems to the other. We will also use the following result from [29], which is a Banach analogue of the well-known theorem of N.K. Bari [30].

Theorem 1. *([29]) Let $\{x_n\}_{n\in N}$ be a q-basis of a Banach space X, and let the system $\{y_n\}_{n\in N}$ be a p-close to $\{x_n\}_{n\in N}$, where $\frac{1}{p} + \frac{1}{q} = 1$. Then the following properties are equivalent:*

a) $\{y_n\}_{n\in N}$ is complete in X;
b) $\{y_n\}_{n\in N}$ is minimal in X;
c) $\{y_n\}_{n\in N}$ is $\omega-$ linearly independent in X;
d) $\{y_n\}_{n\in N}$ forms a basis for X;
e) $\{y_n\}_{n\in N}$ forms a basis in X, isomorphic to the system $\{x_n\}_{n\in N}$;
f) $\{y_n\}_{n\in N}$ forms a q-basis for X.

Let $X_1 = X \oplus C^m$ and $\{\hat{u}_n\}_{n\in N} \subset X_1$ be some minimal system, and $\left\{\widehat{\vartheta}_n\right\}_{n\in N} \subset X_1^* = X^* \oplus C^m$ is its biorthogonal system:

$$\hat{u}_n = (u_n; \alpha_{n1}, ..., \alpha_{nm})\,; \qquad \widehat{\vartheta}_n = (\vartheta_n; \beta_{n1}, ..., \beta_{nm})\,.$$

Let $J = \{n_1, ..., n_m\}$ be some set of m natural numbers. Assume

$$\delta = det \left\|\beta_{n_i j}\right\|_{\ i,j=\overline{1,m}}.$$

The following theorem was proved in [31] (see also [32]).

Theorem 2. *Let the system $\{\hat{u}_n\}_{n\in N}$ form a basis for X_1. For the system $\{u_n\}_{n\in N_J}$, where $N_J = N\backslash J$, to be a basis in X, it is necessary and sufficient that the condition $\delta \neq 0$ be satisfied. In this case, the system biorthogonal to $\{u_n\}_{n\in N_J}$ is defined by the equality*

$$\vartheta_n^* = \frac{1}{\delta}\begin{vmatrix} \vartheta_n & \vartheta_{n_1} & \ldots & \vartheta_{n_m} \\ \beta_{n1} & \beta_{n_1 1} & \ldots & \beta_{n_m 1} \\ \ldots & \ldots & \ldots & \ldots \\ \beta_{nm} & \beta_{n_1 m} & \ldots & \beta_{n_m m} \end{vmatrix}.$$

In particular, if X is a Hilbert space and $\{\hat{u}_n\}_{n\in N}$ is a Riesz basis in X_1, then under the condition $\delta \neq 0$, the system $\{u_n\}_{n\in N_J}$ also forms a Riesz basis for X.

For $\delta = 0$ the system $\{u_n\}_{n\in N_J}$ is neither complete nor minimal in X.

We will need some results from [27,28]. In [27] it was proved that the eigenvalues of problem (1),(2) are asymptotically simple and have the form $\lambda_n = \rho_n^2, n = 0, 1, 2, ...$, where the following asymptotic formula holds for the numbers ρ_n:

$$\rho_n = \pi n + O\left(\frac{1}{n}\right), \tag{3}$$

and for the eigenfunctions and associated functions $y_n(x)$ of problem (1),(2) corresponding to the eigenvalues $\lambda_n, n = 0, 1, 2, ...$, the asymptotic formula

$$y_n(x) = \sin \pi n x + O\left(\frac{1}{n}\right), \tag{4}$$

is valid, moreover, the problem can have only a finite number of associated functions, and the eigenvalues are numbered taking into account their multiplicities.

The conjugate spectral problem has the form

$$-z'' + \overline{q(x)}z = \lambda z, \quad x \in (0,1)\,, \tag{5}$$

$$\left.\begin{array}{c} z(1) = 0, \\ z'(1) + \left(\overline{a}\lambda + \overline{b}\right) z(0) = 0. \end{array}\right\} \tag{6}$$

The spectral problem (1),(2) is reduced to a spectral problem $L\hat{y} = \lambda\hat{y}$ for the operator L, acting in the space $L_p(0,1) \oplus C$. The operator L is defined as follows:

$$D(L) = \left\{\hat{y} = (y(x), ay(1)), y(x) \in W_p^2(0,1), l(y) \in L_p(0,1), y(0) = 0\right\},$$

$$\forall \hat{y} \in D(L) : L\hat{y} = \left(l(y), y'(0) - by(1)\right).$$

It was proved in [28] that the operator L is densely defined in $L_p(0,1) \oplus C$ as a closed operator with a compact resolvent. The eigenvalues of the operator L and problem (1),(2) coincide, and each eigen(or associated) function $y(x)$ of problem (1),(2) corresponds to an eigen(or associated) vector $\hat{y} = (y(x), ay(1))$ of the operator L. The adjoint operator L^* is defined as the operator generated in the space $L_q(0,1) \oplus C$, $\frac{1}{p} + \frac{1}{q} = 1$, by problem (5),(6). The eigenfunctions and associated functions of problem (5),(6) satisfy the asymptotic formulas

$$z_n(x) = 2\sin \pi n x + O\left(\frac{1}{n}\right), \quad n = 0, 1, 2, ..., \tag{7}$$

where the eigenfunctions and associated functions $z_n(x)$ are normalized so that the biorthogonality conditions are satisfied

$$\langle \hat{y}_n, \hat{z}_k \rangle = \int_0^1 y_n(x)\overline{z_k(x)}dx + a^2 y_n(1)\overline{z_k(0)} = \delta_{nk},$$

where $\hat{z}_k = (z_k(x), \bar{a}z_k(0))$ are the eigenvectors and associated vectors of the adjoint operator L^*, and δ_{nk} is the Kronecker symbol.

The following theorems are also true.

Theorem 3. *([28]) The root vectors of the operator L form a complete and minimal system in the space $L_p(0,1) \oplus C, 1 < p < \infty$.*

Theorem 4. *([28]) The system $\{y_n(x)\}_{n=0,n\neq n_0}^{\infty}$ of eigenfunctions and associated functions of problem (1),(2) with one rejected eigenfunction $y_{n_0}(x)$, corresponding to a simple eigenvalue λ_{n_0}, forms a complete and minimal system in the space $L_p(0,1), 1 < p < \infty$. The corresponding biorthogonal system is $\{\vartheta_n(x)\}_{n=0,n\neq n_0}^{\infty}$, where*

$$\vartheta_n(x) = z_n(x) - \frac{z_n(0)}{z_{n_0}(0)} z_{n_0}(x). \tag{8}$$

3. Main results

3.1. Basicity in spaces $L_p(0,1) \oplus C$ and $L_p(0,1)$.

Let $e_n(x) = sin\pi nx, n \in \mathbb{N}$ and introduce the following system in space $L_p(0,1) \oplus C$:

$$\hat{e}_0 = (0,1), \quad \hat{e}_n = (e_n(x), 0), n \in \mathbb{N}.$$

The following theorem is true.

Theorem 5. *The system* $\{\hat{y}_n\}_{n=0}^{\infty}$ *, eigenvectors and associated vectors of the operator* L *forms a basis for* $L_p(0,1) \oplus C$, $1 < p < \infty$, *isomorphic to the system* $\{\hat{e}_n\}_{n=0}^{\infty}$.

Proof. From the formula (4) it follows

$$y_n = e_n + O\left(\frac{1}{n}\right), \quad y_n(1) = O\left(\frac{1}{n}\right).$$

On the other hand

$$\hat{y}_n = \left(y_n\left(x\right),\ ay_n\left(1\right)\right) = \hat{e}_n + \ O\left(\frac{1}{n}\right).$$

Therefore, for any $r > 1$:

$$\sum_{n=0}^{\infty} \|\hat{y}_n - \hat{e}_n\|^r < +\infty, \tag{9}$$

i.e. the system $\{\hat{y}_n\}_{n=0}^{\infty}$ is $r-$ close to the system $\{\hat{e}_n\}_{n=0}^{\infty}$, and by Theorem 3 the system $\{\hat{y}_n\}_{n=0}^{\infty}$ is complete and minimal in $L_p(0,1) \oplus C$.

Let $1 < p \leq 2$, and $q-$ be its conjugate number: $\frac{1}{p} + \frac{1}{q} = 1$. By the Hausdorff-Young inequality [33] for any function $f \in L_p(0,1)$

$$\left(\sum_{n=1}^{\infty} |\langle f, e_n\rangle|^q\right)^{\frac{1}{q}} \leq C\|f\|_{L_p}.$$

Then for any element $\hat{f} = (f,\ \beta) \in L_p(0,1) \oplus C$ we have

$$\left(\sum_{n=0}^{\infty} \left|\left\langle \hat{f}, \hat{e}_n\right\rangle\right|^q\right)^{\frac{1}{q}} \leq |\beta| + \left(\sum_{n=1}^{\infty} |\langle f, e_n\rangle|^q\right)^{\frac{1}{q}} \leq |\beta| + \|f\|_{L_p} \leq C_1 \left\|\hat{f}\right\|_{L_p \oplus C}.$$

Consequently, the system $\{\hat{e}_n\}_{n=0}^{\infty}$ is a q - basis in $L_p(0,1) \oplus C$. Now, choosing $r = p$ in (9) we get that all the conditions of Theorem 2 are satisfied, therefore, the system $\{\hat{y}_n\}_{n=0}^{\infty}$ forms a basis for $L_p(0,1) \oplus C$, equivalent to the system $\{\hat{e}_n\}_{n=0}^{\infty}$.

Let, now $p > 2$. Then $1 < q < 2$ and the embedding

$$L_p(0,1) \subset L_q(0,1)$$

or

$$L_p(0,1) \oplus C \subset L_q(0,1) \oplus C$$

holds, and for $\hat{f} \in L_p(0,1) \oplus C$ we have

$$\left(\sum_{n=0}^{\infty} \left|\left\langle \hat{f}, \hat{e}_n\right\rangle\right|^p\right)^{\frac{1}{p}} \leq \ \mathrm{c}\left\|\hat{f}\right\|_{L_q \oplus C} \leq \mathrm{c}_1 \left\|\hat{f}\right\|_{L_p \oplus C}$$

i.e. the system $\{\hat{e}_n\}_{n=0}^{\infty}$ is a p-basis in $L_p\left(0,1\right) \oplus C$. Choosing $r = q$ we get that all the conditions of Theorem 2 are satisfied, which means that in this case the system $\{\hat{y}_n\}_{n=0}^{\infty}$ forms a basis for $L_p\left(0,1\right) \oplus C$, equivalent to the system$\{\hat{e}_n\}_{n=0}^{\infty}$. Theorem is proved.

Corollary 1. *In the case p=2 the system* $\{\hat{y}_n\}_{n=0}^{\infty}$ *forms a Riesz basis for* $L_2(0,1) \oplus C$.

Theorem 6. *In order for the system* $\{y_n(x)\}_{n=0,\ n\neq n_0}^{\infty}$ *of root functions of problem* (1) *and* (2) *with one remote function* $y_{n_0}(x)$ *to form a basis for* $L_p(0,1)$, $1 < p < \infty$, *it is necessary and sufficient that the condition* $z_{n_0}(0) \neq 0$ *be satisfied. If* $z_{n_0}(0) = 0$, *then the system* $\{y_n(x)\}_{n=0,\ n\neq n_0}^{\infty}$ *is not complete and minimal, and even more so is not a basis in* $L_p(0,1)$.

The *proof* follows from Theorem 5 followed by the application of Theorems 2 and 4.

Theorem 7. *The eigenfunctions and associated functions* $\{y_n(x)\}_{n=0,\ n\neq n_0}^{\infty}$ *of problem* (1) *and* (2) *with one remote eigenfunction* $y_{n_0}(x)$, *corresponding to a simple eigenvalue* λ_{n_0} *forms a basis for* $L_p(0,1)$, $1 < p < \infty$, *isomorphic to the trigonometric system* $\{sin\pi nx\}_{n=1}^{\infty}$.

Proof. If λ_{n_0} is a simple eigenvalue, then it corresponds to one eigenfunction $y_{n_0}(x)$ and $z_{n_0}(x)$ is the corresponding eigenfunction of the adjoint problem (5), (6). It should be noted that for all eigenfunctions $z_n(x)$ of the adjoint problem, the condition $z_n(0) \neq 0$ is satisfied. Indeed, let $z_n(0) = 0$, then from the second boundary condition (6) we obtain $z_n^{'}(1) = 0$, and this together with the first boundary condition $z_n(1) = 0$ means that $z_n(x)$ is the solution of Cauchy problem

$$-z^{''} + q(x)\,z = \lambda z,$$

$$z(1) = z^{'}(1) = 0,$$

which has only the trivial solution $z(x) \equiv 0$. And this contradicts the fact that $z_n(x)$ is an eigenfunction. Thus $z_{n_0}(1) \neq 0$. Then, by Theorem 6, the system $\{y_n(x)\}_{n=0,\ n\neq n_0}^{\infty}$ forms a basis for $L_p(0,1)$. It follows from the asymptotic formulas (4) that $\forall r \in (1;+\infty)$

$$\sum_{n=n_0+1}^{\infty} \|y_n - e_n\|^r < +\infty$$

i.e. the system $\{y_n(x)\}_{n=0,\ ,\ n\neq n_0}^{\infty}$ is $r-$ close to the system $\{e_n\}_{n=1}^{\infty}$ ($e_n(x) = \sin\pi nx$). Choosing $r = min\{p,q\}$, and taking into account that the system $\{e_n\}_{n=1}^{\infty}$ is an $r^{'}-$basis in $L_p(0,1)$ for the system $\{y_n(x)\}_{n=0,\ n\neq n_0}^{\infty}$ ($r' = max\{p,q\}$, $\frac{1}{r} + \frac{1}{r'} = 1$), we find that all conditions of Theorem 1 are satisfied and, therefore, it is isomorphic to the system $\{\sin\pi nx\}_{n=1}^{\infty}$. Theorem is proved.

Corollary 2. *Under the conditions of Theorem 7, the system* $\{y_n(x)\}_{n=0,\ n\neq n_0}^{\infty}$ *forms a* $r-$ *basis for* $L_p(0,1)$, $1 < p < \infty$, *where* $r = max\{p,q\}$.

Corollary 3. *In the case* $p = 2$ *the system* $\{y_n(x)\}_{n=0,\ n\neq n_0}^{\infty}$ *forms a Riesz basis for* $L_2(0,1)$.

3.2. Basicity in spaces

$L_{p,\omega}\ (0,1) \oplus C$ and $L_{p,\omega}(0,1)$.

Denote by $L_{p,\omega}\ (0,1)$ the weighted Lebesgue space with the norm

$$\|f\|_{L_{p,\omega}} = \left(\int_0^1 |f\,(x)|^p \omega\,(x)\,dx\right)^{\frac{1}{p}},$$

where the weight function $\omega\,(x)$ belongs to the Mackenhaupt class A_p, i.e. satisfies the condition

$$\sup_{I\subset(0,1)} \left(\frac{1}{|I|}\int_I \omega\,(x)\,dx\right)\left(\frac{1}{|I|}\int_I (\omega\,(x))^{-\frac{1}{p-1}}dx\right)^{p-1} < +\infty.$$

It was proved in [34] that if $\omega\,(x) \in A_p$, then there exists a number $r \in (1,p)$ such that $\omega\,(x) \in A_r$. Using this fact, we prove the following

Lemma 1. *Let the weight function $\omega\,(x)$ belong to the class A_p, $1 < p < \infty$. Then there exists a number p_0: $1 < p_0 < p$, such that a continuous embedding $L_{p,\omega}\ (0,1) \subset L_{p_0}\ (0,1)$ holds.*

Proof. Let $f \in L_{p,\omega}\ (0,1)$. Assume $p_0 = \frac{p}{r}$. Then $|f\,(x)|^{p_0} = |f\,(x)|^{p_0}\omega^{\frac{p_0}{p}}(x)\omega^{-\frac{p_0}{p}}(x)$ and from belonging of the function $|f\,(x)|^{p_0}\omega^{\frac{p_0}{p}}(x)$ to the class $L_{\frac{p}{p_0}}\ (0,1)$, and also from belonging of the function $\omega^{-\frac{p_0}{p}}(x)$ to the class $\left(L_{\frac{p}{p_0}}\ (0,1)\right)^* = L_{\frac{p}{p-p_0}}\ (0,1)$, and using the Hölder inequality, we obtain

$$\|f\|_{L_{p_0}\ (0,1)} = \left(\int_0^1 |f\,(x)|^{p_0}dx\right)^{\frac{1}{p_0}} = \left(\int_0^1 |f\,(x)|^{p_0}\omega^{\frac{p_0}{p}}(x)\omega^{-\frac{p_0}{p}}(x)dx\right)^{\frac{1}{p_0}} \leq$$

$$\leq \left(\int_0^1 |f\,(x)|^p\omega\,(x)\,dx\right)^{\frac{1}{p}}\left(\int_0^1 \omega^{-\frac{p_0}{p-p_0}}(x)\,dx\right)^{\frac{p-p_0}{pp_0}} = \|f\|_{L_{p,\omega}\ (0,1)}\left(\int_0^1 \omega^{-\frac{1}{r-1}}(x)\,dx\right)^{\frac{r-1}{p}} =$$

$$= K_{p,r}(\omega)\|f\|_{L_{p,\omega}\ (0,1)}.$$

Since $\omega^{-1} \in L_{\frac{1}{r-1}}(0,1)$, then the quantity $K_{p,r}(\omega) = \left(\int_0^1 \omega^{-\frac{1}{r-1}}(x)\,dx\right)^{\frac{r-1}{p}}$ has a finite value. Consequently, $f \in L_{p_0}\ (0,1)$.

Corollary 4. *If $f \in L_{p,\omega}\ (0,1)$, then $\forall s \in (0,p_0]$, i.e. $\forall s \in \left(0,\frac{p}{r}\right] : f \in L_s\ (0,1)$.*

Lemma 2. *Let $\omega \in A_p\,(0,1)$. Then each of the systems $\{sin\pi nx\}_{n=1}^{\infty}$ and $\{cos\pi nx\}_{n=0}^{\infty}$ forms a basis for $L_{p,\omega}\ (0,1)$.*

Proof. Denote by $\widetilde{\omega}\,(x)$ the even extension of the function $\omega\,(x)$ to $[-1,1]$, i.e. for $x \in [-1,0]$ $\widetilde{\omega}\,(x) = \omega\,(-x)$, or $x \in [0,1]$ $\widetilde{\omega}\,(x) = \omega\,(x)$. Then it is evident that $\widetilde{\omega}\,(x) \in A_p(-1,1)$. Let $f \in L_{p,\omega}\ (0,1)$. Let's extend it to $[-1,1]$ in an odd way, i.e.

$$\tilde{f}\,(x) = \begin{cases} f\,(x)\,, & x \in [0,1]\,, \\ -f\,(-x)\,, & x \in [-1,0]\,. \end{cases}$$

Then $\tilde{f}(x) \in L_{p,\widetilde{\omega}}\ (-1,1)$. We expand this function in the basis $\left\{e^{\ i\pi nx}\right\}_{n=-\infty}^{+\infty}$:

$$\tilde{f}\,(x) = \sum_{n=-\infty}^{+\infty} a_n e^{\ i\pi nx}, a_n = \frac{1}{2}\int_{-1}^{1} \tilde{f}\,(x) e^{-i\pi nx} dx.$$

It is obvious that

$$a_n = \frac{1}{2}\int_0^1 f\,(x) e^{-i\pi nx} dx - \frac{1}{2}\int_{-1}^{0} f\,(-x) e^{-i\pi nx} dx =$$

$$= \frac{1}{2}\int_0^1 f\,(x)\,(e^{-i\pi nx} - e^{i\pi nx}) dx = \frac{1}{i}\int_0^1 f\,(x)\,\sin\pi nx\ dx.$$

In addition a_{-n}=-a_n, $a_0 = 0$. Taking into account these relations, we get

$$\sum_{n=-m}^{m} a_n e^{i\pi nx} = \sum_{n=1}^{m} a_n (e^{i\pi nx} - e^{-i\pi nx}) =$$

$$= 2i\sum_{n=1}^{m} a_n \sin\pi nx\ = \sum_{n=1}^{m} \langle f,\ 2\sin\pi nt\ \rangle \sin\pi nx\ .$$

Hence

$$\left\| \tilde{f}\,(x) - \sum_{n=-m}^{m} a_n e^{i\pi nx} \right\|_{L_{p,\omega}\ (-1,1)} = \left\| \tilde{f}\,(x) - \sum_{n=1}^{m} \langle f,\ 2\sin\pi nt\ \rangle \sin\pi nx\ \right\|_{L_{p,\omega}\ (-1,1)} =$$

$$= 2^{\frac{1}{p}} \left\| f\,(x) - \sum_{n=1}^{m} \langle f,\ 2\sin\pi nt\ \rangle \sin\pi nx\ \right\|_{L_{p,\omega}\ (0,1)} .$$

The left side of the last equality tends to zero as $m \to \infty$, which means that the right side tends to zero as $m \to \infty$, and it means that the system $\{\sin\pi nx\ \}_{n=1}^{\infty}$ forms a basis for $L_{p,\omega}\ (0,1)$.

The basicity of the system $\{cos\pi nx\ \}_{n=0}^{\infty}$ in $L_{p,\omega}\ (0,1)$ is proved similarly. To do this, it suffices to take an even extension of the function $f\,(x)$ to $[-1,1]$.

Theorem 8. *The system* $\{\hat{y}_n\}_{n=0}^{\infty}$ *of root vectors of the operator* L *forms a basis for* $L_{p,\omega}\ (0,1) \oplus C$ *isomorphic to the system* $\{\hat{e}_n\}_{n=0}^{\infty}$.

Proof. From the continuity of the embedding

$$L_{p,\omega}\ (0,1) \oplus C \subset L_{p_0}\ (0,1) \oplus C,$$

and also from the minimality of the system $\{\hat{y}_n\}_{n=0}^{\infty}$ (according to Theorem 3) in the space $L_{p_0}\ (0,1) \oplus C$ it follows that this system is also minimal in $L_{p,\omega}\ (0,1) \oplus C$. It follows from asymptotic formulas (4) that

$$y_n\,(x) = \ e_n\,(x) + \varepsilon_n\,(x) \tag{10}$$

where for $\varepsilon_n(x)$ uniformly with respect to $x\in[0,1]$ the estimate

$$|\varepsilon_n(x)| \leq \frac{const}{n}, \tag{11}$$

is valid. Taking into account estimate (11), from (10) we obtain

$$\|\hat{y}_n - \hat{e}_n\|_{L_{p,\omega}\ (0,1)\oplus C} = \left(\int_0^1 |\varepsilon_n(\mathrm{x})|^{\mathrm{p}} \omega(\mathrm{x})\,\mathrm{dx}\right)^{\frac{1}{\mathrm{p}}} \leq \frac{const}{n}.$$

Consequently, $\forall \tau \in (1;+\infty)$

$$\sum_{n=0}^{\infty} \|\hat{y}_n - \hat{e}_n\|^{\tau} < +\infty, \tag{12}$$

i.e. the system $\{\hat{y}_n\}_{n=0}^{\infty}$ is $\tau-$close to the system $\{\hat{e}_n\}_{n=0}^{\infty}$ for any $\tau \in (1;+\infty)$. On the other hand, according to Corollary 3, a continuous embedding

$$L_{p,\omega}\ (0,1)\ \oplus C \subset L_s\ (0,1)\ \oplus C,$$

holds, $\forall s \in (1,\ p_0]$. Then, choosing $1 < s < min\{2, p_0\}$ and applying the Hausdorff-Young inequality for the system $\{e_n(x)\}_{n=1}^{\infty}$ ($e_n(x) = \sin\pi n x$), we obtain $\forall \hat{f} = (f(x),\ \beta) \in L_{p,\omega}(0,1) \oplus C$

$$\left(\sum_{n=0}^{\infty} \left|\left\langle \hat{f}, \hat{e}_n \right\rangle\right|^{\mathrm{s}'}\right)^{\frac{1}{\mathrm{s}'}} \leq |\beta| + \left(\sum_{n=1}^{\infty} |\langle f, e_n\rangle|^{\mathrm{s}'}\right)^{\frac{1}{\mathrm{s}'}} \leq$$

$$\leq |\beta| + c_2\|f\|_{L_s} \leq c_3\left\|\hat{f}\right\|_{L_s\ (0,1)\ \oplus C} \leq c_4\|f\|_{L_{p,\omega}\ \oplus C}.$$

The latter means that the system $\{\hat{e}_n\}_{n=0}^{\infty}$ forms an s'–basis for $L_{p,\omega}\ (0,1)$, where $\mathrm{s}' = s/(s-1)$. Choosing $\tau = s$, in (12) we obtain that all the conditions of Theorem 1 are satisfied, therefore, the system $\{\hat{y}_n\}_{n=0}^{\infty}$ forms a basis for $L_{p,\omega}\ (0,1) \oplus C$ isomorphic to the system $\{\hat{e}_n\}_{n=0}^{\infty}$.

Similarly to the previous section, we prove that the following theorems and corollaries are true.

Theorem 9. *For the basicity of the system $\{y_n(x)\}_{n=0,\ n\neq n_0}^{\infty}$ of eigenfunctions and associated functions of problem* (1), (2) *with one remote function $y_{n_0}(x)$ in $L_{p,\omega}\ (0,1)$ it is necessary and sufficient that the condition $z_{n_0}(0) \neq 0$ be satisfied. For $z_{n_0}(0) = 0$ the system $\{y_n(x)\}_{n=0,\ n\neq n_0}^{\infty}$ does not form a basis in the space $L_{p,\omega}\ (0,1)$. Moreover, in this case the system $\{y_n(x)\}_{n=0,\ n\neq n_0}^{\infty}$ is neither complete nor minimal in $L_{p,\omega}\ (0,1)$.*

Theorem 10. *The system $\{y_n(x)\}_{n=0,\ n\neq n_0}^{\infty}$ corresponding to eigenfunctions and associated functions of problem* (1), (2) *with one removed function $y_{n_0}(x)$, corresponding to a simple eigenfunction value λ_{n_0}, forms a basis for $L_{p,\omega}\ (0,1)$, $1 < p < \infty$, isomorphic to the trigonometric system $\{sin\pi nx\}_{n=1}^{\infty}$.*

Corollary 5. *Under the conditions of Theorem 10, the system $\{y_n(x)\}_{n=0,\ n\neq n_0}^{\infty}$ forms an s–basis in $L_{p,\omega}\ (0,1)$ for some $s > 2$.*

Acknowledgment

This work is supported by the Scientific and Technological Research Council of Turkey [TÜBİTAK 2542 -Bilateral Cooperation Program with Azerbaijan National Academy of Sciences (ANAS)], Project Number: 118F447 and Project Number: 19042020 (for Azerbaijan National Academy of Sciences).

References

[1] Atkinson F.B. Discrete and continuous boundary value problems. Moscow, Mir, 1968, 749 p.

[2] Tikhonov A.N., Samarskii A.A. Equations of Mathematical Physics. Moscow, Nauka, 1977, 766 p.

[3] Kapustin N.Yu., Moiseev E.I. Spectral problems with the spectral parameter in the boundary condition. Diff. Uravn., 1997, v. 33, No. 1, pp. 115-119.

[4] Gomilko A.M., Pivovarchik V.N. On bases of eigenfunctions of boundary problem associated with small vibrations of damped non smooth inhomogeneous string. Asympt. Anal. 1999., v. 20 No. 3-4, pp. 301-315.

[5] Kerimov N.B., Mirzoev V.S. On the Basis Properties of One Spectral Problem with a Spectral Parameter in a Boundary Condition. Sib. Math. J. 2003, v.44, No 5, pp.1041-1045.

[6] Kerimov N.B., Poladov R.G. On basicity in L_p of the system of eigenfunctions of one boundary value problem II. Proc. IMM NAS Azerb., 2005, v. 23, pp. 65-76.

[7] Kerimov N.B., Aliyev Z.S. On the basis properties of eigenfunctions of a spectral problem with a spectral parameter in the boundary condition. Dokl. RAN, 2007, v. 412, No 1, pp. 18-21.

[8] Marchenkov D.B. On the convergence of spectral expansions of functions for problems with a spectral parameter in a boundary condition. Differential Equations, 2005, v. 41, No. 10, pp. 1496-1500.

[9] Marchenkov D.B. Basis property of the system of eigenfunctions corresponding to a problem with a spectral parameter in the boundary condition. Differential Equations, 2006, v. 42, No. 6, pp. 905-908.

[10] Gasymov T.B., Mammadova Sh.J. On convergence of spectral expansions for one discontinuous problem with spectral parameter in the boundary condition. Trans. Of NAS of Azerb. 2006,vol. XXVI, No. 4, pp. 103-116.

[11] Gasymov T.B., Huseynli A.A. The basis properties of eigenfunctions of a discontinuous differential operator with a spectral parameter in boundary condition // Proc. of IMM of NAS of Azerb., v. XXXV(XLIII), 2011, pp. 21-32.

[12] Shahriari M., Akbarfam J.A., Teschl G. Uniqueness for inverse Sturm-Liouville problems with a finite number of transmission conditions. J. Math. Anal. Appl., v. 395, 2012, pp. 19-29.

[13] Shahriari M. Inverse Sturm-Liouville Problem with Eigenparameter Dependent Boundary and Transmission Conditions. Azerb. J. Math., v.4, No. 2, 2014, 16-30.

[14] Kerimov N.B., Maris E.A. On the basis properties and convergence of expansions in terms of eigenfunctions for a spectral problem with a spectral parameter in the boundary condition. Proceedings of IMM of NAS of Azerbaijan, 2014, v.40, Special Issue, pp. 245-258.

[15] Bilalov B.T., Gasymov T.B. On bases for direct decomposition. Doklady Mathematics. v. 93, No. 2, 2016, pp 183-185.

[16] Bilalov B.T., Gasymov T.B. On basicity a system of eigenfunctions of second order discontinuous differential operator. Ufa Mathematical Journal, 2017, v. 9, No 1, pp. 109-122.

[17] Maharramova G.V. On the completeness of eigenfunctions of a second-order differential operator, Journal of Contemporary Applied Mathematics, v. 8, No. 2, 2018, pp. 45-55.

[18] Gasymov T.B., Maharramova G.V. The stability of the basis properties of multiple systems in a Banach space with respect to certain transformations. Caspian Journal of Applied Mathematics, Ecology and Economics. V. 6, No 2, 2018, December, pp. 66-77.

[19] Gasymov T.B, Maharramova G.V., Mammadova N.G. Spectral properties of a problem of vibrations of a loaded string in Lebesgue spaces. Transactions of NAS of Azerbaijan, Issue Mathematics, 38(1), 62-68(2018), Series of Physical-Technical and Mathematical Sciences.

[20] Gasymov T.B, Maharramova G.V., Jabrailova A.N. Spectral properties of the problem of vibration of a loaded string in Morrey type spaces. Proceedings of the Institute of Mathematics and Mechanics, National Academy of Sciences of Azerbaijan, v. 44, No 1, 2018, pp. 116-122.

[21] Bilalov B.T., Gasymov T.B., Maharramova G.V. Basis Property of Eigenfunctions in Lebesgue Spaces for a Spectral Problem with a Point of Discontinuity. Diff. Uravn., 2019, v.55, No 12, pp.1544-1553.

[22] Maharramova G.V. Properties of Eigenvalues and Eigenfunctions of a Spectral Problem with Discontinuity Point. Caspian Journal of Applied Mathematics, Ecology and Economics V. 7, No 1, 2019, July, pp. 114-125.

[23] Gasymov T.B., Maharramova G.V., T.F. Kasimov, Completeness and minimality of eigenfunctions of a spectral problem in spaces $L_p \oplus C$ and L_p. Journal of Contemporary Applied Mathematics, v.10, No 2, 2020, pp. 84-100.

[24] Gasymov T.B., Akhtyamov A.M., Ahmedzade N.R. On the basicity of eigenfunctions of a second-order differential operator with a discontinuity point in weighted Lebesgue spaces. Proceedings of the Institute of Mathematics and Mechanics, National Academy of Sciences of Azerbaijan, v. 46, No 1, 2020, pp. 32–44.

[25] Maris E.A., Goktaş S. On the spectral properties of a Sturm-Liouville problem with eigenparameter in the boundary condition. Hacet. J. Math. Stat., v. 49, No. 4, 2020, pp. 1373 – 1382.

[26] Goktaş S., Maris E.A. The uniform convergence of Fourier series expansions of a Sturm-Liouville problem with boundary condition which contains the eigenparameter. Commun. Fac. Sci. Univ. Ank. Ser. A1 M ath. Stat., v. 70, No. 1, 2021, pp. 205- 215.

[27] Kasimov T.F. On the Completeness and Minimality of Eigenfunctions of a Non-self-adjoint Spectral Problem With Spectral Parameter in the Boundary Condition, Caspian Journal of Applied Mathematics, Ecology and Economics, V. 8, No 2, 2020, December, pp. 61-75.

[28] Kasimov T.F. Asymptotics of the Eigenvalues and Eigenfunctions of a Non-Self-Adjoint Problem with a Spectral Parameter in the Boundary Condition. Journal of Contemporary Applied Mathematics V. 11, No 2, 2021, December, pp. 11-22.

[29] Bilalov, B.T., Bases of exponentials, cosines, and sines formed by eigenfunctions of differential operators, Differ. Uravn., v. 39, No. 5, 2003, pp.652-657.

[30] Gohberg I.C., Krein M.G. Introduction to the Theory of Linear Nonselfadjoint Operators in Hilbert Space, AMS, 1969, 394p.

[31] Gasymov T.B. On necessary and sufficient conditions of basicity of some defective systems in Banach spaces. Trans. NAS Azerb., ser. phys.-tech. math. sci., math.mech., v.26, No. 1, 2006, pp. 65-70.

[32] Gasymov T.B., Garayev T.Z. On necessary and sufficient conditions for obtaining the bases of Banach spaces. Proc. of IMM of NAS of Azerb., v. XXVI(XXXIV), 2007, pp. 93-98.

[33] Sigmund A. Trigonometric series. v 2, M.: Mir, 1965, 537 p.

[34] Hunt R., Muckenhoupt B., Wheeden R. Weighted norm inequalities for the conjugate function and Holder transform. Transactions of the American Mathematical Society, v. 176, February 1973, pp. 227-251.

Telman Gasymov
Baku State University, Baku, Azerbaijan
Institute of Mathematics and Mechanics of NAS of Azerbaijan, Baku, Azerbaijan
E-mail: telmankasumov@rambler.ru

Ulkar Hashimova
Baku State University, Baku, Azerbaijan
E-mail: hashimovaulkar.uh@gmail.com

Reyhan Taghiyeva
Baku State University, Baku, Azerbaijan
E-mail: reyhanabasli2015@gmail.com

Received 07 March 2022
Accepted 07 April 2022