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On Absolute and Uniform Convergence of Biorthogonal Expansion in Root Functions of a Second Order Discontinuous Differential Operator

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Abstract. A second order differential operator is considered. Theorems on absolute and uniform convergence of expansions of discontinuous functions by the root functions of the given are proved. Uniform convergence rate of these expansions is also studied.

Key Words and Phrases: discontinuous operator, biorthogonal expansions, root functions, uniform convergence.

1. Introduction

In V.A.Il'ins paper [1], a second order discontinuous operator was investigated and constructive necessary and sufficient conditions of Riesz basicity of the system of root functions of the considered operator arc established. These investigations played an important roll in studying absolute and uniform convergence of biorthogonal expansions of functions from the class W_2^1 by the root functions of the Schredinger operator with multipoint boundary conditions [2,3]. In the present paper, we consider a second order discontinuous operator, issues of absolutes and uniform convergence and also on uniform convergence rate of biorthogonal expansions of discontinuous functions by root functions are studied.

2. The Basic Concepts and Formulation of Results

Let $G = \bigcup\limits_{l=1}^{m} G_l$, $G_l = (\xi_{l-1}, \xi_l)$, $a = \xi_0 < ... < \xi_m = b$, where $-\infty < a < b < +\infty$.

By $\widetilde{W}_r^n(G)$, $1 \le r \le \infty$, denote a class of functions possessing the property: if $f(x) \in \widetilde{W}_r^n(G)$, then for each G_l, $l = \overline{1,m}$ there exists the function $f_l(x)$ from the Sobolev space $W_r^n(G_l)$ such that $f(x) = f_l(x)$ for $\xi_{l-1} < x < \xi_l$.

On the set G consider the formal operator

$$Lu = u'' + q(x)u \qquad (1)$$

with complex-valued coefficients $q(x) \in L_1(G)$.

Under the system of the root functions of the operator we'll understand an arbitrary system $\{u_k\}_{k=1}^{\infty}$ of complex-valued non identical zero functions from $\widetilde{W}_r^2(G)$, satisfying almost everywhere in G the equation

$$Lu_k + \lambda_k u_k = \theta_k u_{k-1},$$

where θ_k either equals zero (in this case the function uk is an eigenfunction) or unit (in this case we require $\lambda_k = \lambda_{k-1}$ and call u_k an associated function of order j, where $\theta_k = \theta_{k-1} = ... = \theta_{k-j+1} = 1$, $\theta_{k-j} = 0$).

Denote $\mu_k = \sqrt{\lambda_k}$, $\arg \mu_k \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Later on, we'll additionally suppose that the elements of the considered system of the root functions $\{u_k(x)\}_{k=1}^{\infty}$ are determined at the points ξ_l, $l = \overline{1,m}$ and are continuous from the left at these points, but at the point $\xi_0 = a$ they are continuous from the right. The expanded function $f(x) \in \widetilde{W}_p^1(G)$, $1 \le p \le \infty$ will also satisfy these requirements.

We'll require that the system $\{u_k(x)\}_{k=1}^{\infty}$ satisfies V.A. Ilin's following conditions: 1) the system $\{u_k(x)\}_{k=1}^{\infty}$ complete and minimal in $L_2(G)$; 2) the inequalities are

$$|Im\mu_k| \le C_0, \tag{2}$$

$$\sum_{\tau \le Re\mu_k \le \tau+1} \le C_1, \quad \forall \tau \ge 0; \tag{3}$$

fulfilled. 3) there exists a constant $C_2 > 0$ such half

$$\|u_k\|_2 \|u_k\|_2 \le C_2, \quad k = 1, 2, ..., \tag{4}$$

where $\{v_k\}_{k=1}^{\infty}$ is a biorthogonall adjoint system to the system $\{u_k(x)\}_{k=1}^{\infty}$ and consists of the root functions of the formally adjoint operator $L^* = \frac{d^2}{dx^2} + \overline{q(x)}$, $x \in G$ (i.e. $L^* v_k + +\overline{\lambda}_k v_k = \theta_{k+1} v_{k+1}$).

For an arbitrary function $f(x) \in \widetilde{W}_p^1(G)$, $p \ge 1$ compose a partial sum of its biorthogonal expansion by the system $\{u_k\}_{k=1}^{\infty}$

$$\sigma_\nu(x, f) = \sum_{\rho_k \le \nu} (f, v_k)\, u_k(x), \nu \ge 1,$$

where

$$(f, v_k) = \int_G f(x)\, \overline{v_k(x)} dx.$$

Assume

$$R_\nu(x, f) = f(x) - \sigma_\nu(x, f)$$

and denote

$$Q_m(f, v_k) = f(b)\, \overline{v_k'(b-0)} - f(a)\, \overline{v_k'(a+0)} +$$

$$+\sum_{l=1}^{m-1}\left[f(\xi_l-0)\overline{v_k'(\xi_l-0)}-f(\xi_l+0)\overline{v_k'(\xi_l+0)}\right].$$

The main results of this section consist of the following theorems:

Theorem 1. Let the system $\{u_k(x)\}_{k=1}^{\infty}$ of the root functions of the operator (1) satisfy conditions 1)-3). Then, a biorthogonal expansion of any function $f(x)\in \widetilde{W}_p^1(G)$, $p\geq 1$ satisfying the condition

$$|Q_m(f,v_k)|\leq C_1(f)\,\|v_k\|_2\,,\quad k=1,2,....,\tag{5}$$

converges absolutely and uniformly on $\overline{G}=[a,b]$ and the following the relations are valid

$$f(x)=\sum_{k=1}^{\infty}(f,v_k)\,u_k(x)\,,\qquad x\in\overline{G},\tag{6}$$

$$\sup_{x\in\overline{G}}|R_\nu(x,f)|\leq const\left\{\nu^{-\delta}\,\|f\|_{\widetilde{W}_p^1(G)}+\nu^{-1}\left[C_1(f)+\|q\|_1\,\|f\|_\infty\right]\right\},\tag{7}$$

$$\sup_{x\in\overline{G}}|R_\nu(x,f)|=o\left(\nu^{-\delta}\right),\nu\to+\infty,\tag{8}$$

where $\delta=\min\left\{\frac{1}{2},\frac{1}{q}\right\}$, $p^{-1}+q^{-1}=1$; symbol "o" depends on the expanded function $f(x)$; const is independent of the expanded function $f(x)$;

$$\|f\|_{\widetilde{W}_p^1(G)}=\|f\|_{L_p(G)}+\|f'\|_{L_p(G)}\,.$$

Theorem 2. Let the function $f(x)$ belong to the class $\widetilde{W}_1^1(G)$ conditions l)-3) and condition(5) be fulfilled for the system of the root functions $\{u_k(x)\}_{k=1}^{\infty}$ of operator (1) and the number series

$$\sum_{n=n_0}^{\infty}n^{-1}\omega_1\left(f',n^{-1}\right)<\infty,\tag{9}$$

converge. Here $n_0>2(b-a)$, $\omega_1(\cdot,\delta)$ is a modulus of continuity in $L_1(G)$.

Then a biorthogonal expansion of the function $f(x)$ by the system $\{u_k(x)\}_{k=1}^{\infty}$ converges absolutely and uniformly on $\overline{G}$, equality (6) and the estimation

$$\sup_{x\in\overline{G}}|R_\nu(x,f)|\leq constK(\nu,f)\,,\tag{10}$$

are valid. Here the const is independent on the expanded function $f(x)$; the function $K(\nu,f)$ is determined by the formula

$$K(\nu,f)=\sum_{n=[\nu]}^{\infty}n^{-1}\omega_1\left(f',n^{-1}\right)+\nu^{-1}\left[C_1(f)+\|q\|_1\,\|f\|_\infty+\|f'\|_1\right],$$

$$\nu > 4\pi / \left(\min_{1 \le l \le m} |G_l| \right).$$

By $\widetilde{H}_1^\alpha(G)$ denote a space of functions $f(x)$ such that for each l $\left(l = \overline{1,m}\right)$ there exists the function $f_l(x)$ of Nikolskiy class $\widetilde{H}_1^\alpha(G)$, such that $f(x) = f_l(x)$ for $\xi_{l-1} < x < \xi_l$. We define the norm in $\widetilde{H}_1^\alpha(G)$ by the equality

$$\|f\|_{\widetilde{H}_1^\alpha(G)} = \|f\|_{1,G}^\alpha = \|f\|_{L_1(G)} + \max_{1 \le l \le m} \sup_{\delta > 0} \frac{\omega_1(f,\delta)_{G_l}}{\delta^\alpha},$$

where

$$\omega_1(f,\delta)_{G_l} = \sup_{0 < h \le \delta} \int_{\xi_{l-1}}^{\xi_l - h} |f(x+h) - f(x)|\, dx.$$

Corollary from theorem 2. If in theorem 2 we additionally require $f(x) \in \widetilde{H}_1^\alpha(G)$, $0 < \alpha < 1$, the estimation

$$\sup_{x \in \overline{G}} |R_\nu(x,f)|_{C(\overline{G})} = O\left(\nu^{-\alpha}\right)$$

is fulfilled.

Remark 1. The condition (5) required from the bilinear functional $Q_m(f, v_k)$, is natural. Usually, the class of functions $f(x)$, for which $Q_m(f, v_k) = 0$, $k = 1, 2, ...$, should be determined from the considered space. In many cases, condition (5) is fulfilled for all the functions $f(x)$ from the space under consideration. For example, in the case when the boundary conditions for $v_k(x)$ are two -point and the principal minor of the matrix that corresponds to boundary conditions are non zero, i. e. if

$$\alpha_i v'(\alpha) + \beta_i v'(b) + c_i v(a) + d_i v(b) = 0,$$

$$i = 1, 2;\;\; \alpha_1 \beta_2 - \alpha_2 \beta_1 \ne 0.$$

Remark 2. Estimation (7) is unimprovable in the considered class of functions.

3. Representation for the Fourier coefficients from the class $\widetilde{W}_p^1(G)$, $1 \le p \le \infty$.

Before proving theorems 1 and 2, we get some representations for the Fourier coefficients of the functions from the class $\widetilde{W}_p^1(G)$, $1 \le p \le \infty$.

Lemma 1. The following representations are valid for the Foruier coefficients of the function $f(x) \in \widetilde{W}_p^1(G)$.

$$(v_k, f) = -\frac{\overline{Q_m(f, v_k)}}{\overline{\mu}_k^2} + \frac{(v_k', f')}{\overline{\mu}_k^2} - \frac{1}{\overline{\mu}_k^2}(v_k, qf) + \frac{\theta_{k+1}}{\overline{\mu}_k^2}(v_{k+1}, f) \tag{11}$$

$$(v_k, f) = -\sum_{j=0}^{n_k} \frac{\overline{Q_m(f, v_{k+j})}}{\overline{\mu}_k^{2(j+1)}} - \sum_{j=0}^{n_k} \frac{(v_{k+j}, qf)}{\overline{\mu}_k^{2(j+1)}} +$$

$$+\sum_{j=0}^{n_k}\sum_{l=1}^{m} \frac{v'_{k+j}(\xi_{l-1})}{\overline{\mu}_k^{2(j+1)}} \left(\cos\overline{\mu}_k (t-\xi_{l-1}), f'(t)\right)_{G_l} -$$

$$-\sum_{j=0}^{n_k}\sum_{l=1}^{m} \frac{v_{k+j}(\xi_{l-1})}{\overline{\mu}_k^{2j+1}} \left(\sin\overline{\mu}_k (t-\xi_{l-1}), f'(t)\right)_{G_l} -$$

$$-\sum_{j=0}^{n_k}\sum_{l=1}^{m} \frac{1}{\overline{\mu}_k^{2(j+1)}} \int\limits_{\xi_{l-1}}^{\xi_l} \overline{q(\tau)} v_{k+j}(\tau) \left(\int\limits_{\tau}^{\xi_l} \overline{f'(t)} \cos\overline{\mu}_k (t-\tau)\, dt \right) d\tau +$$

$$+\sum_{j=0}^{n_k}\sum_{l=1}^{m} \frac{\theta_{k+j+1}}{\overline{\mu}_k^{2(j+1)}} \int\limits_{\xi_{l-1}}^{\xi_l} v_{k+j+1}(\tau) \left(\int\limits_{\tau}^{\xi_l} \overline{f'(t)} \cos\overline{\mu}_k (t-\tau)\, dt \right) d\tau, \qquad (12)$$

therewith n_k is the order of the associated function v_k; $\theta_{k+n_k+1} = 0$;

$$v_{k+j}^{(i)}(\xi_{l-1}) \overset{def}{=} v_{k+j}^{(i)}(\xi_{l-1}+0),$$

$$l = \overline{1, m}, \quad i = \overline{0, 1}; \quad (f, g)_{G_l} = \int\limits_{G_l} f(t)\, \overline{g(t)} dt.$$

Proof. It is seen from the definition of the function $v_k(x)$ that

$$v''_k(x) + \overline{q(x)} v_k(x) + \overline{\lambda}_k v_k(x) = \theta_{k+1} v_{k+1}(x), \quad x \in G_l, l = 1, m.$$

Therefore, multiplying by $\overline{f}(x)$ and integrating with respect to x from ξ_{l-1} to ξ_l, from the last equality we get

$$\int\limits_{\xi_{l-1}}^{\xi_l} \overline{f(x)} v_k(x)\, dx = -\frac{1}{\overline{\lambda}_k} \int\limits_{\xi_{l-1}}^{\xi_l} \overline{f(x)} v''_k(x)\, dx -$$

$$-\frac{1}{\overline{\lambda}_k} \int\limits_{\xi_{l-1}}^{\xi_l} \overline{q(x)} v_k(x)\, \overline{f(x)} dx + \frac{\theta_{k+1}}{\overline{\lambda}_k} \int\limits_{\xi_{l-1}}^{\xi_l} \overline{f(x)} v_{k+1}(x)\, dx.$$

Conducting integration by parts in the integral containing $v''_k(\xi)$, we find

$$\int\limits_{\xi_{l-1}}^{\xi_l} \overline{f(x)} v_k(x)\, dx = -\frac{1}{\overline{\lambda}_k} \left[\overline{f(\xi_l - 0)} v'_k(\xi_l - 0) - \overline{f(\xi_{l-1}+0)} v'_k(\xi_l + 0) \right] +$$

$$+\frac{1}{\overline{\lambda}_k}\int\limits_{\xi_{l-1}}^{\xi_l}\overline{f'(x)}v'_k(x)\,dx-\frac{1}{\overline{\lambda}_k}(v_k,qf)+\frac{\theta_{k+1}}{\overline{\lambda}_k}\int\limits_{\xi_{l-1}}^{\xi_l}\overline{f(x)}v_{k+1}(x)\,dx.$$

Summing these equalities over l from 1 to m and taking into account $\lambda_k=\mu_k^2$, we arrive at representation (11).

Now, derive formula (12). It follows from (11) that if we consider this formula as a recurrent relation for the Fourier coefficients, then

$$(v_k,f)=-\frac{\overline{Q_m(f,v_k)}}{\overline{\mu}_k^2}+\frac{1}{\overline{\mu}_k^2}\left(v'_k,f'\right)-\frac{1}{\overline{\mu}_k^2}(v_k,qf)+$$

$$+\frac{\theta_{k+1}}{\overline{\mu}_k^2}\left[-\frac{\overline{Q_m(f,v_{k+1})}}{\overline{\mu}_k^2}+\frac{\left(v'_{k+1},f'\right)}{\overline{\mu}_k^2}-\frac{1}{\overline{\mu}_k^2}(v_{k+1},qf)+\frac{1}{\overline{\mu}_k^2}(v_{k+2},f)\right]$$

Continuing this substitution process to the eigenfunction v_{k+n_k}, for (v_k,f) we get the following representation

$$(v_k,f)=-\sum_{j=0}^{n_k}\frac{\overline{Q_m(f,v_{k+j})}}{\overline{\mu}_k^{2(j+1)}}+\sum_{j=0}^{n_k}\frac{\left(v'_{k+j},f'\right)}{\overline{\mu}_k^{2(j+1)}}-\sum_{j=0}^{n_k}\frac{(v_{k+j},qf)}{\overline{\mu}_k^{2(j+1)}}. \tag{13}$$

Transform the expression $\left(v'_{k+j},f'\right)$. For that we use the mean value formula for the function $v'_{k+j}(t)$ for $t\in(\xi_{-1},\xi_l)$:

$$v'_{k+j}(t)=-\overline{\mu}_k v_{k+j}(\xi_{l-1}+0)\sin\overline{\mu}_k(t-\xi_{l-1})+v'_{k+j}(\xi_{l-1}+0)\times$$

$$\times\cos\overline{\mu}_k(t-\xi_{l-1})-\int\limits_{\xi_{l-1}}^{t}\overline{q}(\tau)\,v_{k+j}(\tau)\cos\overline{\mu}_k(t-\tau)\,d\tau+$$

$$+\theta_{k+j+1}\int\limits_{\xi_{l-1}}^{t}v_{k+j+1}(\tau)\cos\overline{\mu}_k(t-\tau)\,d\tau,$$

$$j=\overline{0,n_k},\qquad \theta_{k+n_k+1}=0.$$

As a result of substitution of the expression $v'_{k+j}(t)$ in $\left(v'_{k+j},f'\right)$ we get

$$\left(v'_{k+j},f'\right)=\sum_{l=1}^{m}\int\limits_{\xi_{l-1}}^{\xi_l}\overline{f'(t)}v'_{k+j}(t)\,dt=\sum_{l=1}^{m}\left\{-\overline{\mu}_k\int\limits_{\xi_{l-1}}^{\xi_l}\overline{f'(t)}\sin\overline{\mu}_k(t-\xi_{l-1})\,dt\times\right.$$

$$\times v_{k+j}(\xi_{l-1}+0)+\int\limits_{\xi_{l-1}}^{\xi_l}\overline{f'(t)}\cos\overline{\mu}_k(t-\xi_{l-1})\,dt\,v'_{k+j}(\xi_{l-1}+0)-$$

$$-\int\limits_{\xi_{l-1}}^{\xi_l}\overline{f'(t)}\left(\int\limits_{\xi_{l-1}}^{t}\overline{q(\tau)}v_{k+j}(\tau)\cos\overline{\mu}_k(t-\tau)\,d\tau\right)dt+$$

$$\left.+\theta_{k+j+1}\int\limits_{\xi_{l-1}}^{\xi_l}\overline{f'(t)}\left(\int\limits_{\xi_{l-1}}^{t}\overline{q(\tau)}v_{k+j+1}(\tau)\cos\overline{\mu}_k(t-\tau)\,d\tau\right)dt\right\}.$$

Change the integration order in double integrals, take into account the obtained expression for $\left(v'_{k+j},f'\right)$ in formula (13), and get the formula

$$(v_k,f)=-\sum_{j=0}^{n_k}\frac{\overline{Q_m(f,v_{k+j})}}{\overline{\mu}_k^{2(j+1)}}-\sum_{j=0}^{n_k}\frac{(v_{k+j},qf)}{\overline{\mu}_k^{2(j+1)}}+$$

$$+\sum_{j=0}^{n_k}\sum_{l=1}^{m}\frac{v'_{k+j}(\xi_{l-1})}{\overline{\mu}_k^{2(j+1)}}\left(\cos\overline{\mu}_k(t-\xi_{l-1}),f'(t)\right)_{G_l}-$$

$$-\sum_{j=0}^{n_k}\sum_{l=1}^{m}\frac{v_{k+j}(\xi_{l-1})}{\overline{\mu}_k^{2j+1}}\left(\sin\overline{\mu}_k(t-\xi_{l-1}),f'(t)\right)_{G_l}-$$

$$-\sum_{j=0}^{n_k}\sum_{l=1}^{m}\frac{1}{\overline{\mu}_k^{2(j+1)}}\int\limits_{\xi_{l-1}}^{\xi_l}\overline{q(\tau)}v_{k+j}(\tau)\left(\int\limits_{\tau}^{\xi_l}\overline{f'(t)}\cos\mu_k(t-\tau)\,dt\right)d\tau+$$

$$+\sum_{j=0}^{n_k}\sum_{l=1}^{m}\frac{\theta_{k+j+1}}{\overline{\mu}_k^{2(j+1)}}\int\limits_{\tau}^{\xi_l}\overline{q(\tau)}v_{k+j+1}(\tau)\left(\int\limits_{\tau}^{\xi_l}\overline{f'(t)}\cos\mu_k(t-\tau)\,dt\right)d\tau,$$

where under $v_{k+j}^{(i)}(\xi_{l-1})$, $i=0,1$, we mean $v_{k+j}^{(i)}(\xi_{l-1}+0)$. Besides $\theta_{k+n_k+1}=0$, n_k - is the order of the associated function $v_k(x)$. Lemma 1 is proved.

Proof of Theorem 1.

It suffices to consider the case $1<p\le 2$. Estimate the series

$$\sum_{k=1}^{\infty}|(f,v_k)|\,|u_k(x)|\,,\quad x\in\overline{G}.$$

Represent this series in the form

$$\sum_{k=1}^{\infty}|(f,v_k)|\,|u_k(x)|=\sum_{0\le Re\mu_k<1}|(f,v_k)|\,|u_k(x)|+\sum_{Re\mu_k\ge 1}|(f,v_k)|\,|u_k(x)|\,.$$

Estimate each of the sums in the right hand side of the last equality

$$\sum_{0\le Re\mu_k<1}|(f,v_k)|\,|u_k(x)|\le\sum_{0\le Re\mu_k<1}\|f\|_1\,\|v_k\|_\infty\,|u_k(x)|\,.$$

Apply here the estimates (2)-(4) and (see [4])

$$\sup_{x\in\overline{G}_l}|u_k(x)|\leq C(l)\|u_k\|_{L_2(\xi_{l-1},\xi_l)} \tag{14}$$

$$\sup_{x\in\overline{G}_l}|v_k(x)|\leq C(l)\|v_k\|_{L_2(\xi_{l-1},\xi_l)}\,. \tag{15}$$

As a result we have:

$$\sum_{0\leq Re\mu_k<1}|(f,v_k)|\,|u_k(x)|\leq C\sum_{0\leq Re\mu_k<1}\|u_k\|_2\|v_k\|_2\|f\|_1\leq$$

$$\leq C\|f\|_1\sum_{0\leq Re\mu_k<1}1\leq C\|f\|_1\,,$$

where C is a positive number. For estimating the series

$$\sum_{Re\mu_k\geq 1}|(f,v_k)|\,|u_k(x)|$$

we apply formula (11) for the Fourier coefficients (f,v_k):

$$\sum_{Re\mu_k\geq 1}|(f,v_k)|\,|u_k(x)|\leq\sum_{|\mu_k|\geq 1}|(f,v_k)|\,|u_k(x)|\leq\sum_{|\mu_k|\geq 1}\frac{\overline{|Q_m(f,v_k)|}}{|\mu_k|^2}|u_k(x)|+$$

$$+\sum_{|\mu_k|\geq 1}\frac{1}{|\mu_k|^2}\left|\left(v_k',f'\right)\right||u_k(x)|+\sum_{|\mu_k|\geq 1}\frac{|u_k(x)|}{|\mu_k|^2}(v_k,qf)+$$

$$+\sum_{|\mu_k|\geq 1}\frac{\theta_{k+1}}{|\mu_k|^2}|(v_{k+1},f)|\,|u_k(x)|\,. \tag{16}$$

For estimating the first sum in the right hand side of relation (16) we apply inequalities (5), (14), (15) and then (2)-(4). As a result, we find

$$\sum_{|\mu_k|\geq 1}\frac{\overline{|Q_m(f,v_k)|}}{|\mu_k|^2}|u_k(x)|\leq C_1(f)\sum_{|\mu_k|\geq 1}\frac{\|v_k\|_2}{|\mu_k|^2}\|u_k\|_2\leq$$

$$\leq C_2(f)\sum_{|\mu_k|\geq 1}\frac{1}{|\mu_k|^2}\leq C_2(f)\sum_{n=1}^{\infty}\left(\sum_{n\leq|\mu_k|\leq n+1}\frac{1}{|\mu_k|^2}\right)\leq$$

$$\leq C_2(f)\sum_{n=1}^{\infty}\frac{1}{n^2}\left(\sum_{n\leq|\mu_k|\leq n+1}1\right)\leq C(f)\,,$$

where $C(f)$ in a number dependent on $f(x)$.

Estimate the second sum in the right hand side of (16)

$$\sum_{|\mu_k|\geq 1} \frac{1}{|\mu_k|^2} \left|\left(f', v_k'\right)\right| |u_k(x)| = \sum_{|\mu_k|\geq 1} \frac{1}{|\mu_k|} \left|\left(f', v_k' \|v_k\|_q^{-1} |\mu_k|^{-1}\right)\right| \|v_k\|_q |u_k(x)|,$$

$$p^{-1} + q^{-1} = 1.$$

Applying estimates (14), (15) and (4), we get that the left hand side of the last relation is majorized from above by the quantity

$$C \sum_{|\mu_k|\geq 1} \frac{1}{|\mu_k|} \left|\left(f', v_k' \|v_k\|_q^{-1} |\mu_k|^{-1}\right)\right|.$$

Since the system $\left\{v_k'(x) \|v_k\|_q^{-1} |\mu_k|^{-1}\right\}$ is a Riesz system (see [8]) and taking into account $f'(x) \in L_p(a,b)$, we can apply the Riesz inequality.

Consequently, the second sum in (16) is bounded from above by the quantity:

$$C \left(\sum_{|\mu_k|\geq 1} \frac{1}{|\mu_k|^p}\right)^{1/p} \left(\sum_{|\mu_k|\geq 1} \left|\left(f', v_k' \|v_k\|_q^{-1} |\mu_k|^{-1}\right)\right|^q\right)^{1/q}.$$

By the condition (4) the sum

$$\sum_{|\mu_k|\geq 1} \frac{1}{|\mu_k|^p}, \quad p > 1,$$

converges by the Riesz inequality

$$\left(\sum_{|\mu_k|\geq 1} \left|\left(f', v_k' \|v_k\|_q^{-1} |\mu_k|^{-1}\right)\right|^q\right)^{1/q} \leq C \left\|f'\right\|_p.$$

Thus, the second series in the right hand side of relation (16) also converges and its sum doesn't exceed the quantity $C \|f'\|_p$, where C is independent of the function $f(x)$.

Estimate the third sum in the right hand side of (16):

$$\sum_{|\mu_k|\geq 1} \frac{|u_k(x)|}{|\mu_k|^2} |(v_k, qf)| \leq \sum_{|\mu_k|\geq 1} \|q\|_1 \|v_k\|_\infty |u_k(x)| |\mu_k|^{-2} \|f\|_\infty.$$

Apply estimates (14), (15), and then (2)-(4). As a result we find

$$\sum_{|\mu_k|\geq 1} \frac{|u_k(x)|}{|\mu_k|^2} |(v_k, qf)| \leq C_1 \|q\|_1 \|f\|_\infty \sum_{|\mu_k|\geq 1} \frac{1}{|\mu_k|^2} \leq$$

$$\leq C_1 \|q\|_1 \|f\|_\infty \sum_{n=1}^{\infty} \frac{1}{n^2} \left(\sum_{n \leq |\mu_k| \leq n+1} 1 \right) \leq C \|q\|_1 \|f\|_\infty ,$$

where C is a constant independent of $f(x)$.

Now, prove the convergence of the last series in the right hand side of (16)

$$\sum_{|\mu_k| \geq 1} \frac{\theta_{k+1}}{|\mu_k|^2} |(v_{k+1}, f)| |u_k(x)| =$$

$$= \sum_{|\mu_k| \geq 1} \frac{1}{|\mu_k|} \left| \left(f, \theta_{k+1} v_{k+1} \|v_{k+1}\|_q^{-1} \right) \right| \frac{1}{|\mu_k|} \|v_{k+1}\|_q |u_k(x)| .$$

Here, applying estimation (14)

$$\|v_{k+1}\|_q \leq C |\mu_k| \|v_k\|_2 , \tag{17}$$

(see [5]) and the Holder inequality, we have:

$$\sum_{|\mu_k| \geq 1} \frac{\theta_{k+1}}{|\mu_k|^2} |(v_{k+1}, f)| |u_k(x)| \leq$$

$$\leq C \sum_{|\mu_k| \geq 1} \frac{1}{|\mu_k|} \left| \left(f, \theta_{k+1} v_{k+1} \|v_{k+1}\|_q^{-1} \right) \right| \leq$$

$$\leq C \left(\sum_{|\mu_k| \geq 1} \frac{1}{|\mu_k|^p} \right)^{1/p} \left(\sum_{|\mu_k| \geq 1} \left| \left(f, \theta_{k+1} v_{k+1} \|v_{k+1}\|_q^{-1} \right) \right|^q \right)^{1/q} , \quad \frac{1}{p} + \frac{1}{q} = 1.$$

By the convergence of the series $\sum\limits_{|\mu_k| \geq 1} \frac{1}{|\mu_k|^p}$, $p > 1$ of the system $\left\{ \theta_{k+1} v_{k+1} \|v_{k+1}\|_q^{-1} \right\}$, $|\mu_k| \geq 1$ (see [9]), we get

$$\sum_{|\mu_k| \geq 1} \frac{\theta_{k+1}}{|\mu_k|^2} |(v_{k+1}, f)| |u_k(x)| \leq const \|f\|_p .$$

Consequently, the series $\sum\limits_{k=1}^{\infty} |(f, v_k)| |u_k(x)|$ converges uniformly with respect to $x \in \overline{G}$. Prove the validity of representation (6). Let the series in the right hand side of (6) converge on $\overline{G}$ to some function $g(x)$. By the uniformity of the convergence the function $g(x)$ will be continuous on each of intervals $[\xi_0, \xi_1]$, $(\xi_{l-1}, \xi_l]$, $l = \overline{2, m}$. From the representation

$$g(x) = \sum_{k=1}^{\infty} (f, v_k) u_k(x) , \quad x \in \overline{G}$$

we get that $(f, v_k) = (g, v_k)$, $k = 1, 2, ...$ Hence, by the completeness of the system $\{v_k(x)\}$ in $L_2(G)$, it follows that $f(x) = g(x)$ almost everywhere at each of intervals $[\xi_0, \xi_1]$, $(\xi_{l-1}, \xi_l]$, $l = \overline{2, m}$. Since the both functions $f(x)$ and $g(x)$ are continuous in these intervals, they coincide everywhere on $\overline{G}$. Consequently, representation (6) is true.

Now, establish estimation (7) for $1 < p \le 2$. It follows from representations (6) and (11) that for each $x \in \overline{G}$

$$|R_\nu(x, f)| = |f(x) - \sigma_\nu(x, f)| \le \sum_{Re\mu_k > \nu} |(f, v_k)| \, |u_k(x)| \le$$

$$\le \sum_{|\mu_k| \ge \nu} |(f, v_k)| \, |u_k(x)| \le \sum_{|\mu_k| \ge \nu} \frac{\overline{|Q_m(f, v_k)|}}{|\mu_k|^2} |u_k(x)| +$$

$$+ \sum_{|\mu_k| \ge \nu} \frac{1}{|\mu_k|^2} \left|(v'_k, f')\right| |u_k(x)| + \sum_{|\mu_k| \ge \nu} \frac{1}{|\mu_k|^2} (v_k, qf) \, |u_k(x)| +$$

$$+ \sum_{|\mu_k| \ge \nu} \frac{\theta_{k+1}}{|\mu_k|^2} |(v_{k+1}, f)| \, |u_k(x)| \le const C_1(f) \, \nu^{-1} +$$

$$+ const \nu^{-1/q} \left\|f'\right\|_p + const \, \|q\|_1 \, \|f\|_\infty \, \nu^{-1} + const \nu^{-1/q} \, \|f\|_p \le$$

$$\le const \left\{ \nu^{-1/q} \, \|f\|_{\widetilde{W}_p^1(G)} + \nu^{-1} \left[C_1(f) + \|q\|_1 \, \|f\|_\infty\right] \right\}.$$

Prove estimation (8) for $1 < p \le 2$. For that, we again use representation (6), (11) and behave as in the estimation of the right hand side of relation (16).

$$|R_\nu(x, f)| = |f(x) - \sigma_\nu(x, f)| = \left| \sum_{Re\mu_k > \nu} (f, v_k) \, u_k(x) \right| \le$$

$$\le \sum_{|\mu_k| \ge \nu} |(f, v_k)| \, |u_k(x)| \le (CC_1(f) + \|q\|_1 \, \|f\|_\infty) \, \nu^{-1} +$$

$$+ C \left(\sum_{|\mu_k| \ge \nu} \frac{1}{|\mu_k|^p} \right)^{1/p} \left(\sum_{|\mu_k| \ge \nu} \left| \left(f', v'_k \, \|v_k\|_q^{-1} \, |\mu_k|^{-1}\right) \right|^q \right)^{1/q} +$$

$$+ \left(\sum_{|\mu_k| \ge \nu} \frac{1}{|\mu_k|^p} \right)^{1/p} \left(\sum_{|\mu_k| \ge \nu} \left| \left(f', \frac{\theta_{k+1} v_{k+1} \, \|v_{k+1}\|_q^{-1}}{|\mu_k|}\right) \right|^q \right)^{1/q} =$$

$$= O\left(\nu^{-1}\right) + O\left(\nu^{-1/q}\right) \left(\sum_{|\mu_k| \ge \nu} \left| \left(f', v'_k \, \|v_k\|_q^{-1} \, |\mu_k|^{-1}\right) \right|^q \right)^{1/q} +$$

$$+O\left(\nu^{-1/q}\right)\left(\sum_{|\mu_k|\geq\nu}\left|\left(f',\theta_{k+1}v_{k+1}\|v_{k+1}\|_q^{-1}\right)\right|^q\right)^{1/q}.$$

Since the residual of the converging series tends to zero, then

$$\left(\sum_{|\mu_k|\geq\nu}\left|\left(f',v_k'\|v_k\|_q^{-1}|\mu_k|^{-1}\right)\right|^q\right)^{1/q}=o(1)$$

$$\left(\sum_{|\mu_k|\geq\nu}\left|\left(f',\theta_{k+1}v_{k+1}\|v_{k+1}\|_q^{-1}\right)\right|^q\right)^{1/q}=o(1),$$

as $\nu\to\infty$.

Consequently, for any $x\in\overline{G}$ and $1<p\leq 2$

$$|R_\nu(x,f)|=O\left(\nu^{-1}\right)+O\left(\nu^{-1/q}\right)o(1)=o\left(\nu^{-1/q}\right),\quad \nu\to+\infty$$

in the case $1<p\leq 2$. Theorem 1 is proved.

The case $p>2$ is reduced to the case $p=2$, or $\widetilde{W}_p^1(G)$ is embedded into $\widetilde{W}_2^1(G)$. Theorem 1 is completely proved.

Proof of theorem 2.

Let $f(x)\in\widetilde{W}_1^1(G)$. Prove the uniform convergence of the series $\sum\limits_{k=1}^{\infty}|(f,v_k)||u_k(x)|$ on $\overline{G}$. For that, we again represent it in the from (see the proof of theorem 1)

$$\sum_{k=1}^{\infty}|(f,v_k)||u_k(x)|=\sum_{0\leq Re\mu_k<\alpha}+\sum_{Re\mu_k\geq\alpha}\leq C_1\|f\|_1+$$

$$+\sum_{Re\mu_k\geq\alpha}|(f,v_k)||u_k(x)|,$$

where $\alpha=\max\limits_{1\leq l\leq m}\frac{4\pi}{|G_l|}$.

For investigating a uniform convergence in $\overline{G}$ of the series

$$\sum_{Re\mu_k\geq\alpha}|(f,v_k)||u_k(x)|$$

we use representation (12) for the Fourier coefficient (f,v_k). This leads us to studying the uniform convergence of the series

$$\sum_{Re\mu_k\geq\alpha}\sum_{j=0}^{n_k}\frac{\overline{|Q_m(f,v_{k+j})|}}{|\mu_k|^{2(j+1)}}|u_k(x)|,\tag{18}$$

$$\sum_{Re\mu_k \geq \alpha} \sum_{j=0}^{n_k} \frac{|(v_{k+j}, qf)|}{|\mu_k|^{2(j+1)}} |u_k(x)|, \tag{19}$$

$$\sum_{Re\mu_k \geq \alpha} \sum_{j=0}^{n_k} \sum_{l=1}^{m} \frac{\left|v'_{k+j}(\xi_{l-1})\right|}{|\mu_k|^{2(j+1)}} |u_k(x)| \left|\left(\cos \overline{\mu}_k (t - \xi_{l-1}), f'(t)\right)_{G_l}\right|, \tag{20}$$

$$\sum_{Re\mu_k \geq \alpha} \sum_{j=0}^{n_k} \sum_{l=1}^{m} \frac{|v_{k+j}(\xi_{l-1})|}{|\mu_k|^{2j+1}} |u_k(x)| \left|\left(\sin \overline{\mu}_k (t - \xi_{l-1}), f'(t)\right)_{G_l}\right|, \tag{21}$$

$$\sum_{Re\mu_k \geq \alpha} \sum_{j=0}^{n_k} \sum_{l=1}^{m} \frac{|u_k(x)|}{|\mu_k|^{2(j+1)}} \times$$

$$\times \left| \int_{\xi_{l-1}}^{\xi_l} \overline{q(\tau)} v_{k+j}(\tau) \left(\int_{\xi_{l-1}}^{\xi_l} \overline{f'(t)} \cos \overline{\mu}_k (t - \tau)\, dt \right) d\tau \right|, \tag{22}$$

$$\sum_{Re\mu_k \geq \alpha} \sum_{j=0}^{n_k} \sum_{l=1}^{m} \frac{|\theta_{k+j+1} u_k(x)|}{|\mu_k|^{2(j+1)}} \times$$

$$\times \left| \int_{\xi_{l-1}}^{\xi_l} v_{k+j+1}(\tau) \left(\int_{\tau}^{\xi_l} \overline{f'(t)} \cos \overline{\mu}_k (t - \tau)\, dt \right) d\tau \right|. \tag{23}$$

By inequalities (2), (3), (6), (14), (17) and (4) the series (18) is majorized by the number series

$$C_1(f) \sum_{Re\mu_k \geq \alpha} \frac{1}{|\mu_k|^2},$$

that converges by condition (3).

By inequalities (2), (3), (15), (17) and (4) series (19) is majorized by the converging number series

$$C \|q\|_1 \|f\|_\infty \sum_{Re\mu_k \geq \alpha} \frac{1}{|\mu_k|^2}.$$

Since (see [4])

$$\left|v'_{k+j}(\xi_{l-1})\right| \leq C |\mu_k| \|v_{k+j}\|_2$$

then by (2), (3), (14), (17) and (4) series (20) is majorized by the number series

$$C \sum_{Re\mu_k \geq \alpha} \frac{1}{|\mu_k|} \sum_{l=1}^{m} \left|\left(\cos \overline{\mu}_k (t - \xi_{l-1}), f'(t)\right)_{G_l}\right|. \tag{24}$$

Obviously, the system $\{\cos\overline{\mu}_k(t-\xi_{l-1})\}_{k=1}^{\infty}$ is a system of eigenfunctions of the operator $-y=\overline{\mu}^2 y$. Therefore, by lemma 7 of the paper [9]

$$\left|\left(\cos\overline{\mu}_k(t-\xi_{l-1}), f'(t)\right)_{G_l}\right| \le C\left[\omega_1\left(f', \frac{1}{Re\mu_k}\right)+\frac{\|f'\|_1}{Re\mu_k}\right]. \tag{25}$$

Taking this into account in (25), we get that series (20) is majorized by the number series

$$C\sum_{Re\mu_k\ge\alpha}\frac{1}{Re\mu_k}\left[\omega_1\left(f', \frac{1}{Re\mu_k}\right)+\frac{\|f'\|_1}{Re\mu_k}\right],$$

that converges by conditions (3) and (10). Allowing for estimation (17), a uniform convergence on $\overline{G}$ of series (21) is proved in the same way. It follows from (25) that

$$\left|\int_{\tau}^{\xi_l}\overline{f'(t)}\cos\overline{\mu}_k(t-\tau)\,dt\right| \le C\left\{\omega_1\left(g_\tau, \frac{1}{Re\mu_k}\right)_{G_l}+\frac{\|f'\|_1}{Re\mu_k}\right\}, \quad l=\overline{1,m},$$

where

$$g_\tau(z)=\begin{cases} f'(z), & z\ge\tau \\ 0, & z<\tau \end{cases} \quad \xi_{l-1}\le\tau\le\xi_l.$$

On the other hand, (see [6]) it is known that for $Re\mu_k\ge\frac{4\pi}{|G_l|}$

$$\omega_1\left(g_\tau, \frac{1}{Re\mu_k}\right)_{G_l} \le const\left\{\omega_1\left(f', \frac{1}{Re\mu_k}\right)+\frac{\|f'\|_1}{Re\mu_k}\right\}.$$

Consequently,

$$\left|\int_{\tau}^{\xi_l}\overline{f'(t)}\cos\mu_k(t-\tau)\,dt\right| \le \left\{\omega_1\left(f', \frac{1}{Re\mu_k}\right)+\frac{\|f'\|_1}{Re\mu_k}\right\}. \tag{26}$$

Taking into account (26), (2), (3), (14), (15), (17) and inequality (4), we majorize series (22) from above by the number series

$$C\sum_{Re\mu_k\ge\alpha}\frac{1}{Re\mu_k}\left[\omega_1\left(f', \frac{1}{Re\mu_k}\right)+\frac{\|f'\|_1}{Re\mu_k}\right],$$

that converges by (3) and (9).

For establishing uniforn convergence of series (23) it suffices to take into account

$$\left|\int_{\xi_{l-1}}^{\xi_l}\overline{q(t)}v_{k+j}(\tau)\left(\int_{\tau}^{\xi_l}\overline{f'(t)}\cos\overline{\mu}_k(t-\tau)\,dt\right)d\tau\right| \le$$

$$\le C\|q\|_1\left\|f'\right\|_1\sup_{G_l}|v_{k+j}(x)|$$

and apply the estimations (14), (15), (17) and (4). As a result, series (23) will be majorized from above by a converging series

$$C\left\|f'\right\|_1 \|q\|_1 \sum_{Re\mu_k\geq\alpha}\frac{1}{|\mu_k|^2}.$$

Thus, the series

$$\sum_{k=1}^{\infty}|(f,v_k)|\,|u_k(x)|$$

converges uniformly on $\overline{G}$.

By unconditional basicity in $L_2(G)$ of the system $\{u_k(x)\}$ the following representation will be walid:

$$f(x)=\sum_{k=1}^{\infty}(f,v_k)\,u_k(x).$$

Now, estimate the residual $R_\nu(x,f)$, $x\in\overline{G}$, $\nu\geq\frac{4\pi}{\min\limits_{1\leq l\leq m}\{|G_l|\}}$:

$$|R_\nu(x,f)|\leq\sum_{Re\mu_k\geq\nu}|(f,v_k)|\,|u_k(x)|.$$

Substituting here the expression (f,v_k), from (12) we conclude that for estimating the residual $R_\nu(x,f)$ it is enough to estimate the residuals of the series (18)-(23). Therefore, in the estimations obtained above for the series (18)-(23) we must put $a=\nu$. Consequently, the inequality

$$|R_\nu(x,f)|\leq CC_1(f)\sum_{Re\mu_k\geq\nu}\frac{1}{|\mu_k|^2}+C\|q\|_1\|f\|_\infty+$$

$$+\sum_{Re\mu_k\geq\nu}\frac{1}{|\mu_k|^2}+C\sum_{Re\mu_k\geq\nu}\frac{1}{Re\mu_k}\left[\omega_1\left(f',\frac{1}{Re\mu_k}\right)+\frac{\|f'\|_1}{Re\mu_k}\right]+$$

$$+C\left\|f'\right\|_\infty\|q\|_1\sum_{Re\mu_k\geq\nu}\frac{1}{|\mu_k|^2}$$

will be valid. Here, taking into account

$$\sum_{Re\mu_k\geq\nu}\frac{1}{|\mu_k|^2}\leq\sum_{Re\mu_k\geq\nu}\frac{1}{(Re\mu_k)^2}=O\left(\nu^{-1}\right),$$

$$\sum_{Re\mu_k\geq\nu}\frac{1}{Re\mu_k}\omega_1\left(f',\frac{1}{Re\mu_k}\right)\leq$$

$$\leq\sum_{n=[\nu]}^{\infty}\sum_{n\leq Re\mu_k\leq n+1}\frac{1}{Re\mu_k}\omega_1\left(f',\frac{1}{Re\mu_k}\right)\leq$$

$$\leq \sum_{n=[\nu]}^{\infty}\frac{1}{n}\omega_1\left(f',\frac{1}{n}\right)\sum_{n\leq Re\mu_k\leq n+1}1\leq C\sum_{n=[\nu]}^{\infty}\frac{1}{n}\omega_1\left(f',\frac{1}{n}\right),$$

we get

$$\sup_{x\in\overline{G}}\left|R_\nu\left(x,f\right)\right|\leq const\left\{\nu^{-1}C_1\left(f\right)+\left\|q\right\|_1\left\|f\right\|_\infty+\left\|f'\right\|_1\left[1+\left\|q\right\|_1\right]\right\}+$$

$$+\sum_{n=[\nu]}^{\infty}n^{-1}\omega_1\left(f',n^{-1}\right)\leq constK\left(\nu,f\right)$$

Theorem 2 proved.

Remark 3. *It is seen from the proof of theorems 1 and 2 that if the system* $\{u_k(x)\}$ *is not biorthogonally adjoint to* $\{u_k(x)\}$ *and the remaining conditions of these theorems are fulfilled, then the*

$$\sum_{k=1}^{\infty}\left(f,v_k\right)u_k\left(x\right)$$

uniformly and absolutely converges on $\overline{G}$ *and estimations (7), (8) and (10) are valid for the residual* $\sum\limits_{Re\mu_k\geq\nu}\left(f,v_k\right)u_k\left(x\right)$.

4. Some applications of the theorems 1 and 2.

1. Consider the operator $Lu=u''$ on $G=\left(0,\frac{1}{2}\right)\cup\left(\frac{1}{2},1\right)$ with the conditions $u\left(0\right)=0$, $u'\left(1\right)=0$, $u\left(\frac{1}{2}+0\right)-u\left(\frac{1}{2}\right)=u\left(1\right)$, $u'\left(\frac{1}{2}+0\right)-u'\left(\frac{1}{2}-0\right)=0$.

For this problem, it is well known [1] that $\mu_k=4\pi k$ for $k=0,1,2,...$ and $\widetilde{\mu}_k=4\pi k/3$ for $k=1,2,4,5,6,7,8...$ (k is not multiply of 3). The eigenvalue $\lambda_0=\mu_0^2=0$ is associated with the single eigenfunction

$$\overset{\circ}{u}_0\left(x\right)=\begin{cases}0 & \text{if } 0\leq x\leq\frac{1}{2},\\ \frac{8}{3} & \text{if } \frac{1}{2}<x\leq 1.\end{cases}$$

Each eigenvalue $\lambda_k=\mu_k^2=\left(4\pi k\right)^2$, $k=1,2,...$ corresponds to a single eigen-function and a single associated function:

$$\overset{\circ}{u}_k\left(x\right)=\begin{cases}0 & \text{if } 0\leq x\leq\frac{1}{2},\\ 32\cos\left(4\pi k\left(1-x\right)\right) & \text{if } \frac{1}{2}<x\leq 1\end{cases}$$

$$\overset{1}{u}_k\left(x\right)=\begin{cases}-\frac{2}{3\pi k}\sin\left(4kx\right) & \text{if } 0\leq x\leq\frac{1}{2},\\ \frac{4}{3\pi k}\left(1-x\right)\sin\left(4\pi k\left(1-x\right)\right) & \text{if } \frac{1}{2}<x\leq 1.\end{cases}$$

Each eigenvalue $\widetilde{\lambda}_k=\left(\widetilde{\mu}_k\right)^2=\left(\frac{4\pi k}{3}\right)^2$, $k=1,2,4,5,6,7,8...$ corresponds to a single eigenfunction

$$\overset{0}{\widetilde{u}}_k\left(x\right)=\begin{cases}\frac{8}{3}\sin\left(\frac{4\pi k}{3}x\right) & \text{if } 0\leq x\leq\frac{1}{2},\\ \frac{8}{3}ctg\frac{2l\pi}{3}\cos\left(\frac{4\pi k}{3}\left(1-x\right)\right) & \text{if } \frac{1}{2}<x\leq 1.\end{cases}$$

The biorthogonally adjoint system consists of root functions of the operator $Lv = v''$ on (0,1) with the conditions $v(0) = 0$ and $v'(1) - v'\left(\frac{1}{2}\right) = 0$.

The corresponding eigen and associated functions have the form

$$\overset{0}{v}_0(x) = x; \quad \overset{0}{v}_k(x) = 4\pi k \sin(4\pi k x);$$

$$\overset{1}{v}_k(x) = -\frac{x}{2}\cos(4\pi k x), \quad k = 1, 2, 3, ...;$$

$$\overset{0}{\widetilde{v}}_k(x) = \sin(4\pi k x/3), \quad k = 1, 2, 4, 5, 6, 7, 8....$$

For the problem in question,

$$Q_2(f, v) = -f(0)\overline{v'(0)} + \left[f\left(\frac{1}{2}\right) - f\left(\frac{1}{2} + 0\right) + f(1)\right]\overline{v'\left(\frac{1}{2}\right)}.$$

Condition (5) implies that

$$f(0) = 0, f\left(\frac{1}{2} + 0\right) - f\left(\frac{1}{2}\right) = f(1). \tag{27}$$

Since conditions 1) - 3) are fulfilled for this example,Theorem 1 holds for any function $f(x)$ from $\widetilde{W}_p^1(G)$, $p > 1$ that satisfies conditions (27). Moreover. Theorem 2 holds for any function $f(x)$ from $\widetilde{W}_1^1(G)$, that satisfies conditions (9), (27).

2. Consider the eigenvalue problem

$$u''(x) + \lambda u(x) = 0, \quad x \in (-1, 0) \cup (0, 1) \tag{28}$$

$$\begin{gathered} u(-1) = u(1) = 0 \\ u(-0) = u(+0) \\ u'(-0) - u'(+0) = \lambda m u(0), m \neq 0. \end{gathered} \tag{29}$$

It is well known [7] that eigenvalue of this problem are simple and form two series

$$\lambda_{1,k} = \mu_{1,k}^2 = (\pi k)^2, \quad k = 1, 2, ...;$$

$$\lambda_{2,k} = \mu_{2,k}^2, \ \ \mu_{2,k} = \pi k + \frac{2}{\pi m k} + O(k^{-2}), k = 0, 1, ...$$

The eigenfunctions have the form

$$u_{2k-1}(x) = \sin \pi k x, \quad k = 1, 2, ...,$$

$$u_{2k}(x) = \begin{cases} \sin \mu_{2,k}(1 + x) & \text{if} \quad x \in [-1, 0], \\ \sin \mu_{2,k}(1 - x) & \text{if} \quad x \in (0, 1], \quad k = 0, 1, ... \end{cases}$$

The system $\{u_k(x)\}_{k=0, k\neq k_0}^{\infty}$, where k_0 is an arbitrary fixed even number, forms a basis in $L_p(-1, 1)$ $1 < p < \infty$ (a Riesz basis for $p = 2$).

The biorthogonally adjoint system $\{u_k(x)\}_{k=0,k\neq k_0}^{\infty}$ has the form

$$v_k(x) = \varphi_k(x) - \frac{\varphi_k(0)}{\varphi_{k_0}(0)}\varphi_{k_0}(x),$$

where

$$\varphi_{2k-1}(x) = \sin \pi k x, \quad k = 1, 2, ...,$$

$$\varphi_{2k}(x) = \begin{cases} c_{2k} \sin \mu_{2,k}(1+x) & \text{if} \quad x \in [-1, 0] \\ c_{2k} \cos \mu_{2,k}(1-x) & \text{if} \quad x \in [0, 1] \end{cases}$$

$$c_{2k} = 1 + O\left(k^{-2}\right).$$

Obviously, the system $\{u_k(x)\}_{k=0,k\neq k_0}^{\infty}$ satisfies conditions Remark 3. For problem (28)-(29) the application of Theorems 1 and 2 (Remark 3) leads to the conditions

$$f(-1) = f(1) = f(0) = f(+0) = 0 = (f, \varphi_{k_0}). \tag{30}$$

Therefore statements of Theorem 1 holds for any function $f(x) \in \widetilde{W}_p^1(G)$, $p > 1$ satisfying conditions (30),while statements of Theorem 2 holds for any function $f(x) \in \widetilde{W}_1^1(G)$ satisfying condition (9) and (30).

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