Caspian Journal of Applied Mathematics, Ecology and Economics
V. 10, No 2, 2022, December
ISSN 1560-4055

Nodal Solutions of Certain Boundary Value Problem for Fourth Order Ordinary Differential Equations

N.A. Neymatov, R.G. Poladov

Abstract. In this paper, we consider a nonlinear boundary value problem for fourth order ordinary differential equations. We show the existence of two different solutions of this problem with a fixed number of simple zeros.

Key Words and Phrases: nonlinear problem, eigenvalue, eigenfunction, bifurcation point, simple zero, global continua

2010 Mathematics Subject Classifications: 34B05, 34B15, 34C23, 47H11, 47J10, 47J15

---

1. Introduction

We consider the following nonlinear boundary value problem

$$\ell(y) \equiv (p(x)y'')'' - (q(x)y')' = \varrho\,\tau(x)g(y(x)),\ x \in (0, l), \tag{1.1}$$

$$\begin{array}{l} y'(0)\cos\alpha - (py'')(0)\sin\alpha = 0, \\ y(0)\cos\beta + Ty(0)\sin\beta = 0, \\ y'(l)\cos\gamma + (py'')(l)\sin\gamma = 0, \\ y(l)\cos\delta - Ty(l)\sin\delta = 0, \end{array} \tag{1.2}$$

$Ty \equiv (py'')' - qy'$, $p \in C^2([0, l]; (0, +\infty))$, $q \in C^1([0, l]; [0, +\infty))$, $\tau \in C([0, l]; (0, +\infty))$, ϱ is a positive parameter, and α, β, γ, δ are real constants such that $0 \le \alpha$, β, γ, $\delta \le \frac{\pi}{2}$ (the cases $\alpha = \gamma = 0$, $\beta = \delta = \pi/2$ and $\alpha = \beta = \gamma = \delta = \pi/2$ are excluded). The nonlinear term $g \in C^0\left([0, l] \times \mathbb{R}^5; \mathbb{R}\right)$ and satisfies the following conditions:

(B1) $sg(s) > 0$ for any $s \in \mathbb{R}$, $s \ne 0$;

(B2) there exist positive constants g_0 and g_∞ such that

$$g_0 = \lim_{|s|\to 0} \frac{g(s)}{s},\ g_\infty = \lim_{|s|\to+\infty} \frac{g(s)}{s}.$$

It is well known that nonlinear boundary value problems for fourth-order ordinary differential equations arise when modeling various problems in mechanics, physics, and various areas of natural science, see [1, 3, 5] and references therein. Note that there are many papers that are devoted to the existence of positive or sign-changing solutions to such problems addressed by using different methods [1-4, 5-16].

Problems similar to the problem (1.1), (1.2) for ordinary differential equations of second order was considered in [6, 7, 10, 11]. In these papers, using the results on the global bifurcation from zero and infinity of nonlinear Sturm-Liouville problems obtained by Rabinowitz [14, 15], Berestycki [4] and Rynne [16], intervals were found such that for each value of the parameter $\varkappa$ from these intervals, the problems under consideration have solutions with fixed usual nodal properties. Note that special cases of the boundary value problem (1.1), (1.2) were considered in [6, 8, 9, 12, 13], where the values of the parameter $\varkappa$ were found, for which the problems have positive and negative solutions [6, 8, 13] or have solutions with a certain number of sign change nodes [9, 12].

In the present paper, we will find an interval for $\varkappa$, in which there are solutions to problem (1.1), (1.2) with a fixed number of oscillations.

The rest of this paper is arranged as follows. In Section 2, we study the behavior of the nonlinear term g on the neighborhoods of zero and infinity using the conditions from B2. In Section 3, using global bifurcation results of [1, 2], we determine the interval of $\varkappa$, in which there exist nodal solutions of nonlinear boundary value problem (1.1), (1.2).

2. Preliminary and some properties of function g

Let BC be the set of functions that satisfy the boundary conditions (1.2), and let $E = C^3([0,1];\mathbb{R}) \cap BC$ be the Banach space with the norm

$$||y||_3 = \sum_{s=0}^{3} ||y^{(s)}||_\infty, \; ||y||_\infty = \max_{x \in [\,0,l]} |y(x)|.$$

By the first part of (B2) we have the following representation

$$g(s) = g_0 + \tilde{g}_0(s), \tag{2.1}$$

where

$$\lim_{|s| \to 0} \frac{\tilde{g}_0(s)}{s} = 0. \tag{2.2}$$

Lemma 2.1. *The following relation holds:*

$$||\tilde{g}_0(y)||_\infty = o(||y||_3) \text{ as } ||y||_3 \to 0. \tag{2.3}$$

Proof. Let

$$\chi(y) = \max_{0 \le |s| \le y} |\tilde{g}_0(s)|.$$

Then $\chi(y)$ is a nondecreasing function on $(0, +\infty)$. Moreover, by (2.2) we have the relation

$$\lim_{y \to 0+} \frac{\chi(y)}{y} = 0. \tag{2.4}$$

Since χ is nondecreasing we get the following inequalities

$$\frac{\tilde{g}_0(y)}{||y||_3} \leq \frac{\chi(|y|)}{||y||_3} \leq \frac{\chi(||y||_\infty)}{||y||_3} \leq \frac{\chi(||y||_3)}{||y||_3},$$

whence, with regards (2.4), implies that

$$\frac{||\tilde{g}_0(y)||_\infty}{||y||_3} \to 0 \text{ as } ||y||_3 \to 0.$$

The proof of this lemma is complete.

By the second part of (B2) we have the following representation

$$g(s) = g_\infty + \tilde{g}_\infty(s), \tag{2.5}$$

where

$$\lim_{|s| \to +\infty} \frac{\tilde{g}_\infty(s)}{s} = 0. \tag{2.6}$$

Lemma 2.2. *One has t following relation:*

$$||\tilde{g}_\infty(y)||_\infty = o(||y||_3) \text{ as } ||y||_3 \to \infty. \tag{2.7}$$

Proof. We define the function $\sigma(y)$, $y \in (0, +\infty)$, as follows:

$$\sigma(y) = \max_{0 \leq |s| \leq y} |\tilde{g}_\infty(s)|.$$

Then $\sigma(y)$ is a nondecreasing function on $(0, +\infty)$. Moreover, it follows from (2.6) that

$$\lim_{y \to +\infty} \frac{\sigma(y)}{y} = 0. \tag{2.8}$$

Since χ is nondecreasing the following relations hold:

$$\frac{\tilde{g}_\infty(y)}{||y||_3} \leq \frac{\sigma(|y|)}{||y||_3} \leq \frac{\sigma(||y||_\infty)}{||y||_3} \leq \frac{\chi(||y||_3)}{||y||_3},$$

Hence, due to (2.8), we obtain

$$\frac{||\tilde{g}_0(y)||_\infty}{||y||_3} \to 0 \text{ as } ||y||_3 \to 0.$$

The proof of this lemma is complete.

Lemma 2.3. *There exist a positive constants c_0 and c_∞ such that*

$$c_0 \leq \frac{g(s)}{s} \leq c_\infty \text{ for } s \in \mathbb{R},\, s \neq 0. \tag{2.9}$$

Proof. Let $\epsilon_0 > 0$ be a sufficiently small fixed number such that

$$g_0 - \epsilon_0 > 0 \text{ and } g_\infty - \epsilon_0 > 0.$$

Then, by (B2), there exist a sufficiently small number $b_0 > 0$ and a sufficiently large number $d_\infty > 0$ such that

$$g_0 - \epsilon_0 < \frac{g(s)}{s} < g_0 + \epsilon_0 \ \text{ for } |s| < d_0, \ s \neq 0 \tag{2.10}$$

and

$$g_\infty - \epsilon_0 < \frac{g(s)}{s} < g_\infty + \epsilon_0 \ \text{ for } |s| > d_\infty. \tag{2.11}$$

Moreover, since $\frac{f(s)}{s}$ is continuous on $[-d_\infty, -d_0] \cup [d_0, d_\infty]$, there are positive constants κ_0 and κ_∞ such that

$$\kappa_0 \leq \frac{f(s)}{s} \leq \kappa_\infty \ \text{ for } d_0 \leq |s| \leq d_\infty. \tag{2.12}$$

Let

$$c_0 = \min\{g_0 - \epsilon_0, \, g_\infty - \epsilon_0, \, \kappa_0\} \text{ and } c_\infty = \min\{g_0 + \epsilon_0, \, g_\infty + \epsilon_0, \, \kappa_\infty\}.$$

Then (2.9) follows directly from (2.10)-(2.12). The proof of this lemma is complete.

3. Main results

We consider the following linear eigenvalue problem

$$\begin{cases} \ell(y)(x) = \lambda \tau(x) y(x), \ x \in (0, l), \\ y \in BC. \end{cases} \tag{3.1}$$

where $\lambda \in \mathbb{C}$ is a spectral parameter. By [3, Theorems 5.4, 5.5] the eigenvalues of problem (3.1) are positive, simple and form an infinitely increasing sequence $\{\lambda_k\}_{k=1}^\infty$. Moreover, for each $k \in \mathbb{N}$ the eigenfunction y_k corresponding to the eigenvalue λ_k has exactly $k-1$ simple zeros.

To study the global bifurcation of solutions of nonlinear perturbations of problem (3.1), in [1] the author constructed classes S_k^ν, $k \in \mathbb{N}$, $\nu \in \{+, -\}$, of functions in E that have the oscillation properties of eigenfunctions (and their derivatives) of the linear problem (3.1). Note that the sets S_k^+, S_k^- and $S_k = S_k^+ \cap S_k^-$ are pairwise disjoint open subsets of E, and if $y \in \partial S_k^\nu$ (∂S_k), then by [1, Lemma 3.1] y has at least one zero of multiplicity 4 in $(0, l)$.

Alongside the boundary-value problem (1.1), (1.2) we shall consider the following nonlinear eigenvalue problem

$$\begin{cases} \ell(y)(x) = \lambda \varrho \, \tau(x) g(y(x)), \ x \in (0, l), \\ y \in BC. \end{cases} \tag{3.2}$$

It is obvious that any solution of (3.2) of the form $(1, y)$ yields a solution y of (1.1), (1.2). Below we will show that for some $k \in \mathbb{N}$ and every $\nu \in \{+, -\}$ there exists a solution to problem (1.1), (1.2) of the form $(1, y)$ with $y \in S_k^\nu$.

Theorem 3.1. *For each $k \in \mathbb{N}$ and each $\nu \in \{+\,,\,-\}$ there exists a continuum $C^{\nu}_{k,\,0}$ of solutions of problem (1.1), (1.2), containing $\left(\frac{\lambda_k}{\varrho g_0}, 0\right)$ is unbounded in $\mathbb{R} \times E$ and lies in $(\mathbb{R} \times S^{\nu}_k) \cup \left\{\left(\frac{\lambda_k}{\varrho g_0}, 0\right)\right\}$.*

Proof. By (2.1) problem (3.2) takes the following form:

$$\begin{cases} \ell(y)(x) = \lambda\varrho\, g_0\, \tau(x)y(x) + \lambda\varrho\, \tau(x)\tilde{g}_0(y(x)), \;\; x \in (0, l), \\ y \in BC. \end{cases} \tag{3.3}$$

It follows from (2.1) that

$$||\lambda\varrho\, \tau\, \tilde{g}_0(y)||_{\infty} = o(||y||_3) \text{ as } ||y||_3 \to 0, \tag{3.4}$$

uniformly in $\lambda \in \Lambda$ for any bounded interval $\Lambda \subset \mathbb{R}$. Hence problem (3.3) is linearizable and the corresponding linear problem

$$\begin{cases} \ell(y)(x) = \lambda\varrho\, g_0\, \tau(x)y(x), \;\; x \in (0, l), \\ y \in BC. \end{cases} \tag{3.5}$$

possesses infinitely many eigenvalues $\tilde{\lambda}_1 < \tilde{\lambda}_2 < \ldots < \tilde{\lambda}_k \to +\infty$, all of which are simple. The eigenfunction $\tilde{y}_k$ corresponding to $\tilde{\lambda}_k$ lies in S_k. Moreover, from (3.5) it can be seen that $\tilde{\lambda}_k = \frac{\lambda_k}{\varrho g_0}$. Then it follows from [1, Theorem 1.1] that for each $k \in \mathbb{N}$ and each $\nu \in \{+\,,\,-\}$ there exists a continuum $C^{\nu}_{k,\,0}$ of solutions of problem (3.3) which contains $(\tilde{\lambda}_k, 0)$, lies in $(\mathbb{R} \times S^{\nu}_k) \cup \{(\tilde{\lambda}_k, 0)\}$ and is unbounded in $\mathbb{R} \times E$. The proof of this theorem is complete.

By (2.5) problem (3.2) can be rewritten in the following form:

$$\begin{cases} \ell(y)(x) = \lambda\varrho\, g_{\infty}\, \tau(x)y(x) + \lambda\varrho\, \tau(x)\tilde{g}_{\infty}(y(x)), \;\; x \in (0, l), \\ y \in BC. \end{cases} \tag{3.6}$$

Then, using (2.7) and following the appropriate reasoning in the proof of Theorem 2.1, from [2, Theorem 3.1] we obtain the following result.

Theorem 3.2. *For each $k \in \mathbb{N}$ and each ν there exist a continuum $C^{\nu}_{k,\,\infty}$ of solutions of problem (3.6) (or (3.2)) which contains $\left(\frac{\lambda_k}{\varrho g_{\infty}}, \infty\right)$ and has the following properties:*

(i) *there exists a neighborhood Q_k of $\left(\frac{\lambda_k}{\varrho g_{\infty}}, \infty\right)$ in $\mathbb{R} \times E$ such that*

$$Q_k \cap \left(C^{\nu}_{k,\,\infty} \backslash \left\{\left(\frac{\lambda_k}{\varrho g_{\infty}}, \infty\right)\right\}\right) \subset \mathbb{R} \times S^{\nu}_k;$$

(ii) *either $C^{\nu}_{k,\,\infty}$ meets $\left(\frac{\lambda'_k}{\varrho g_{\infty}}, \infty\right)$ through $\mathbb{R} \times S^{\nu'}_{k'}$ for some $(k', \nu') \neq (k, \nu)$, or $C^{\nu}_{k,\,\infty}$ meets $(\lambda, 0)$ for some $\lambda \in \mathbb{R}$, or projection of $C^{\nu}_{k,\,\infty}$ on $\mathbb{R} \times \{0\}$ is unbounded.*

Now we can prove the following very important result.

Theorem 3.3. *For each $k \in \mathbb{N}$ and each ν one has the relation:*

$$C^{\nu}_{k,\,0} = C^{\nu}_{k,\,\infty}. \tag{3.7}$$

Proof. Since g satisfies both conditions B2, by Lemma 3.1 of [1] it follows from [1, Lemma 1.1] that

$$C^{\nu}_{k,\infty} \backslash \left\{ \left(\frac{\lambda_k}{\varrho g_\infty}, \infty \right) \right\} \subset \mathbb{R} \times S^{\nu}_k \tag{3.8}$$

(see also [16, Theorem 3.3]). Consequently, the first part of assertion (ii) of Theorem 3.2 cannot hold. Moreover, it follows from [16, Theorem 3.3] that if $C^{\nu}_{k,\infty}$ meets $\mathbb{R} \times \{0\}$, for some $\lambda \in \mathbb{R}$, then $\lambda = \frac{\lambda_k}{\varrho g_0}$. Similarly, if $C^{\nu}_{k,0}$ meets $\mathbb{R} \times \{\infty\}$, for some $\lambda \in \mathbb{R}$, then $\lambda = \frac{\lambda_k}{\varrho g_\infty}$.

If the third part of assertion (ii) of this theorem holds, then there exists a sequence $\{(\lambda^*_n, y^*_n)\}^{\infty}_{n=1} \subset C^{\nu}_{k,\infty} \backslash Q_k$ such that

$$\lambda^*_n \to \pm\infty \text{ as } n \to \infty. \tag{3.9}$$

Since $(\lambda^*_n, y^*_n) \in \mathbb{R} \times S^{\nu}_k$ it follows from (3.2) that λ^*_n is kth eigenvalue of the linear problem

$$\begin{cases} \ell(y)(x) = \lambda \varrho\, \tau(x) h_n y(x), \ x \in (0, l), \\ y \in BC, \end{cases} \tag{3.10}$$

where

$$h_n(x) = \begin{cases} \frac{g(y^*_n(x))}{y^*_n(x)} & \text{if } y^*_n(x) \neq 0, \\ g_0 & \text{if } y^*_n(x) = 0. \end{cases} \tag{3.11}$$

On the base (3.11) from (2.9) we get

$$0 < c_0 \leq h_n(x) \leq c_\infty, \ x \in [0, l]. \tag{3.12}$$

By (3.12) it follows from [1, relations (4.3) and (4.4)] that the eigenvalues of problems (3.10) are bounded from below uniformly with respect to $n \in \mathbb{N}$, and, consequently, the relation $\lambda^*_n \to -\infty$ is impossible as $n \to \infty$. If $\lambda^*_n \to +\infty$, then for any sufficiently large $n \in \mathbb{N}$ by (3.12) Theorems 5.4 and 5.5 of [3] implies that the number of zeros of the function $y^*_n(x)$ will be sufficiently large, which contradicts the condition $y^*_n \in S^{\nu}_k$. Therefore, the third part of assertion (ii) of Theorem 3.2 does not hold.

Thus, the second part of assertion (ii) of Theorem 3.2 holds. Consequently, $C^{\nu}_{k,\infty}$ meets $(\frac{\lambda_k}{\varrho g_0}, 0)$ and $C^{\nu}_{k,0}$ meets $(\frac{\lambda_k}{\varrho g_\infty}, \infty)$, which, by (3.8), implies that $C^{+}_{k,0} = C^{+}_{k,\infty}$ and $C^{-}_{k,0} = C^{-}_{k,\infty}$ for any $k \in \mathbb{N}$. The proof of this theorem is complete.

For simplicity, we introduce the notation:

$$C^{+}_{k} = C^{+}_{k,0} = C^{+}_{k,\infty}, \ C^{-}_{k} = C^{-}_{k,0} = C^{-}_{k,\infty}, \ k \in \mathbb{N}.$$

By Theorem 3.1 and 3.3, for each $k \in \mathbb{N}$ and each $\nu \in \{+, -\}$ the continuum C^{ν}_k lies in $(\mathbb{R} \times S^{\nu}_k) \cup \left\{ \left(\frac{\lambda_k}{\varrho g_0}, 0 \right) \right\} \cup \left\{ \left(\frac{\lambda_k}{\varrho g_\infty}, \infty \right) \right\}$, and meets $\left(\frac{\lambda_k}{\varrho g_0}, 0 \right)$ and $\left(\frac{\lambda_k}{\varrho g_\infty}, \infty \right)$ in $\mathbb{R} \times E$. It follows from here that if

$$\frac{\lambda_k}{\varrho g_0} < 1 < \frac{\lambda_k}{\varrho g_\infty} \text{ or } \frac{\lambda_k}{\varrho g_\infty} < 1 < \frac{\lambda_k}{\varrho g_0} \tag{3.13}$$

for some $k \in \mathbb{N}$, then the continua C_k^+ and C_k^- intersect the hyperplane $\{1\} \times E$ in $\mathbb{R} \times E$. Consequently, for this k there exist solutions $\hat{y}_k^+$ and $\hat{y}_k^-$ of problem (1.1), (1.2) such that $\hat{y}_k^+ \in S_k^+$ and $\hat{y}_k^- \in S_k^-$.

It is obvious that the conditions from (3.13) are equivalent to the following conditions

$$\frac{\lambda_k}{g_0} < \varrho < \frac{\lambda_k}{g_\infty} \text{ or } \frac{\lambda_k}{g_\infty} < \varrho < \frac{\lambda_k}{g_0}, \tag{3.14}$$

respectively. Therefore, we have proved the following main theorem of this paper.

Theorem 3.4. *Let for some $k \in \mathbb{N}$ condition (3.14) holds. Then there exist solutions $\hat{y}_k^+$ and $\hat{y}_k^-$ of problem (1.1), (1.2) such that $\hat{y}_k^+ \in S_k^+$ and $\hat{y}_k^- \in S_k^-$.*

References

[1] Z.S. Aliyev, Global bifurcation of solutions of certain nonlinear eigenvalue problems for ordinary differential equations of fourth order, *Sb. Math.* **207**(12) (2016), 1625-1649.

[2] Z.S. Aliyev, N.A. Mustafayeva, Bifurcation of solutions from infinity for certain nonlinear eigenvalue problems of fourth-order ordinary differential equations, *Electron. J. Differ. Equ.* **2018** (2018), No. 98, 1-19.

[3] D.O. Banks, G. J. Kurowski, A Prüfer transformation for the equation of a vibrating beam subject to axial forces, *J. Differential Equations* **24**(1) (1977), 57-74.

[4] H. Berestycki, On some nonlinear Sturm-Liouville problems, *J. Differential Equations* **26**(3) (1977), 375-390.

[5] B.B. Bolotin, *Vibrations in technique: Handbook in 6 volumes, The vibrations of linear systems. I*, Engineering Industry, Moscow (1978) [in Russian].

[6] G. Dai, X.Han, Global bifurcation and nodal solutions for fourth-order problems with sign-changing weight, *Appl. Math. Comp.* **219** (2013), 9399-9407.

[7] G. Dai, R. Ma, Bifurcation from intervals for Sturm-Liouville problems and its applications, *Electron. J. Differ. Equ.* 2014(3) (2014), 1-10.

[8] J.R.L. Webb, G. Infante, D. Franco, Positive solutions of nonlinear fourth-order boundary-value problems with local and non-local boundary conditions, *Proc. Roy. Soc. Edinburgh Sect. A* **138**(2) (2008), 427-446.

[9] A.C. Lazer, P.J. McKenna, Global bifurcation and a theorem of Tarantello, *J. Math. Anal. Appl.* **181**(3) (1994), 648-655.

[10] R. Ma and G. Dai, Global bifurcation and nodal solutions for a Sturm-Liouville problem with a nonsmooth nonlinearity, *J. Funct. Anal.* **265** (2013), 1443-1459.

[11] R. Ma, B. Thompson, A note on bifurcation from an interval, *Nonlinear Anal.* **62**(4) (2005), 743-749.

[12] R. Ma, B. Thompson, Nodal solutions for a nonlinear fourth-order eigenvalue problem, *Acta Math. Sin. (Engl. Ser.)* **24**(1) (2008), 27-34

[13] R. Ma, Jia Xu, Bifurcation from interval and positive solutions of a nonlinear fourth-order boundary value problem, *Nonlinear Anal.* **72**(1) (2010), 113-122.

[14] P.H. Rabinowitz, Some global results for nonlinear eigenvalue problems, *J. Funct. Anal.* **7**(3) (1971), 487-513.

[15] P. H. Rabinowitz, On bifurcation from infinity, *J. Differential Equations* **14**(3) (1973), 462-475.

[16] B.P. Rynne, Bifurcation from zero or infinity in Sturm-Liouville problems which are not linearizable, *J. Math. Anal. Appl.* **228**(1) (1998), 141-156.

Nazim A. Neymatov
Ganja State University, Ganja, Azerbaijan
E-mail: nneymatov@yandex.ru

Rovshan G. Poladov
Baku State University, Baku, Azerbaijan
E-mail: r_poladov@mail.ru

Received 27 August 2022
Accepted 29 September 2022