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Basis Property in L_p of Root Functions of Some Fourth-Order Eigenvalue Problem

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Abstract. In this paper, we consider the eigenvalue problem for ordinary differential equations of fourth order, two of the boundary conditions depend on the spectral parameter. We find sufficient conditions under which the system of root functions of this problem forms a defect basis in L_p, $1 < p < \infty$, with a defect number 2.

Key Words and Phrases: eigenvalue problem, spectral parameter, eigenvalue, eigenfunction, defect basis

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1. Introduction

We consider the following eigenvalue problem

$$\ell(y) \equiv y^{(4)}(x) - (q(x)y')' = \lambda y(x)), \ x \in (0, l), \tag{1.1}$$

$$y(0) = y'(0) = 0, \tag{1.2}$$

$$y''(1) - (a\lambda + b)y'(1) = 0, \tag{1.3}$$

$$Ty(1) - c\lambda y(1) = 0, \tag{1.4}$$

where $\lambda \in \mathbb{C}$ is a spectral parameter, $Ty \equiv (py'')' - qy'$, q is nonnegative absolutely continuous function on $[0, 1]$, a, b and c are real constants such that $a > 0$, $b < 0$ and $c > 0$.

The location of eigenvalues on the real axis, the structure of the root subspaces, oscillation properties of eigenfunctions of problem (1.1)-(1.4) and their derivatives was investigated in [10]. Recall that this problem describes the flexural vibrations of a homogeneous rod, in the cross sections of which the longitudinal force acts, the left end is fixed rigidly, and the right end resiliently fastened and on this end an inertial mass is concentrated (see [9, 10]). Obviously, to study this problem of mechanics, we need to study the basic properties of the root functions of problem (1.1)-(1.4) in various function spaces.

Note that the spectral properties, including the basis properties of root functions in the Lebesgue spaces of linear eigenvalue problems for ordinary differential equations of fourth order, were considered in [1-5, 11, 12] (see also references therein). In the present paper, we study the basis properties in the space $L_p(0,1)$, $1<p<\infty$, of root functions of the problem (1.1)-(1.4). We show that, under certain conditions, the system of root functions of this problem, after removing two functions, forms a basis in the space $L_p(0,1)$, $1<p<\infty$, i.e. the system of root functions forms a defect basis in $L_p(0,1)$, $1<p<\infty$, with a defect number 2.

2. Preliminary

Let $H = L_2(0,1) \oplus \mathbb{C}^2$ be a Hilbert space with the scalar product

$$(\hat{y}, \hat{\vartheta})_H = \int_0^1 y(x)\overline{\vartheta(x)}dx + |a|^{-1} m\bar{s} + |c|^{-1} n\bar{t},$$

where

$$\hat{y} = \{y, m, n\} \in H, \ v = \{\vartheta, s, t\} \in H.$$

We define in the space H an operator

$$\hat{L}\hat{y} = \hat{L}\{y, m, n\} = \{\ell(y), y''(1) - by'(1), Ty(1)\}$$

with the domain

$$D(\hat{L}) = \{y = \{y, m, n\} \in H : y \in W_4^2(0,1), \ell(y) \in L_2(0,1), y(0) = y'(0) = 0, \\ m = ay'(1), n = cy(1)\}$$

which is dense in H [12].

Obviously, the operator $L$is well defined. By direct verification, we conclude that problem (1.1)-(1.4) is equivalent to the following spectral problem

$$L\hat{y} = \lambda\hat{y}, \ \hat{y} \in D(L). \tag{2.1}$$

This means that the eigenvalues λ_k, $k \in \mathbb{N}$ of problem (1.1)-(1.4) and problem (2.1) coincide, and there is a one-to-one correspondence between the root functions of these problems

$$y_k \leftrightarrow \hat{y}_k = \{y_k, m_k, n_k\}, \ m_k = ay_k'(1), \ n_k = cy_k(1), k \in \text{N}.$$

The problem (1.1)-(1.4) is strongly regular in the sense of [12]. Therefore, the spectrum of this operator is discrete.

Note that the operator L is a non-self-adjoint, closed, compact resolvent operator in H [6, 7].

Since the operator L is not self-adjoint in H, let us introduce the operator $J : H \to H$ by

$$J\hat{y} = \{y, m, n\} = \{y, m, -n\}.$$

The operator J is unitary and symmetric in H. The spectrum of this operator consists of two eigenvalues: -1 with multiplicity 1 and 1 with infinite multiplicity [8]. Hence this operator generates a Pontryagin space $\Pi_1 = L_2(0,1) \oplus C^2$ (see [6]) whose inner product is defined as

$$(\hat{y}, \hat{\vartheta})_1 = (\{y, m, n\}, \{\vartheta, s, t\})_1 = (J\hat{y}, \hat{\vartheta})_H = \int\limits_0^1 y(x)\overline{\vartheta(x)}dx + a^{-1}m\bar{s} - c^{-1}n\bar{t}.$$

Suppose that the operator L^* is an adjoint to the operator L in H.

By [10, Theorem 2.1] the operator L is self-adjoint in Π_1. In view of [7, Section 3, Proposition 5^0], we have

$$L^* = JLJ. \tag{2.2}$$

By virtue of [7, § 4, Theorem 4.2] the system of root vectors $\{\hat{y}_k\}_{k=1}^{\infty}$, $\hat{y}_k = \{y_k, m_k, n_k\}$, $m_k = ay'_k(1)$, $n_k = cy_k(1)$, of the operator L forms an unconditional basis in H.

We introduce the following boundary condition

$$y(1)\cos\delta - Ty(1)\sin\delta = 0, \tag{2.3}$$

where $\delta \in \left[0, \frac{\pi}{2}\right]$.

Along with problem (1.1)-(1.4) we shall consider the eigenvalue problem (1.1)-(1.3), (2.3). It is known [1] that the eigenvalues of problem (1.1)-(1.3), (2.3) are real and positive, and form an infinitely increasing sequence $\{\lambda_{k,\delta}\}_{k=1}^{\infty}$. Moreover, (i) the eigenfunction $y_{k,\delta}(x)$ corresponding to the eigenvalue $\lambda_{k,\delta}$ has exactly $k-1$ simple zeros in the interval $(0,1)$ for $a\lambda_{k,\delta} + b \leq 0$, and either $k-2$ or $k-1$ simple zeros for $a\lambda_{k,\delta} + b \leq 0$; (ii) the function $y'_{k,\delta}(x)$ has exactly $k-1$ simple zeros in the interval $(0,1)$.

It follows from the maximal-minimal property of eigenvalues that

$$\lambda_{1,\frac{\pi}{2}} < \lambda_{1,\pi} < \lambda_{2,\frac{\pi}{2}} < \lambda_{2,\pi} < \ldots < \lambda_{k,\frac{\pi}{2}} < \lambda_{k,\pi} < \ldots . \tag{2.4}$$

Theorem 2.1 [10, Theorem 3.2]. *For each fixed $\lambda \in \mathbb{C}$ problem (1.1)-(1.3) has a unique non-trivial solution $y(x,\lambda)$ with a constant factor. Moreover, $y(x,\lambda)$ for each fixed $x \in [0,1]$ is entire function of λ.*

Let $B_k = (\lambda_{k-1,\pi}, \lambda_{k,\pi})$, $k \in \mathbb{N}$, where $\lambda_{0,\pi} = -\infty$.

According to above arguments, Theorem 2.1 and relation (2.3), it is clear that the function

$$G(\lambda) = \frac{Ty(1,\lambda)}{y(1,\lambda}$$

is a meromorphic function of finite order defined on the set

$$B = (\mathbb{C}\backslash\mathbb{R}) \cup \left(\bigcup_{k=1}^{\infty} B_k\right).$$

The eigenvalues $\lambda_{k,\frac{\pi}{2}}$ and $\lambda_{k,\pi}$ of problem (1.1)-(1.3) are zeros and poles of this function, respectively.

Lemma 2.1. [10, Lemmas 3.1, 3.2]. *The following relations hold:*

$$\frac{dG(\lambda)}{d\lambda} = -\frac{1}{y^2(1,\lambda)}\left\{\int\limits_0^1 y^2(x,\lambda)dx + ay'^2(1,\lambda)\right\}, \lambda \in B,$$

$$\lim_{\lambda\to-\infty} G(\lambda) = -\infty.$$

Lemma 2.2. [10, Lemma 3.3]. *For the function $G(\lambda)$, the following representation holds:*

$$G(\lambda) = G(0) + \sum_{k=1}^{\infty}\frac{\lambda\varsigma_k}{\lambda_k(\pi)(\lambda-\lambda_k(\pi))},$$

where $\varsigma_k = \underset{\lambda=\lambda_k(\pi)}{res} G(\lambda)$ and $\varsigma_k < 0$, $k \in \mathbb{N}$.

Using Theorem 2.1 and Lemmas 2.1-2.3 in the paper [10] for problem (1.1)-(1.4) the following result is obtained.

Theorem 2.2 [10, Theorem 4.1]. *The eigenvalues of problem (1.1)-(1.4) are real and simple, and form an infinite increasing sequence $\{\lambda_k\}_{k=1}^{\infty}$ such that*

$$\lambda_1 \in (-\infty, 0) \text{ and } \lambda_k \in (\lambda_{k-1,\frac{\pi}{2}}, \lambda_{k,\pi}) \text{ for } k = 2, 3, \ldots . \tag{2.5}$$

For each $k \in \mathbb{N}$ let y_k be the eigenfunction corresponding to the eigenvalue λ_k of problem (1.1)-(1.4).

Theorem 2.3 *For sufficiently large $k \in \mathbb{N}$ the following asymptotic formulas hold:*

$$\sqrt[4]{\lambda_k} = \left(k - \frac{3}{2}\right)\pi + O\left(\frac{1}{k}\right), \tag{2.6}$$

$$\frac{y_k(1)}{y'_k(1)} = \frac{b}{c}\sqrt[4]{\lambda_k}\left(1 + \frac{1}{c\sqrt[4]{\lambda_k}} + O\left(\frac{1}{k^2}\right)\right). \tag{2.7}$$

The proof of this theorem is similar to that of [5, Theorems 4.2 and formula (5.18)].

3. Basis property of eigenfunctions of problem (1.1)-(1.4)

Suppose that the system $\{\hat{\vartheta}_k\}_{k=1}^{\infty}$, $\hat{\vartheta}_k = \{\vartheta_k, s_k, t_k\}$, is an adjoint system to the system $\{\hat{y}_k\}_{k=1}^{\infty}$, $\hat{y}_k = \{y_k, m_k, n_k\}$. Since all eigenvalues of problem (1.1)-(1.4) are real and simple, according to Lemma 2.1 we get

$$\delta_k = (\hat{y}_k, \hat{y}_k)_1 = ||y_k||_2^2 + a{y'_k}^2(1) - cy_k^2(1) \neq 0. \tag{3.1}$$

Then, analogously to [5, Lemma 5.1], we can show that

$$\hat{\vartheta}_k = \delta_k^{-1} J \hat{y}_k, \; k \in \mathrm{N}. \tag{3.2}$$

Now suppose that r and l $(r \neq l)$ are arbitrary natural numbers. By (2.1), (2.2) and (3.1), (3.2) we have the following relation

$$\Delta_{r,l} = \begin{vmatrix} s_r & s_l \\ t_r & t_l \end{vmatrix} = \delta_r^{-1}\delta_l^{-1} \begin{vmatrix} m_r & m_l \\ -n_r & -n_l \end{vmatrix} =$$

$$-\delta_r^{-1}\delta_l^{-1} \begin{vmatrix} ay_r'(1) & ay_l'(1) \\ cy_r(1) & cy_l(1) \end{vmatrix} =$$

$$= -ac\delta_r^{-1}\delta_l^{-1} \begin{vmatrix} y_r'(1) & y_l'(1) \\ y_r(1) & y_l(1) \end{vmatrix} = -acy_r'(1)y_l'(1) \begin{vmatrix} 1 & 1 \\ \frac{y_r(1)}{y_r'(1)} & \frac{y_l(1)}{y_l'(1)} \end{vmatrix} =$$

$$= -acy_r'(1)y_l'(1) \left\{ \frac{y_l(1)}{y_l'(1)} - \frac{y_r(1)}{y_r'(1)} \right\}. \tag{3.3}$$

Remark 3.1. It follows from [3, Theorem 4.1] that the condition $\Delta_{r,l} \neq 0$ is necessary and sufficient in order to the system of eigenfunctions $\{y_k(x)\}_{k=1,\,k\neq r,l}^{\infty}$ of problem (1.1)-(1.4) to form a basis in the space $L_p(0,1)$, $1 < p < \infty$, (an unconditional basis for $p = 2$).

The main result of this paper is the following theorem.

Theorem 3.1. *Suppose that r and l $(r < l)$ are arbitrarily large enough natural numbers. Then the system of eigenfunctions $\{y_k(x)\}_{k=1,\,k\neq r,l}^{\infty}$ of problem (1.1)-(1.4) forms a basis in space $L_p(0,1)$, $1 < p < \infty$, which is an unconditional basis in $L_2(0,1)$.*

Proof. If r and l $(r < l)$ are arbitrarily large enough natural numbers, then by (2.6) and (2.7) from (3.3) we get

$$-\Delta_{r,l} = acy_r'(1)y_l'(1) \left\{ \frac{y_l(1)}{y_l'(1)} - \frac{y_r(1)}{y_r'(1)} \right\} =$$

$$acy_r'(1)y_l'(1)\frac{b}{c} \left\{ \sqrt[4]{\lambda_l}\left(1 + \frac{1}{c\sqrt[4]{\lambda_l}} + O\left(\frac{1}{l^2}\right)\right) - \sqrt[4]{\lambda_r}\left(1 + \frac{1}{c\sqrt[4]{\lambda_r}} + O\left(\frac{1}{r^2}\right)\right) \right\} =$$

$$= -acy_r'(1)y_l'(1)\frac{b}{c} \left\{ \sqrt[4]{\lambda_l} + \frac{1}{c} + O\left(\frac{1}{l}\right) - \sqrt[4]{\lambda_r} - \frac{1}{c} + O\left(\frac{1}{r}\right) \right\} =$$

$$= acy_r'(1)y_l'(1)\frac{b}{c} \left\{ \left(l - \frac{3}{2}\right)\pi + O\left(\frac{1}{l}\right) - \left(r - \frac{3}{2}\right)\pi + O\left(\frac{1}{r}\right) \right\} >$$

$$> acy'_r(1)y'_l(1)\frac{b}{c}\left\{(l-r)\pi - \frac{K}{l} - \frac{K}{r}\right\} > 0,$$

where $K > 0$ is some constant. Now the assertion of this theorem follows from Remark 3.1 by virtue of the last relation. The proof of this theorem is complete.

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