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On the Completeness and Minimality of Eigenfunctions of the Indefinite Sturm-Liouville Problem with Conjugation Condition

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Abstract. In this work we consider the following spectral problem:

$$-y'' = \lambda\omega(x)\,y, \quad x \in (-1,0)\cup(0,1),$$

$$\left.\begin{array}{c} y(-1) = y(1) = 0, \\ y(-0) = ay(+0) \\ y'(-0) = by'(+0) \end{array}\right\}$$

where a weight function $\omega(x)$ is in the following form:

$$\omega(x) = \begin{cases} -\alpha^2, & x \in (-1,0), \\ 1, & x \in (0,1), \end{cases}$$

$\alpha > 0$ is a given number, λ is a spectral parameter, a and b are arbitrary complex numbers. The theorem on the completeness and minimality of the eigenfunctions and associated functions of the spectral problem in the spaces $L_p(-1,1)$ is proved.

Key Words and Phrases: completeness, minimality, eigenfunctions, indefinite Sturm-Liouville problem.

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1. Introduction

Consider the spectral problem for the differential equation

$$-y'' = \lambda\omega(x)\,y, \quad x \in (-1,0)\cup(0,1), \tag{1}$$

with boundary conditions

$$y(-1) = y(1) = 0, \tag{2}$$

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and with conjugation conditions

$$\left\{ \begin{array}{l} y\,(-0) = ay\,(+0)\,, \\ y'\,(-0) = by'\,(+0)\,, \end{array} \right. \tag{3}$$

where $\omega(x)-$ is a sign-alternating weight function,

$$\omega\,(x) = \left\{ \begin{array}{cc} -\alpha^2, & x \in (-1,0)\,, \\ 1, & x \in (0,1)\,, \end{array} \right.$$

$\alpha > 0$ is a given number, λ is a spectral parameter, a and b are non-zero arbitrary complex numbers. Our goal in this work is to find asymptotic formulas for eigenvalues, to prove theorems on the completeness and minimality of eigenfunctions and associated functions of problem (1)-(3) in the spaces $L_p\,(-1,1)$. Previously, such problems were studied in the case $a = b = 1$, i.e. at the discontinuity point of the weight function, as a conjugation condition the continuity of the solution and its derivative are required. In works [1, 2, 3, 4, 5, 6], numerous applications of such problems are given, results are obtained in the case $p = 2, \quad \alpha = 1$. The results of these works are based on the theory of self-adjoint operators. Considering the case $p \neq 2$, in work [7] the methods of [8] are used, and also the methods of the theory of functions of a complex variable, in particular, the results of Paley-Wiener [9] and Levinson [10] on nonharmonic Fourier series are used. We also note the works [11, 12, 13], where ordinary differential operators of arbitrary order with a indefinite weight function are studied, asymptotic formulas for eigenvalues are found, and questions of convergence of expansions in eigenfunctions are investigated.

Recently, interest in spectral problems with a indefinite weight function has increased in connection with attempts to solve the Dirichlet problems for the Lavrent'ev–Bitsadze equation by the method of separation of variables. It is known [14, p. 303] that the problem of transition through the sound barrier of steady two-dimensional irrotational flows of an ideal gas in nozzles, when supersonic waves adjoin the nozzle walls near the minimum cross section, is reduced to the Dirichlet problem for equations of mixed type. In [15, 16], the Dirichlet problem for a mixed-type equation with one internal line of power degeneracy and degeneracy at the boundary in a rectangular domain was studied, a uniqueness criterion was established using spectral analysis methods, and the solution was constructed as the sum of a series over a system of eigenfunctions. In [17], for the first time the Dirichlet problem was studied for the Lavrent'ev–Bitsadze equation with two type-change internal lines in a rectangular domain. A uniqueness criterion is established and the solution of the problem is constructed as the sum of a series in a biorthogonal system of two mutually conjugate spectral conjugation problems for a second-order ordinary differential operator with a discontinuous coefficient at the highest derivative. The uniqueness of the solution of the stated problem is proved based on completeness of the biorthogonal system in the space $L_2\,(-1,1)$.

In [18, 19, 20] the problem for a discontinuous second-order differential operator with a constant coefficient at the highest derivative and with a spectral parameter under conjugation conditions was studied, a system of eigenfunctions was found and investigated for completeness and basicity in the spaces $L_p \oplus C$ and L_p.

2. Asymptotics of eigenvalues

Let $\lambda = \rho^2$. We also denote the linear forms included in the boundary conditions (2), (3) as follows:

$$\left.\begin{array}{ll} U_{11}(y) = y(-1), & U_{12}(y) \equiv 0 \\ U_{21}(y) \equiv 0, & U_{22}(y) = y(1) \\ U_{31}(y) = y(-0), & U_{32}(y) = -ay(+0) \\ U_{41}(y) = y'(-0), & U_{42}(y) = -by'(+0) \end{array}\right\} \tag{4}$$

After these denotations, problem (1)-(3) can be rewritten in the following form:

$$y'' + \rho^2\omega(x)\,y = 0, \quad x \in (-1,0) \cup (0,1), \tag{5}$$

$$\left.\begin{array}{l} U_1(y) = U_{11}(y) + U_{12}(y) = 0 \\ U_2(y) = U_{21}(y) + U_{22}(y) = 0 \\ U_3(y) = U_{31}(y) + U_{32}(y) = 0 \\ U_4(y) = U_{41}(y) + U_{42}(y) = 0 \end{array}\right\} \tag{6}$$

It is known that equation (4) has a fundamental system of solutions $y_{11}(x) = e^{\alpha\rho x}$, $y_{12}(x) = e^{-\alpha\rho x}$, on the interval $(-1,0)$, and $y_{21}(x) = e^{i\rho x}$, $y_{22}(x) = e^{-i\rho x}$ on the interval $(0,1)$. Then the general solution of equation (1) (or (4)) has the form

$$y(x) = \begin{cases} c_{11}y_{11}(x) + c_{12}y_{12}(x), x \in (-1,0) \\ \\ c_{21}y_{21}(x) + c_{22}y_{22}(x),\ x \in (0,1) \end{cases}$$

Let us choose the constants c_{ik} so that the function $y(x)$ satisfies the boundary conditions (5). Then, to find the numbers c_{ik} we get the following system of equations:

$$\left.\begin{array}{l} c_{11}U_{11}(y_{11}) + c_{12}U_{11}(y_{12}) + c_{21}U_{12}(y_{21}) + c_{22}U_{12}(y_{22}) = 0 \\ c_{11}U_{21}(y_{11}) + c_{12}U_{21}(y_{12}) + c_{21}U_{22}(y_{21}) + c_{22}U_{22}(y_{22}) = 0 \\ c_{11}U_{31}(y_{11}) + c_{12}U_{31}(y_{12}) + c_{21}U_{32}(y_{21}) + c_{22}U_{32}(y_{22}) = 0 \\ c_{11}U_{41}(y_{11}) + c_{12}U_{41}(y_{12}) + c_{21}U_{42}(y_{21}) + c_{22}U_{42}(y_{22}) = 0 \end{array}\right\}$$

This system of equations has a nontrivial solution if and only if the main determinant (characteristic determinant) $\mathit{\Delta}(\rho) = det\|U_{\nu i}(y_{ik})\|_{\nu=\overline{1,4};i,k=1,2}$ of this system is zero. Thus, the number $\lambda = \rho^2$ is an eigenvalue of the spectral problem (1)-(3) if and only if the number ρ is a solution of the following equation

$$\mathit{\Delta}(\rho) = \begin{vmatrix} e^{-\alpha\rho} & e^{\alpha\rho} & 0 & 0 \\ 0 & 0 & e^{i\rho} & e^{-i\rho} \\ 1 & 1 & -a & -a \\ \alpha\rho & -\alpha\rho & -bi\rho & bi\rho \end{vmatrix} = 4i\Delta_0(\rho) = 0,$$

where

$$\mathit{\Delta}_0(\rho) = \alpha a \ sin\rho \ ch\alpha\rho + b \ cos\rho \ sh\alpha\rho.$$

Let us divide the complex ρ-plane into the following sectors:

$$S_k = \left\{ \rho = re^{i\theta} : \frac{(k-1)\,\pi}{2} \le \theta \le \frac{k\pi}{2} \right\}, \; k = 0, 1, 2, 3.$$

We also denote by Q_δ the domain of the ρ-plane, obtained from it by throwing out circles with the same radius $\delta > 0$ and with centers at zeros $\Delta(\rho)$. The following theorem is true.

Theorem 1. *The characteristic determinant $\Delta(\rho)$ of the spectral problem* (1)-(3) *has the following properties:*

1) *There exists a positive number M_δ such that in the domain $S_k \cap Q_\delta$ for sufficiently large $|\rho|$ the inequality*

$$|\Delta(\rho)| \ge M_\delta \, |\rho| \, e^{\pm r sin\theta} e^{\pm \alpha r cos\theta}; \tag{7}$$

is satisfied, where the constant M_δ is independent of ρ, but depends only on the number $\delta > 0$; in addition, the signs in the exponents on the right side of this inequality are chosen depending on the sectors S_k as follows: "+", "+" for $\rho \in S_0$; "+", "-" for $\rho \in S_1$; "-", "-" for $\rho \in S_2$; "-", "+" for $\rho \in S_3$.

2) *The zeros of the function $\Delta(\rho)$ are asymptotically simple and have the following asymptotics*

$$\rho_{1n} = \pi n - \gamma + O\left(e^{-2\pi n \alpha}\right), \; n \to \infty,$$

$$\rho_{2n} = -\frac{i}{\alpha}\left(\pi n + \gamma + \frac{\pi}{2} + O\left(e^{-\alpha \pi n}\right)\right), \; n \to \infty.$$

Proof. **1)** Let us estimate the function $\Delta_0(\rho)$ in each sector S_k. Let $\rho \in S_0$. Then the inequalities

$$Re(\,i\rho) \le 0 \le Re(-\,i\rho), Re(\,-\alpha\rho) \le 0 \le Re(\alpha\,\rho),$$

hold. Let us reduce the function $\Delta_0(\rho)$ to the following form:

$$\Delta_0(\rho) = e^{-i\rho} e^{\alpha\rho} \left(\alpha a \left(1 - e^{-2i\rho}\right)\left(1 + e^{-2\alpha\rho}\right) + b\left(1 + e^{-2i\rho}\right)\left(1 - e^{-2\alpha\rho}\right)\right).$$

All exponents inside the brackets on the right side of this equality have a non-positive real part in the exponent, therefore they are bounded. Moreover, if $\rho \in Q_\delta$, then the expression in brackets is bounded from below by some positive number M_δ in absolute value. Therefore we have

$$|\Delta_0(\rho)| \ge M_\delta \left|e^{-i\rho} e^{\alpha\rho}\right| = M_\delta e^{r sin\theta} e^{\alpha r cos\theta}.$$

Hence we obtain the validity of inequality (7) for $\rho \in S_0 \cap Q_\delta$. Other cases are considered in a similar way.

2) Define the number γ as follows:

$$cos\gamma = \frac{\alpha a}{\sqrt{\alpha^2 a^2 + b^2}},\ sin\gamma = \frac{b}{\sqrt{\alpha^2 a^2 + b^2}},\ Re\gamma \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

Then the function $\Delta_0(\rho)$ with the help of elementary transformations can be represented in the following form:

$$\Delta_0(\rho) = \frac{1}{2}\sqrt{\alpha^2 a^2 + b^2} e^{\alpha\rho} \left(sin(\rho+\gamma) + e^{-2\alpha\rho} sin(\rho-\gamma)\right). \tag{8}$$

Based on the Rouché's theorem, we obtain that the zeros of the function $\Delta_0(\rho)$, situated in the strip $|Im\rho| \le h$ are asymptotically situated in a small neighborhood of the zeros of the function $sin(\rho+\gamma)$, and for large values of $|\rho|$ near each zero of the function $sin(\rho+\gamma)$ there is one zero of the function $\Delta_0(\rho)$. Hence we obtain the asymptotics of the zeros $\Delta_0(\rho)$, situated in the strip $|Im\rho| \le h$:

$$\rho_{1n} = \pi n - \gamma + O\left(e^{-2\alpha\pi n}\right),\ n \to \infty.$$

On the other hand, replacing ρ by $i\rho$ in formula (8), we obtain

$$\Delta_0(i\rho) = \frac{i}{2}\sqrt{\alpha^2 a^2 + b^2} e^{\rho} \left(cos(\alpha\rho-\gamma) - e^{-2\rho} cos(\alpha\rho+\gamma)\right). \tag{9}$$

Applying the Rouché's theorem again, from formula (9) we obtain that the zeros of the function $\Delta_0(\rho)$, situated in the strip $|Re\rho| \le h$ are asymptotically situated in a small neighborhood of the zeros of the function $cos(\alpha\rho-\gamma)$, and for large values of $|\rho|$ near each zero of the function $cos(\alpha\rho-\gamma)$ there is one zero of the function $\Delta_0(\rho)$. Hence we obtain the asymptotics of the zeros $\Delta_0(\rho)$, situated in the strip $|Re\rho| \le h$ are:

$$\rho_{2n} = -\frac{i}{\alpha}\left(\pi n + \gamma + \frac{\pi}{2} + O\left(e^{-\alpha\pi n}\right)\right),\ n \to \infty.$$

◄

3. Construction of the Green's function of the spectral problem

The Green's function of problem (1)-(3) is defined as the kernel of the integral representation of the solution of the nonhomogeneous equation

$$-y''(x) - \rho^2\omega(x)\, y(x) = f(x), \tag{10}$$

Satisfying the boundary conditions (2),(3). Let us look for a solution of the problem (10),(2),(3) in the form

$$y(x) = \begin{cases} y_1(x), & x \in [-1, 0], \\ y_2(x), & x \in [0, 1], \end{cases} \tag{11}$$

where

$$\begin{cases} y_1(x) = c_{11}y_{11}(x) + c_{12}y_{12}(x) + \int_{-1}^{0} g_1(x,\xi,\rho)f(\xi)d\xi, \\ \\ y_2(x) = c_{21}y_{21}(x) + c_{22}y_{22}(x) + \int_{0}^{1} g_2(x,\xi,\rho)f(\xi)d\xi. \end{cases} \tag{12}$$

$$g_1(x,\xi,\rho) = \begin{cases} -\frac{1}{4\alpha\rho}(e^{\alpha\rho(x-\xi)} - e^{-\alpha\rho(x-\xi)}), & -1 \le x < \xi \le 0, \\ \\ \frac{1}{4\alpha\rho}(e^{\alpha\rho(x-\xi)} - e^{-\alpha\rho(x-\xi)}), & -1 \le \xi < x \le 0, \end{cases} \tag{13}$$

$$g_2(x,\xi,\rho) = \begin{cases} \frac{1}{2i\rho}(e^{i\rho(x-\xi)} - e^{-i\rho(x-\xi)}), & 0 \le x < \xi \le 1, \\ \\ -\frac{1}{2i\rho}(e^{i\rho(x-\xi)} - e^{-i\rho(x-\xi)}), & 0 \le \xi < x \le 1. \end{cases} \tag{14}$$

We require that the function $y(x)$, defined by formulas (11)-(14), satisfies the boundary conditions (2) and conjugation conditions (3). Then, to determine the numbers c_{jk} we obtain the following system of equations:

$$\begin{cases} U_\nu\,(y) = \sum_{j,k=1}^{2} c_{jk}U_{\nu j}(y_{jk}) + \int_{-1}^{0} U_{\nu 1}(g_1)f(\xi)d\xi + \int_{0}^{1} U_{\nu 2}(g_2)f(\xi)d\xi = 0, \\ \nu = \overline{1,4}. \end{cases} \tag{15}$$

Having determined the numbers c_{jk} from system (15) and substituting their values into (12), for the solution of equation (10) that satisfies (2), (3), we obtain the following formula:

$$y(x) = \begin{cases} y_1(x) = \int_{-1}^{0} G_{11}(x,\xi,\rho)f(\xi)d\xi + \int_{0}^{1} G_{12}(x,\xi,\rho)f(\xi)d\xi, & x \in [-1,0]\,, \\ y_2(x) = \int_{-1}^{0} G_{21}(x,\xi,\rho)f(\xi)d\xi + \int_{0}^{1} G_{22}(x,\xi,\rho)f(\xi)d\xi, & x \in [0,1]\,, \end{cases} \tag{16}$$

Here

$$G_{ik}(x,\xi,\rho) = \frac{1}{\Delta\,(\rho)} H_{ik}(x,\xi,\rho), \;\; i,k = 1,2, \tag{17}$$

$$H_{ik}(x,\xi,\rho) = \begin{vmatrix} \delta_{ik}g_k\,(x,\xi) & \delta_{1k}y_{11}\,(x) & \delta_{1k}y_{12}\,(x) & \delta_{2k}y_{21}\,(x) & \delta_{2k}y_{22}\,(x) \\ U_{1k}\,(g_k)\,(\xi) & U_{11}\,(y_{11}) & U_{11}\,(y_{12}) & U_{12}\,(y_{21}) & U_{12}\,(y_{22}) \\ U_{2k}\,(g_k)\,(\xi) & U_{21}\,(y_{11}) & U_{21}\,(y_{12}) & U_{22}\,(y_{21}) & U_{22}\,(y_{22}) \\ U_{3k}\,(g_k)\,(\xi) & U_{31}\,(y_{11}) & U_{31}\,(y_{12}) & U_{32}\,(y_{21}) & U_{32}\,(y_{22}) \\ U_{4k}\,(g_k)\,(\xi) & U_{41}\,(y_{11}) & U_{41}\,(y_{12}) & U_{42}\,(y_{21}) & U_{42}\,(y_{22}) \end{vmatrix},$$

δ_{ik} is the Kronecker symbol. Denote $I_1 = (-1,0)$, $I_2 = (0,1)$ and let $\chi_1\,(x)\,,\ \chi_2\,(x)$ be the characteristic functions of these intervals, respectively. The Green's function of problem

(1)-(3) is defined as follows:

$$G(x,\xi,\rho)=\sum_{i,k=1}^{2}\chi_i\,(x)\,\chi_k\,(\xi)\,G_{ik}(x,\xi,\rho). \tag{18}$$

Then the solution of equation (10) that satisfies conditions (2), (3) can be represented as

$$y(x)=\int_{-1}^{1}G(x,\xi,\rho)f(\xi)d\xi. \tag{19}$$

According to denotation (4) and formulas (13), (14) we have

$$\begin{array}{ll}
U_{11}(g_1)=-\frac{1}{4\alpha\rho}\left(e^{-\alpha\rho(1+\xi)}-e^{\alpha\rho(1+\xi)}\right), & U_{12}(g_2)=0,\\
U_{21}(g_1)=0, & U_{22}(g_2)=-\frac{1}{4i\rho}\left(e^{i\rho(1-\xi)}-e^{-i\rho(1-\xi)}\right),\\
U_{31}(g_1)=\frac{1}{4\alpha\rho}\left(e^{-\alpha\rho\xi}-e^{\alpha\rho\xi}\right), & U_{32}(g_2)=\frac{a}{4i\rho}\left(e^{-i\rho\xi}-e^{i\rho\xi}\right),\\
U_{41}(g_1)=\frac{1}{4}\left(e^{-\alpha\rho\xi}+e^{\alpha\rho\xi}\right), & U_{42}(g_2)=-\frac{b}{4}\left(e^{-i\rho\xi}+e^{i\rho\xi}\right).
\end{array}$$

Taking into account these values, as well as the values $U_{\nu s}\,(y_{sk})$ in formulas $H_{kj}(x,\xi,\rho)$, we obtain

$$H_{11}(x,\xi,\rho)=\begin{vmatrix}
\frac{\pm1}{4\alpha\rho}\left(e^{\alpha\rho(x-\xi)}-e^{-\alpha\rho(x-\xi)}\right) & e^{\alpha\rho\, x} & e^{-\alpha\rho\, x} & 0 & 0\\
-\frac{1}{4\alpha\rho}\left(e^{-\alpha\rho(1+\xi)}-e^{\alpha\rho(1+\xi)}\right) & e^{-\alpha\rho} & e^{\alpha\rho} & 0 & 0\\
0 & 0 & 0 & e^{i\rho} & e^{-i\rho}\\
\frac{1}{4\alpha\rho}\left(e^{-\alpha\rho\xi}-e^{\alpha\rho\xi}\right) & 1 & 1 & -a & -a\\
\frac{1}{4}\left(e^{-\alpha\rho\xi}+e^{\alpha\rho\xi}\right) & \alpha\rho & -\alpha\rho & -bi\rho & bi\rho
\end{vmatrix},\quad x,\ \xi\in I_1,$$

here the sign " $+$ " is taken in the case of $-1\le\xi<x\le 0$, and the sign " $-$ " in the case of $-1\le x<\xi\le 0$;

$$H_{12}(x,\xi,\rho)=\begin{vmatrix}
0 & e^{\alpha\rho x} & e^{-\alpha\rho x} & 0 & 0\\
0 & e^{-\alpha\rho} & e^{\alpha\rho} & 0 & 0\\
\frac{1}{4i\rho}\left(e^{i\rho(1-\xi)}-e^{-i\rho(1-\xi)}\right) & 0 & 0 & e^{i\rho} & e^{-i\rho}\\
\frac{a}{4i\rho}\left(e^{-i\rho\xi}-e^{i\rho\xi}\right) & 1 & 1 & -a & -a\\
\frac{b}{4}\left(e^{-i\rho\xi}+e^{i\rho\xi}\right) & \alpha\rho & -\alpha\rho & -bi\rho & bi\rho
\end{vmatrix},\quad x\in I_1,\ \ \xi\in I_2;$$

$$H_{21}(x,\xi,\rho)=\begin{vmatrix} 0 & 0 & 0 & e^{i\rho x} & e^{-i\rho x} \\ -\frac{1}{4\alpha\rho}\left(e^{-\alpha\rho(1+\xi)}-e^{\alpha\rho(1+\xi)}\right) & e^{-\alpha\rho} & e^{\alpha\rho} & 0 & 0 \\ 0 & 0 & 0 & e^{i\rho} & e^{-i\rho} \\ \frac{1}{4\alpha\rho}\left(e^{-\alpha\rho\xi}-e^{\alpha\rho\xi}\right) & 1 & 1 & -a & -a \\ \frac{1}{4}\left(e^{-\alpha\rho\xi}+e^{\alpha\rho\xi}\right) & \alpha\rho & -\alpha\rho & -bi\rho & bi\rho \end{vmatrix},\ x\in I_2,\ \ \xi\in I_1;$$

$$H_{22}(x,\xi,\rho)=\begin{vmatrix} \frac{\pm 1}{4i\rho}\left(e^{i\rho(x-\xi)}-e^{-i\rho(x-\xi)}\right) & 0 & 0 & e^{i\rho x} & e^{-i\rho x} \\ 0 & e^{-\alpha\rho} & e^{\alpha\rho} & 0 & 0 \\ \frac{1}{4i\rho}\left(e^{i\rho(1-\xi)}-e^{-i\rho(1-\xi)}\right) & 0 & 0 & e^{i\rho} & e^{-i\rho} \\ \frac{a}{4i\rho}\left(e^{-i\rho\xi}-e^{i\rho\xi}\right) & 1 & 1 & -a & -a \\ \frac{b}{4}\left(e^{-i\rho\xi}+e^{i\rho\xi}\right) & \alpha\rho & -\alpha\rho & -bi\rho & bi\rho \end{vmatrix},\quad x,\ \xi\in I_2,$$

here the sign " $+$ " is taken in the case of $-1\le\xi<x\le 0$, and the sign " $-$ " in the case of $-1\le x<\xi\le 0$;

Theorem 2. *For the components $G_{ik}(x,\xi,\rho)$ of the Green's function of problem* (1)-(3) *in the domain Q_δ for sufficiently large values $|\rho|$ uniformly in the variables $x\in_i,\ \xi\in_k$ the estimate*

$$|G_{ik}(x,\xi,\rho)|\le\frac{C_\delta}{|\rho|}, \tag{20}$$

is true, where the positive number C_δ is independent of ρ, but depends only on the number δ.

Proof. Let us perform the following transformations on the determinants $H_{ik}(x,\xi,\rho)$: in the determinant $H_{11}(x,\xi,\rho)$ in the case $-1\le\xi<x\le 0$ multiply the second and third columns by $-\frac{1}{4\alpha\rho}e^{-\alpha\rho\xi}$, $-\frac{1}{4\alpha\rho}e^{\alpha\rho\xi}$ respectively and add to the first column, then we get

$$H_{11}(x,\xi,\rho)=\begin{vmatrix} -\frac{1}{2\alpha\rho}e^{-\alpha\rho(x-\xi)} & e^{\alpha\rho x} & e^{-\alpha\rho x} & 0 & 0 \\ -\frac{1}{2\alpha\rho}e^{-\alpha\rho(1+\xi)} & e^{-\alpha\rho} & e^{\alpha\rho} & 0 & 0 \\ 0 & 0 & 0 & e^{i\rho} & e^{-i\rho} \\ -\frac{1}{2\alpha\rho}e^{\alpha\rho\xi} & 1 & 1 & -a & -a \\ \frac{1}{2}e^{\alpha\rho\xi} & \alpha\rho & -\alpha\rho & -bi\rho & bi\rho \end{vmatrix}=$$

$$= \frac{1}{2}e^{-i\rho}e^{\alpha\rho}\begin{vmatrix} -e^{-\alpha\rho(x-\xi)} & e^{\alpha\rho x} & e^{-\alpha\rho(1+x)} & 0 & 0 \\ -e^{-\alpha\rho(1+\xi)} & e^{-\alpha\rho} & 1 & 0 & 0 \\ 0 & 0 & 0 & e^{2i\rho} & 1 \\ -e^{\alpha\rho\xi} & 1 & e^{-\alpha\rho} & -a & -a \\ e^{\alpha\rho\xi} & 1 & -e^{-\alpha\rho} & -\frac{b}{\alpha}i & \frac{b}{\alpha}i \end{vmatrix}$$

(in the case $-1 \le x < \xi \le 0$ similar actions are performed by multiplying the second and third columns by $\frac{1}{4\alpha\rho}e^{-\alpha\rho\xi}$, $\frac{1}{4\alpha\rho}e^{\alpha\rho\xi}$ respectively); in the determinant $H_{12}(x,\xi,\rho)$ multiply the fourth and fifth columns by $\frac{1}{4i\rho}e^{-i\rho\xi}$, $\frac{1}{4i\rho}e^{i\rho\xi}$ respectively and add to the first column, then we get

$$H_{12}(x,\xi,\rho) = \frac{1}{2}\begin{vmatrix} 0 & e^{\alpha\rho x} & e^{-\alpha\rho x} & 0 & 0 \\ 0 & e^{-\alpha\rho} & e^{\alpha\rho} & 0 & 0 \\ e^{i\rho(1-\xi)} & 0 & 0 & e^{i\rho} & e^{-i\rho} \\ -e^{i\rho\xi} & 1 & 1 & -a & -a \\ e^{i\rho\xi} & -\alpha i & \alpha i & -b & b \end{vmatrix} =$$

$$= \frac{1}{2}e^{-i\rho}e^{\alpha\rho}\begin{vmatrix} 0 & e^{\alpha\rho x} & e^{-\alpha\rho(1+x)} & 0 & 0 \\ 0 & e^{-\alpha\rho} & 1 & 0 & 0 \\ e^{i\rho(1-\xi)} & 0 & 0 & e^{i\rho} & 1 \\ -e^{i\rho\xi} & 1 & e^{-\alpha\rho} & -a & -ae^{i\rho} \\ e^{i\rho\xi} & 1 & \alpha i e^{-\alpha\rho} & -b & be^{i\rho} \end{vmatrix};$$

in the determinant $H_{21}(x,\xi,\rho)$ multiply the second and third columns by $-\frac{1}{4\alpha\rho}e^{-\alpha\rho\xi}$, $-\frac{1}{4\alpha\rho}e^{\alpha\rho\xi}$

respectively and add to the first column, then we get

$$H_{21}(x,\xi,\rho)=\frac{1}{2}\begin{vmatrix} 0 & 0 & 0 & e^{i\rho x} & e^{-i\rho x} \\ -e^{-\alpha\rho(1+\xi)} & e^{-\alpha\rho} & e^{\alpha\rho} & 0 & 0 \\ 0 & 0 & 0 & e^{i\rho} & e^{-i\rho} \\ -e^{\alpha\rho\xi} & 1 & 1 & -a & -a \\ e^{\alpha\rho\xi} & 1 & -1 & -\frac{b}{\alpha}i & \frac{b}{\alpha}i \end{vmatrix}=$$

$$=\frac{1}{2}e^{-i\rho}e^{\alpha\rho}\begin{vmatrix} 0 & 0 & 0 & e^{i\rho x} & e^{i\rho(1-x)} \\ -e^{-\alpha\rho(1+\xi)} & e^{-\alpha\rho} & 1 & 0 & 0 \\ 0 & 0 & 0 & e^{i\rho} & 1 \\ -e^{\alpha\rho\xi} & 1 & e^{-\alpha\rho} & -a & -ae^{i\rho} \\ e^{\alpha\rho\xi} & 1 & -e^{-\alpha\rho} & -\frac{b}{\alpha}i & \frac{b}{\alpha}ie^{i\rho} \end{vmatrix};$$

in the determinant $H_{22}(x,\xi,\rho)$ in the case of $0\le\xi<x\le1$ multiply the fourth and fifth columns by $\frac{1}{4i\rho}e^{-i\rho\xi}$, $\frac{1}{4i\rho}e^{i\rho\xi}$, respectively and add to the first column, then we get

$$H_{22}(x,\xi,\rho)=$$

$$=\frac{1}{2}\begin{vmatrix} e^{i\rho(x-\xi)} & 0 & 0 & e^{i\rho x} & e^{-i\rho x} \\ 0 & e^{-\alpha\rho} & e^{\alpha\rho} & 0 & 0 \\ e^{i\rho(1-\xi)} & 0 & 0 & e^{i\rho} & e^{-i\rho} \\ -e^{i\rho\xi} & 1 & 1 & -a & -a \\ e^{i\rho\xi} & -\alpha i & \alpha i & -b & b \end{vmatrix}=\frac{1}{2}e^{-i\rho}e^{\alpha\rho}\begin{vmatrix} e^{i\rho(x-\xi)} & 0 & 0 & e^{i\rho x} & e^{i\rho(1-x)} \\ 0 & e^{-2\alpha\rho} & 1 & 0 & 0 \\ e^{i\rho(1-\xi)} & 0 & 0 & e^{i\rho} & 1 \\ -e^{i\rho\xi} & 1 & 1 & -a & -ae^{i\rho} \\ e^{i\rho\xi} & -\alpha i & \alpha i & -b & be^{i\rho} \end{vmatrix},$$

(in the case $0\le x<\xi\le1$ similar actions are performed by multiplying the second and third columns by $-\frac{1}{4i\rho}e^{-i\rho\xi}$, $-\frac{1}{4i\rho}e^{i\rho\xi}$, respectively)

Thus, in the formulas obtained for $H_{ik}(x,\xi,\rho)$ in all determinants on the right side of the last equalities, all exponents have a non-positive real part in the exponent, these determinants for $\rho\in S_0$ are uniformly bounded in variables $x\in I_i,\xi\in I_k$. A similar property is established in other sectors S_k. It follows that for functions $H_{ik}(x,\xi,\rho)$ for sufficiently large values of $|\rho|$ uniformly in the variables $x\in I_i,\xi\in I_k$ the estimate

$$|H_{ik}(x,\xi,\rho)|\le Ce^{\pm r\sin\theta}e^{\pm\alpha r\cos\theta},\tag{21}$$

holds, the signs here are taken in accordance with the rule specified in Theorem 1. Now, taking into account inequalities (7) and (21) in formula (17), we obtain the validity of inequality (20). ◄

4. Completeness and minimality of eigenfunctions in the space L_p

Recall that a system $\{u_n\}_{n\in N}$ of a Banach space X is called complete in X, if the closure of the linear span of this system coincides with the entire space X, and minimal if no element of this system is included in the closed linear span of the remaining elements of this system. Recall also that a system is complete in X if and only if there is no nonzero linear continuous functional that annihilates all elements of this system. A system is minimal in X if and only if it has a biorthogonal system.

Denote by $W_p^2(-1,\,0)\cup(0,1)$ the space of functions from $L_p(-1,\,1)$, whose restrictions to each of the intervals $(-1,\,0)$ and $(0,\,1)$ belong to the Sobolev spaces $W_p^2(-1,\,0)$ and $W_p^2(0,\,1)$ respectively. Let us define an operator L in space $L_p(-1,\,1)$ as follows:

$$D(L)=\big\{y\in W_p^2(-1,\,0)\cup(0,1): y(-1)-y(1)=$$

$$y(-0)-ay(+0)=y'(-0)-by'(+0)=0\big\}$$

and for $y\in D(L)$

$$Ly=-\frac{1}{\omega(x)}y''.$$

Obviously, L is a densely defined closed operator in $L_p(-1,\,1)$ with a compact resolvent. The eigenvalues of the operator L are the numbers $\lambda_{in}=(\rho_{in})^2\,, i=1,2; n\in N$. Denote by $\{y_{in}\}_{i=1,2;n\in N}$ the system of corresponding eigenfunctions and associated functions.

Theorem 3. *System $\{y_{in}\}_{i=1,2;n\in N}$ of eigenfunctions and associated functions of the operator L is complete in space $L_p(-1,\,1)$ $1<p<\infty$.*

Proof. To prove the completeness of the system $\{y_{in}\}_{i=1,2;n\in N}$ in $L_p(-1,1)\,,\ 1<p<\infty$, let us estimate the norms of the resolvent of the operator L for sufficiently large values of $|\rho|$.

Let $\rho\in Q_\delta,\ |\rho|\geq r_0$. Then, taking into account inequalities (20) in formula (18), we obtain that the Green's function uniformly in variables $x,\xi\in[-1,1]$ satisfies the inequality

$$|G(x,\xi,\rho)|\leq\frac{C_\delta}{|\rho|},\ \rho\in Q_\delta,\ |\rho|\geq r_0.$$

Taking into account this estimate in formula (19) for the function $y(x)$, we obtain the following estimate:

$$|y(x)|\leq\frac{C_\delta}{|\rho|}\,\|f\|_{L_p}\,,\ \rho\in Q_\delta,\ |\rho|\geq r_0.$$

Moreover, this inequality is satisfied uniformly in $x \in [-1, 1]$. As a consequence, hence we get

$$\|y\|_{L_p} \leq \frac{C_\delta}{|\rho|} \|f\|_{L_p}, \ \rho \in Q_\delta, \ |\rho| \geq r_0.$$

The last inequality means that the for the resolvent $R(\lambda) = (L - \lambda I)^{-1}$ of the operator L the following estimate

$$\left\|R\left(\rho^2\right)\right\| \leq \frac{C_\delta}{|\rho|}, \ \rho \in Q_\delta, \ |\rho| \geq r_0, \tag{22}$$

holds. Now suppose that the system of root functions of the operator L is not complete in $L_p(-1, 1)$. Then there exists a function $g \in L_q(-1, 1)$, $q = p/(p-1)$, orthogonal to all root subspaces of the operator L, i.e.

$$\langle Q_{in} f, g\rangle = 0, \forall f \in L_p(-1, 1), \ i = 1, 2; n \in N.$$

Hence it follows that $Q_{in}^* g = 0$, $i = 1, 2; n \in N$; here Q_{in} denotes the Riesz projectors of the operator L, i.e.

$$Q_{in} = \frac{1}{2\pi i} \oint_{\gamma_{in}(\delta)} R(\lambda)\, d\lambda,$$

where $\gamma_{in}(\delta)$ are the images of the circles $\gamma_{in}(\delta) = \{\rho : |\rho - \rho_{in}| = \delta\}$ under the mapping $\lambda = \rho^2$. In this case it is obvious that Q_{in}^*, $i = 1, 2; n \in N$, are the Riesz projectors of the adjoint operator L^*. This implies that $R(\lambda, L^*)\, g$ is an entire function in the λ-plane. On the other hand, according to estimate (22), the inequality

$$\|R(\lambda, L^*)\| \leq \frac{C_\delta}{|\lambda|^{\frac{1}{2}}}, \ \lambda \in \Omega_\delta, \ |\lambda| \geq R_0, \tag{23}$$

is true, where Ω_δ denotes the image of the set Q_δ under the mapping $\lambda = \rho^2$. Then, according to the maximum principle, inequality (23) is satisfied in the entire λ-plane and in turn, we obtain $R(\lambda, L^*)\, g \to 0, |\lambda| \to \infty$. The latter, by Liouville's theorem, the entire function $R(\lambda, L^*)\, g$ is constant. Then, differentiating this function and taking into account the formula $\frac{d}{d\lambda} R(\lambda, L^*) = R(\lambda, L^*)^2$ we obtain that $R(\lambda, L^*)^2 g = 0$. But, since for $\lambda \in \rho(L^*)$ the operator $R(\lambda, L^*)$ is unique, then we obtain that $g = 0$. And this means that the system $\{y_{in}\}_{i=1,2; n \in N}$ of eigenfunctions and associated functions of the operator L is complete in $L_p(-1, 1)$.

Theorem is proved. ◄

Denote by $\{z_{in}\}_{i=1,2; n \in N}$ the system of eigenfunctions and associated functions of the adjoint operator L^*. The operator L^* is the operator generated by the adjoint spectral problem

$$z'' + \lambda \omega(x)\, z = 0, \quad x \in (-1, 0) \cup (0,\, 1)$$

$$\left.\begin{array}{l} z(-1) = z(1) = 0, \\ z(-0) = -\frac{\alpha^2}{\bar{b}} z(+0), \\ z'(-0) = -\frac{\alpha^2}{\bar{a}} z'(+0). \end{array}\right\}$$

Then the system $\{z_{in}\}_{i=1,2;n\in N}$ (after appropriate normalization) is biorthogonal to the system $\{y_{in}\}_{i=1,2;n\in N}$. Taking this fact into account, we obtain the following corollaries from Theorem 3.

Corollary 1. *System $\{y_{in}\}_{i=1,2;n\in N}$ of eigenfunctions and associated functions of the operator L is complete and minimal in $L_p(-1,1)$, $1<p<\infty$.*

Corollary 2. *System $\{z_{in}\}_{i=1,2;n\in N}$ of eigenfunctions and associated functions of the operator L^* is complete and minimal in $L_p(-1,1)$, $1<p<\infty$.*

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