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On Some Embedding Theorems of Besov-Morrey Spaces with Dominant Mixed Derivatives

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Abstract. In this paper introduced and studied view embedding theory some differential properties of functions from Besov-Morrey spaces with dominant mixed derivatives.

Key Words and Phrases: Besov-Morrey spaces with dominant mixed derivatives, embedding theorems, Hölder condition.
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1. Introduction

The fact some mixed derivatives of f entering the definition of the norm of W_p^l, H_p^l and $B_{p,\theta}^l$ leads to the necessity of consideration of the function spaces of another type in which the key role is played by mixed derivatives.

In this paper introduced and studied the Besov-Morrey spaces with dominant mixed derivatives.

$$S_{p,\theta,\varphi,\beta}^l B(G_\varphi)$$

and help of method of integral representation differential and difference-differential properties of functions from this space.

Here $G \subset R^n$, $1 \le p < \infty$, $1 \le \theta \le \infty$, $\varphi = (\varphi_1(t_1), \varphi_2(t_2), \ldots, \varphi_n(t_n))$, $\varphi_j(t_j) > 0$, $\varphi_j'(t_j) > 0$, $(t_j > 0)$ be continuously differentiable functions, $\lim\limits_{t_j \to +0} \varphi_j(t_j) = 0$, $\lim\limits_{t_j \to +\infty} \varphi_j(t_j) = K_j \le \infty$, $j \in e_n = \{1, 2, ..., n\}$. We denote the set of such vector-functions φ by Ψ.

Note that the spaces with parameters constructed and studied in C.B. Morrey's papers [6], and after these results were developed and generalized in the papers of V.P. Il'in [4], Y.V.Netrusov [12], A. Mazzucato [5], V.S. Guliyev [3], A.M. Najafov [7-11] and other mathematicians.

For any $x \in R^n$ we assume

$$G_{\varphi(t)}(x) = G \cap I_{\varphi(t)}(x) = G \cap \left\{ y : |y_j - x_j| < \frac{1}{2}\varphi_j(t_j), j \in e_n \right\},$$

and let $m_j > 0$, $k_j \ge 0$ are integers and $m_j > l_j - k_j > 0$, $l_j > 0$, $j \in e_n$.

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Definition 1. *Denote by $S^l_{p,\theta,\varphi,\beta}B(G_\varphi)$ the Banach space of locally summable functions on G with finite norm*

$$\|f\|_{S^l_{p,\theta,\varphi,\beta}B(G_\varphi)} = \sum_{e\subseteq e_n}\left\{\int\limits_{0^e}^{t_0^e}\left[\frac{\|\triangle^{m^e}(\varphi(t),G_{\varphi(t)}))D^{k^e}f\|_{p,\varphi,\beta}}{\prod\limits_{j\in e}(\varphi_j(t_j))^{(l_j-k_j)}}\right]^\theta\prod_{j\in e}\frac{d\varphi_j(t_j)}{\varphi_j(t_j)}\right\}^{\frac{1}{\theta}}, \quad (1)$$

where

$$\|f\|_{p,\varphi,\beta;G} = \|f\|_{L_{p,\varphi,\beta}(G)} = \sup_{\substack{x\in G,\\ t_j>0, j\in e_n}}\left(|\varphi([t]_1)|^{-\beta}\|f\|_{p,G_{\varphi(t)}(x)}\right), \quad (2)$$

$|\varphi([t]_1)|^{-\beta} = \prod\limits_{j\in e_n}\varphi_j([t_j]_1)^{-\beta_j}$, $\beta_j \in [0,1]$, $[t_j]_1 = \min\{1,t_j\}$, $1\le\theta\le\infty$, $l^e = (l_1^e, l_2^e,\ldots,l_n^e)$, $l_j^e = l_j (j\in e)$, $l_j^e = 0$ $(j\in e_n - e = e')$,

$$\triangle^{m^e}(\varphi(t))f(x) = \left(\prod_{j\in e}\triangle_j^{m_j}(\varphi_j(t_j))\right)f(x),$$

and $t_0 = (t_{01},\ldots,t_{0n})$ is a fixed positive vector, $t_0^e = (t_{01}^e, t_{02}^e,\ldots,t_{0n}^e)$, $t_{0j}^e = t_{0j}$ $(j\in e)$, $t_{0j}^e = 0$ $(j\in e')$, and

$$\int\limits_{a^e}^{b^e} f(x)dx^e = \left(\prod_{j\in e}\int\limits_{a_j}^{b_j}dx_j\right)f(x),$$

i.e., integration is carried out only with respect to the variables x_j whose indices belong to e.

The spaces $S^l_{p,\theta,\varphi,\beta}B\left(G_\varphi\right)$ in case $\varphi_j(t_j) = t_j^{\varkappa_j}$,$\beta_j = \frac{a_j}{p}$ $(j\in e_n)$, coincides with the space $S^l_{p,\theta,a,\varkappa}B\left(G\right)$ introduced and studied in [11], in the case $\beta_j = 0$ $(j\in e_n)$, coincides with the space $S^l_{p,\theta}B\left(G\right)$ introduced and studied by A.J. Dzhabrailov [2], in the case $\theta=\infty$, coincides with the space Nikolskii-Morrey with dominant mixed derivatives $S^l_{p,\varphi,\beta}H\left(G_\varphi\right)$.

In the case for any $t_j>0$ $(j\in e_n)$, there exists a constant $C>0$ it holds the embedding

$$L_{p,\varphi,\beta}(G)\hookrightarrow L_p(G), \qquad S^l_{p,\theta,\varphi,\beta}B\left(G_\varphi\right)\hookrightarrow S^l_{p,\theta}B\left(G_\varphi\right),$$

i.e.,

$$\|f\|_{p,G}\le C\|f\|_{p,\varphi,\beta;G}, \ \ \|f\|_{S^l_{p,\theta}B(G_\varphi)}\le C\|f\|_{S^l_{p,\theta,\varphi,\beta}B(G_\varphi)}. \quad (3)$$

Definition 2. [10] *An open set $G\subset R^n$ is said to satisfy condition of flexible φ-horn type, if for some $\omega\in(0,1]^n, T\in(0,\infty)^n$ for any $x\in G$ there exists a vector -function*

$$\rho(\varphi(t),x) = (\rho_1(\varphi_1(t_1),x),\ldots,\rho_n(\varphi_n(t_n),x)), \ 0\le t_j\le T_j, \ (j\in e_n)$$

with the following properties:

1) for all $j \in e_n$, $\rho_j(\varphi_j(t_j), x)$ is absolutely continuous on $[0, T_j]$, $|\rho_j(\varphi_j(t_j), x)| \le 1$ for almost all $t_j \in [0, T_j]$, $j \in e_n$,

2) $\rho_j(0, x) = 0$;

$$x + V(x, \omega) = x + \bigcup_{\substack{0 \le t_j \le T_j, \\ j \in e_n}} [\rho_j(\varphi_j(t_j), x) + \varphi_j(t_j)\omega_j T_j] \subset G.$$

In particular, $\varphi_j = t_j$ $(j \in e_n)$ is the set $V(x, \omega)$ and $x + V(x, \omega)$ will be said to be a set of flexible horn type introduced in [9], if $t_j = t$, $(j \in e_n)$, $\varphi(t) = t^\lambda$ $(t^\lambda = t^{\lambda_1}, t^{\lambda_2}, \dots, t^{\lambda_n})$ is the set $V(x, \omega)$ and $x + V(x, \omega)$ will be said to be a set of flexible λ-horn type introduced in [1].

Theorem 1. *Let $1 \le p < \infty$, $1 \le \theta \le \infty$, $f \in S^l_{p,\theta}B(G_\varphi)$, $\varphi \in \Psi$. Then one can construct the sequence $h_s = h_s(x)$ $(s = 1, 2, \dots)$ of infinitely differentiable finite functions in R^n such that*

$$\lim_{s \to \infty} \|f - h_s\|_{S^l_{p,\theta}B(G)} = 0.$$

Lemma 1. *Let $1 \le p \le q \le r \le \infty$, $0 < \eta_j, t_j \le T_j \le 1$ $(j \in e_n)$, $\nu = (\nu_1, \dots, \nu_n)$, $\nu_j \ge 0$ $(j \in e_n)$ are integers, $\triangle^{m^e}(\varphi(t))f \in L_{p,\varphi,\beta}(G)$ and let*

$$\mu_j = l_j - \nu_j - (1 - \beta_j p)\left(\frac{1}{p} - \frac{1}{q}\right),$$

$$B^e_\eta(x) = \prod_{j \in e'} (\varphi_j(T_j))^{-2-\nu_j} \int_{0^e}^{\eta^e} L_e(x, t) \prod_{j \in e} (\varphi_j(t_j))^{-\nu_j - 2} \prod_{j \in e} \frac{\varphi_j'(t_j)}{\varphi_j(t_j)} dt^e \tag{4}$$

$$B^e_{\eta,T}(x) = \prod_{j \in e'} (\varphi_j(T_j))^{-2-\nu_j} \int_{\eta^e}^{T^e} L_e(x, t) \prod_{j \in e} (\varphi_j(t_j))^{-\nu_j - 2} \prod_{j \in e} \frac{\varphi_j'(t_j)}{\varphi_j(t_j)} dt^e, \tag{5}$$

$$\begin{aligned} L_e(x, t) = \int_{R^n} \int_{-\infty^e}^{+\infty^e} M_e\left(\frac{y}{\varphi(t^e + T^{e'})}, \frac{\rho(\varphi(t^e + T^e), x)}{\varphi(t^e + T^{e'})}\right) \times \\ \times J_e\left(\frac{U}{\varphi(t)}, \frac{\rho(\varphi(t), x)}{\varphi(t)}, \frac{1}{2}\rho'(\varphi(t), x)\right) \times \\ \times \triangle^{m^e}(\varphi(\delta)u) f(x + y + u^e) du^e dy. \end{aligned} \tag{6}$$

Then for any $\overline{x} \in U$ the following inequalities are valid

$$\sup_{\overline{x} \in U} \left\| B^e_\eta \right\|_{q, U_{\psi(\xi)}(\overline{x})} \le C_1 \left\| \prod_{j \in e} (\varphi_j(t_j))^{-l_j} \triangle^{m^e}(\varphi(t)) f \right\|_{p,\varphi,\beta;G} \times$$

$$\times \prod_{j\in e'} \left(\varphi_j\left(T_j\right)\right)^{-\nu_j-(1-\beta_j p)\left(\frac{1}{p}-\frac{1}{q}\right)-1} \prod_{j\in e_n} \left(\psi_j\left([\xi_j]_1\right)\right)^{\beta_j \frac{p}{q}} \prod_{j\in e} \left(\varphi_j\left(\eta_j\right)\right)^{\mu_j} (\mu_j>0), \tag{7}$$

$$\sup_{\overline{x}\in U} \left\|B^e_{\eta,T}\right\|_{q,U_{\psi(\xi)}(\overline{x})} \leq C_2 \left\| \prod_{j\in e} (\varphi_j(t_j))^{-l_j} \triangle^{m^e}(\varphi(t))f \right\|_{p,\varphi,\beta;G}$$

$$\times \prod_{j\in e'} \left(\varphi_j\left(T_j\right)\right)^{-\nu_j-(1-\beta_j p)\left(\frac{1}{p}-\frac{1}{q}\right)-1} \prod_{j\in e_n} \left(\psi_j\left([\xi_j]_1\right)\right)^{\beta_j \frac{p}{q}} \times$$

$$\times \begin{cases} \prod\limits_{j\in e} (\varphi_j(T_j))^{\mu_j}, & \textit{for } \mu_j>0 \\ \prod\limits_{j\in e} \ln \frac{\varphi_j(T_j)}{\varphi_j(\eta_j)}, & \textit{for } \mu_j=0, \\ \prod\limits_{j\in e} \left(\varphi_j\left(\eta_j\right)\right)^{\mu_j}, & \textit{for } \mu_j<0, \end{cases} \tag{8}$$

here $U_{\psi(\xi)}\left(\overline{x}\right)=\left\{x: |x_j-\overline{x}_j|<\frac{1}{2}\psi_j\left(\xi_j\right), j\in e_n\right\}$, *and* $\psi\in\Psi$, C_1 *and* C_2 *are constants independent of* f, ξ, η *and* T.

Proof. Applying sequentially the Minkowskii generalized inequality for any $\overline{x}\in U$

$$\left\|B^e_\eta\right\|_{q,U_{\psi(\xi)}(\overline{x})} \leq \prod_{j\in e'} (\varphi_j(T_j))^{-2-\nu_j} \int\limits_{0^e}^{\eta^e} \left\|L_e\left(\cdot, t^e+T^{e'}\right)\right\|_{q,U_{\psi(\xi)}(\overline{x})} \times$$

$$\times \prod_{j\in e} \left(\varphi_j\left(t_j\right)\right)^{-2-\nu_j} \prod_{j\in e} \frac{\varphi'_j\left(t_j\right)}{\varphi_j\left(t_j\right)} dt^e. \tag{9}$$

From the Hölder inequality ($q\leq r$) we have

$$\left\|L_e\left(\cdot, t^e+T^{e'}\right)\right\|_{q,U_{\psi(\xi)}(\overline{x})} \leq \left\|L_e\left(\cdot, t^e+T^{e'}\right)\right\|_{r,U_{\psi(\xi)}(\overline{x})} \prod_{j\in e_n} \left(\psi_j\left(\xi_j\right)\right)^{\frac{1}{q}-\frac{1}{r}}. \tag{10}$$

Further, we will assume that there exists a function $|M_e(x,y)|\leq C|M^1_e(x)|$, for all $y\in R^n$. Let χ be a characteristic function of the set $S\left(M_e\right)$. Again applying the Hölder inequality ($\frac{1}{r}+(\frac{1}{p}-\frac{1}{r})+(\frac{1}{s}-\frac{1}{r})=1$) for representing function in the form (6) in the case $1\leq p\leq r\leq\infty$, $s\leq r$, $s\leq r$ ($\frac{1}{s}=1-\frac{1}{p}+\frac{1}{r}$), we get

$$\left\|L_e\left(\cdot, t^e+T^{e'}\right)\right\|_{r,U_{\psi(\xi)}(\overline{x})} \leq$$

$$\leq \sup_{x\in U_{\psi(\xi)}(\overline{x})} \left(\int\limits_{R^n} \left| \int\limits_{-\infty^e}^{+\infty^e} |J_e||\triangle^{m^e} f(x+y+u^e) du^e \right|^p \chi\left(\frac{y}{\varphi\left(t^e+T^{e'}\right)}\right) dy \right)^{\frac{1}{p}-\frac{1}{r}} \times$$

$$\times \sup_{y\in V}\left(\int\limits_{U_{\psi(\xi)}(\overline{x})}\left|\int\limits_{-\infty^e}^{+\infty^e}|J_e||\triangle^{m^e}f(y+u^e)|du^e\right|^p dx\right)^{\frac{1}{r}}\times$$

$$\times\left(\int\limits_{R^n}\|M_e^1\left(\frac{y}{\varphi(t^e+T^{e'})}\right)\|^S dy\right)^{\frac{1}{s}}. \tag{11}$$

For any $x\in U$ we have

$$\int\limits_{R^n}\left|\int\limits_{-\infty^e}^{+\infty^e}|J_e||\triangle^{m^e}f(x+y+u^e)|du^e\right|^p\chi\left(\frac{y}{\varphi\left(t^e+T^{e'}\right)}\right)dy\le$$

$$\le\int\limits_{(U+V)_{\varphi(t^e+T^{e'})}(\overline{x})}\left|\int\limits_{-\infty^e}^{+\infty^e}|J_e||\triangle^{m^e}f(y+u^e)|du^e\right|^p dy\le$$

$$\le\int\limits_{G_{\varphi(t^e+T^{e'})}(\overline{x})}\left|\int\limits_{-\infty^e}^{+\infty^e}|J_e||\triangle^{m^e}f(y+u^e)|du^e\right|^p dy\le$$

$$\le\prod_{j\in e}\left(\varphi_j\left(t_j\right)\right)^{l_jp}\int\limits_{G_{\varphi(t^e+T^{e'})}(\overline{x})}\left|\int\limits_{-\infty^e}^{+\infty^e}|J_e||\prod_{j\in e}\left(\varphi_j\left(t_j\right)^{-l_j}\triangle^{m^e}f(y+u^e)\right|du^e\right|^p dy\le$$

$$\le\prod_{j\in e}\left(\varphi_j\left(t_j\right)\right)^{l_jp}\left\|\int\limits_{-\infty^e}^{+\infty^e}|J_e||\prod_{j\in e}\left(\varphi_j\left(t_j\right)\right)^{-l_j}\triangle^{m^e}f(y+u^e)|du^e\right\|^p_{p,G_{\varphi(t^e+T^{e'})}(\overline{x})}\le$$

$$\le\prod_{j\in e}\left(\varphi_j\left(t_j\right)\right)^{l_jp}\prod_{j\in e}\left(\varphi_j\left(t_j\right)\right)^p\|\prod_{j\in e}\left(\varphi_j\left(t_j\right)\right)^{-l_j}\triangle^{m^e}f\|^p_{p,G_{\varphi(t^e+T^{e'})}(\overline{x})}\le$$

$$\le C_1\prod_{j\in e'}\left(\varphi_j\left(T_j\right)\right)^{\beta_jp}\prod_{j\in e}\left(\varphi_j\left(t_j\right)\right)^p\prod_{j\in e}\left(\varphi_j\left(t_j\right)\right)^{l_jp}\prod_{j\in e}\left(\varphi_j\left(t_j\right)\right)^{\beta_jp}$$

$$\times\|\prod_{j\in e}\left(\varphi_j\left(t_j\right)\right)^{-l_j}\triangle^{m^e}\left(\varphi(t)\right)f\|_{p,\varphi,\beta}\cdot\prod_{j\in e}\left(\varphi_j\left(t_j\right)\right)^{\beta_jp}. \tag{12}$$

For $y\in V\ \left(\varphi_j\left(t_j\right)\le\Psi_j\left(t_j\right), j\in e_n\right)$

$$\int\limits_{U_{\psi(\xi)}}\left|\int\limits_{-\infty^e}^{+\infty^e}|J_e||\triangle^{m^e}f(x+y+u^e)|du^e\right|^p dx\le$$

$$\leq \int\limits_{G_{\varphi(\xi)}} \left| \int\limits_{-\infty^e}^{+\infty^e} |J_e||\triangle^{m^e}(\varphi(\delta)u)f(x+u^e)|du^e \right|^p dx \leq$$

$$\leq \prod_{j\in e}(\varphi_j(t_j))^{l_jp} \int\limits_{G_{\varphi(\xi)}} \left| \int\limits_{-\infty^e}^{+\infty^e} |J_e|| \prod_{j\in e}(\varphi_j(t_j))^{-l_j} \triangle^{m^e}(\varphi(\delta)u)f(x+u^e)|du^e \right|^p dx \leq$$

$$\leq \prod_{j\in e}(\varphi_j(t_j))^{l_jp} \| \prod_{j\in e}(\varphi_j(t_j))^{-l_j} \triangle^{m^e}(\varphi(t))f\|^p_{p,G_{\Psi(\xi)}(\overline{x})} \leq$$

$$\leq C_2 \prod_{j\in e}(\varphi_j(t_j))^{pl_j} \prod_{j\in e}(\varphi_j(t_j))^p \prod_{j\in e_n}(\varphi_j([\xi_j]_1))^{\beta_jp} \| \prod_{j\in e}(\varphi_j(t_j))^{-l_j} \triangle^{m^e}(\varphi(t))f\|^p_{p,G_{\Psi(\xi)}(\overline{x})} \leq$$

$$\leq C_1 \prod_{j\in e}(\varphi_j(t_j))^{pl_j} \prod_{j\in e}(\varphi_j(t_j))^p \prod_{j\in e_n}(\Psi_j([\xi_j]_1))^{\beta_jp} \| \prod_{j\in e}(\varphi_j(t_j))^{-l_j} \triangle^{m^e}(\varphi(t))f\|^p_{p,G_{\Psi(\xi)}(\overline{x})} \tag{13}$$

and

$$\int\limits_{R^n} \left| M_e^1 \left(\frac{y}{\varphi(t^e+T^{e'})} \right) \right|^s dy = \left\| M_e^1 \right\|_s \prod_{j\in e} \varphi_j(t_j) \prod_{j\in e'} \varphi_j(T_j). \tag{14}$$

From inequalities (10)-(14) it follows that

$$\|L_e\|_{q,U_{\psi(\xi)}(\overline{x})} \leq C_1 \left\| \prod_{j\in e}(\varphi_j(t_j))^{-l_j} \triangle^{m^e}(\varphi(t))f \right\|_{p,\varphi,\beta} \times$$

$$\times \prod_{j\in e'} (\varphi_j(T_j))^{1-(1-\beta_jp)\left(\frac{1}{p}-\frac{1}{q}\right)} \prod_{j\in e} (\varphi_j(t_j))^{1-(1-\beta_jp)\left(\frac{1}{p}-\frac{1}{q}\right)+l_j} \times$$

$$\times \prod_{j\in e_n} (\psi_j([\xi_j]_1))^{\left(\frac{1}{q}-\frac{1}{r}\right)} \prod_{j\in e_n} (\psi_j([\xi_j]_1))^{\frac{\beta_jp}{q}}. \tag{15}$$

Substituting inequalities in (9) for $(r=q)$, for $\mu_j>0$ $(j\in e)$ we obtain (7). Inequality (8) is proved in the same way.

Corollary 1. *From inequality* (7) *for* $\beta_j^1 = \frac{\beta_jp}{q}$, $j\in e_n$ *it follows that:*

$$\left\| B_\eta^e \right\|_{q,\psi,\beta^1;U} \leq C_2 \left\| \prod_{j\in e}(\varphi_j(t_j))^{-l_j} \triangle^{m^e}(\varphi(t))f \right\|_{p,\varphi,\beta;G}, \tag{16}$$

C_2 *is the constant independent of* f.

2. Main results

We prove two theorems on the properties of functions from the space $S^l_{p,\theta,\varphi,\beta}B\left(G_\varphi\right)$.

Theorem 2. *Let $G \subset R^n$ satisfy the condition of flexible φ-horn [10], $1 \le p \le q \le \infty$ and let $\nu = (\nu_1, \nu_2, .., \nu_n)$, $\nu_j \ge 0$ be entire $j \in e_n$, $\mu_j > 0$ $(j \in e_n)$, and let $f \in S^l_{p,\theta,\varphi,\beta}B(G_\varphi)$.*

Then the following embedding holds

$$D^\nu : S^l_{p,\theta_1,\varphi,\beta}B(G_\varphi) \hookrightarrow L_{q,\psi,\beta^1}(G)$$

i.e., for $f \in S^l_{p,\theta,\varphi,\beta}B\left(G_\varphi\right)$ there exists a generalized derivatives $D^\nu f$ and the following inequalities are true

$$\|D^\nu f\|_{q,G} \le$$

$$\le C_1 \sum_{e \subseteq e_n} \prod_{j \in e_n} (\varphi_j(T_j))^{s_{e,j}} \left\{ \int\limits_{0^e}^{t_0^e} \left[\frac{\|\triangle^{m^e}\left(\varphi(t), G_{\varphi(t)}\right) D^{k^e} f\|_{p,\alpha,\beta}}{\prod\limits_{j \in e} (\varphi_j(t_j))^{l_j - k_j}} \right]^\theta \prod_{j \in e} \frac{d\varphi_j(t_j)}{\varphi_j(t_j)} \right\}^{\frac{1}{\theta}}, \quad (17)$$

$$\|D^\nu f\|_{q,\psi^1,\beta;G} \le C^2 \|f\|_{S^l_{p,\theta,\varphi,\beta}B(G_\varphi)}, \; p \le q < \infty. \quad (18)$$

In particular, if

$$\mu_{j,0} = l_j - \nu_j - (1 - \beta_j p)\frac{1}{p} > 0, \; (j \in e_n),$$

then $D^\nu f(x)$ is continuous in the domain G, and

$$\sup_{x \in G} |D^\nu f(x)| \le$$

$$\le C^2 \sum_{e \subseteq e_n} \prod_{j \in e_n} (\varphi_j(T_j))^{s_{e,j,0}} \left\{ \int\limits_{0^e}^{t_0^e} \left[\frac{\|\triangle^{m^e}\left(\varphi(t), G_{\varphi(t)}\right) D^{k^e} f\|_{p,\alpha,\beta}}{\prod\limits_{j \in e} (\varphi_j(t_j))^{l_j - k_j}} \right]^\theta \prod_{j \in e} \frac{d\varphi_j(t_j)}{\varphi_j(t_j)} \right\}^{\frac{1}{\theta}}, \quad (19)$$

where

$$s_{e,j,0} = \begin{cases} \mu_{j,0}, & j \in e, \\ -\nu_j - (1 - \beta_j p)\frac{1}{p}, & j \in e' \end{cases}$$

$0 \le T_j \le \min\{1, t_{oj}\}$ *$(j \in e_n)$, and C_1, C_2 are the constants indepent of f, C^1 independent of $T = (T_1, T_2, \ldots, T_n)$.*

Proof. Under the conditions of our theorem, there exist generalized derivatives $D^\nu f$. Indeed, if $\mu_j > 0$, $\{j \in e_n\}$, then for $f \in S^l_{p,\theta,\varphi,\beta}B(G_\varphi) \to S^l_{p,\theta}B(G_\varphi)$ there exist generalized derivatives $D^\nu f \in L_p(G)$, and for almost each point $x \in G$ the integral representation [13]

$$D^\nu f(x) = \sum_{e \subseteq e_n} \prod_{j \in e'} (\varphi_j(T_j))^{\nu_j - 2} \int\limits_{0^e}^{T^e} \int\limits_{-\infty^e}^{+\infty^e} \int\limits_{R^n} \prod_{j \in e'} (\varphi_j(T_j))^{-\nu_j - 2}$$

$$\times M_e^{(\nu)}\left(\frac{y}{\varphi\left(t^e+T^{e'}\right)},\frac{\rho\left(\varphi\left(t^e+T^{e'},x\right)\right)}{\varphi\left(t^e+T^{e'}\right)}\right)J_e\left(\frac{u}{\varphi(t)},\frac{\rho(\varphi(t),x)}{\varphi(t)},\frac{1}{2}\rho'(\varphi(t),x)\right)\times$$

$$\times\triangle^{m^e}(\varphi(\delta)u)f(x+y+u^e)du^e dydt \tag{20}$$

with the kernels is valid and $0\leq T_j\leq\min\{1,t_{j,0}\},j\in e_n$, $M_e(\cdot,y)\in C_0^\infty(R^n)$, $\xi_e(\cdot,y,z)\in C_0^\infty(R^{|e|})$, where $R^{|e|}=R_1^e\times R_2^e\times R_n^e$, where $R_j^e=R=(-\infty,+\infty)$, $j\in e$; $R_j^e=1$ $j\in e'$.

Based on Minkowski inequality we have

$$\|D^\nu f\|_{q,G}\leq\sum_{e\subseteq e_n}\|B_T^e\|_{q,G}\,. \tag{21}$$

By means of inequalities (7) for $U=G$, $\eta_j=T_j,(j\in e)$, $p\leq\theta$ we get inequality (17).

By means are inequalities (8) for $\eta_j=T_j,(j\in e)$, and (6), $p\leq\theta$ we get inequality (18).

Now let conditions $\mu_{j,0}=\mu_j(q=\infty)>0$, $(j\in e_n)$, then based around identity (20), for $q=\infty$, $p\leq\theta$ we get

$$\left\|D^\nu f-f_{\varphi(T)}^{(\nu)}\right\|_{\infty,G}\leq$$

$$\leq C\sum_{\varnothing\neq e\subseteq e_n}\prod_{j\in e}(\varphi_j(T_j))^{s_{e,j,0}}\left\{\int\limits_{0^e}^{t_0^e}\left[\frac{\|\triangle^{m^e}(\varphi(t))\,D^{k^e}f\|_{p,\varphi,\beta}}{\prod\limits_{j\in e}(\varphi_j(t_j))^{l_j-k_j}}\right]^\theta\prod_{j\in e}\frac{d\varphi_j(t_j)}{\varphi_j(t_j)}\right\}^{\frac{1}{\theta}}.$$

As $T_j\to 0$, $j\in e$, then $\left\|D^\nu f-f_{\varphi(T)}^{(\nu)}\right\|_{\infty,G}\to 0$. Since $f_{\varphi(T)}^{(\nu)}(x)$ is continuous on G the convergence on $L_\infty(G)$ coincides with the uniform convergence. Then the limit function $D^\nu f(x)$ is continuous on G. Theorem 2 is proved.

Let γ be an n-dimensional vector.

Theorem 3. *Let all the conditions of Theorem 2 be satisfied. Then for $\mu_j>0$ $(j\in e_n)$ the generalized derivatives $D^\nu f$ satisfies on G the generalized Hölder condition, i.e. the following inequality is valid:*

$$\|\Delta(\gamma,G)\,D^\nu f\|_{q,G}\leq C\,\|f\|_{S^l_{p,\varphi,\beta}B(G_\varphi)}\prod_{j\in e_n}(\sigma_j(|\gamma_j|)), \tag{22}$$

where

$$\sigma_j(|\gamma_j|)=\begin{cases}\max\left\{\left(\varphi_j(|\gamma_j^*|)\right)^{\mu_j},\left(\varphi_j(|\gamma_j^*|)\right)^{\mu_{j-1}}\right\}, & \text{for } j\in e,\\ (\varphi_j(T_j))^{\mu_j-l_j}, & \text{for } j\in e',\end{cases}$$

If $\mu_{j,0}>0$ $(j\in e_n)$, then

$$\sup_{x\in G}|\Delta(\gamma,G)D^\nu f(x)|\leq C\|f\|_{S^l_{p,\varphi,\beta}B(G)}\prod_{j\in e_n}(\sigma_{j,0}(|\gamma_j|)). \tag{23}$$

where $\sigma_{j,0}$ satisfies the same conditions as σ_j, but with μ_j replaced $\mu_{j,0}$.

Proof. By Lemma 8.6 from [1] there exists a domain $G_\omega \subset G\,(\omega = (\omega_1, \omega_2, \ldots, \omega_n)$, $\omega_j = \lambda_j \rho(x)$, $\lambda_j > 0$ $(j \in e_n)$, $\rho(x) = dist\,(x, \partial G)$, $x \in G)$.

Suppose that $|\gamma_j| < \omega_j$, $j \in e_n$, then for any $x \in G_\omega$ the segment connecting the points $x, x+\gamma$ is contained in G. Consequently, for all the points of this segment, identity (20) with the same kernels are valid. After same transformations, from (20) we get

$$|\Delta(\gamma, G)\, D^\nu f(x)| \leq C_1 \sum_{e \subseteq e_n} \prod_{j \in e'} (\varphi_j(T_j))^{-\nu_j - 2} \times$$

$$\int_0^{|\gamma_1^e|} \cdots \int_0^{|\gamma_n^e|} \prod_{j \in e} (\varphi_j(t_j))^{-\nu_j - 2} \prod_{j \in e} \frac{\varphi_j'(t_j)}{\varphi_j(t_j)} \times$$

$$\int_{-\infty^e}^{+\infty^e} \int_{R^n} \left| M_e^{(\nu)} \left(\frac{y}{\varphi(t^e + T^{e'})}, \frac{\rho\left(\varphi\left(t^e + T^{e'}, x\right)\right)}{\varphi(t^e + T^{e'})} \right) \times \right.$$

$$\left. \times J_e \left(\frac{u}{\varphi(t)}, \frac{\rho(\varphi(t), x)}{\varphi(t)}, \frac{1}{2}\rho'(\varphi(t), x) \right) \right| \times$$

$$\left| \Delta(\gamma, G) \triangle^{m^e}(\varphi(\delta)u) f(x + y + u^e) \right| du^e dy dt + \prod_{j \in e'} (\varphi_j(T_j))^{-\nu_j - 3} \times$$

$$\prod_{j \in e_n} |\gamma_j| \int_{|\gamma_1^e|}^{T_1^e} \cdots \int_{|\gamma_n^e|}^{T_n^e} \prod_{j \in e} (\varphi_j(t_j))^{-\nu_j - 3} \prod_{j \in e} \frac{\varphi_j'(t_j)}{\varphi_j(t_j)} \times$$

$$\int_{-\infty^e}^{+\infty^e} \int_{R^n} \left| M_e^{(\nu)} \left(\frac{y}{\varphi(t^e + T^{e'})}, \frac{\rho\left(\varphi\left(t^e + T^{e'}, x\right)\right)}{\varphi(t^e + T^{e'})} \right) \right| \times$$

$$\left. \times J_e \left(\frac{u}{\varphi(t)}, \frac{\rho(\varphi(t), x)}{\varphi(t)}, \frac{1}{2}\rho'(\varphi(t), x) \right) \right| \times$$

$$\int_0^1 \cdots \int_0^1 \left| \triangle^{m^e}(\varphi(\delta)u) f(x + y + u^e + \gamma v) \right| dv dy du^e dt =$$

$$= C_1 \sum_{e \subseteq e_n} \left(B_e^1(x, \gamma) + B_e^2(x, \gamma) \right), \tag{24}$$

where $|\gamma_j^e| = |\gamma|$ $(j \in e)$, $0 < T_j \leq t_{0,j}$ $j \in e_n$. We also assume that $|\gamma_j| < T_j (j \in e_n)$, and consequently, $|\gamma_j| < \min(\omega_j, T_j)(j \in e_n)$. If $x \in G \setminus G_\omega$, then

$$\Delta(\gamma, G)\, D^\nu f(x) = 0.$$

Based around (24) we have

$$\|\Delta(\gamma,G)\,D^{\nu}f\|_{q,G}\le C^{1}\sum_{e\subseteq e_n}\Big(\left\|B_e^1(\cdot,\gamma)\right\|_{q,G_\omega}+$$

$$+\left\|B_e^2(\cdot,\gamma)\right\|_{q,G_\omega}\Big) \tag{25}$$

By means of inequality (7), for $U=G$, $\eta_j=|\gamma_j|$ $(j\in e)$ we have

$$\left\|B_e^1(\cdot,\gamma)\right\|_{q,G_\omega}\le C_1\left\|\prod_{j\in e}(\varphi_j(t_j))^{-l_j}\triangle^{m^e}(\varphi(t))f\right\|_{p,\varphi,\beta;G}\prod_{j\in e'}(\varphi_j(T_j))^{\mu_j-l_j}. \tag{26}$$

and by means of inequality (8) for $U=G$, $\eta_j=|\gamma_j|$ $(j\in e_n)$ we have

$$\left\|B_e^2(\cdot,\gamma)\right\|_{q,G_\omega}\le C_2\left\|\prod_{j\in e}(\varphi_j(t_j))^{-l_j}\triangle^{m^e}(\varphi(t))f\right\|_{p,\varphi,\beta;G}\prod_{j\in e'}\varphi_j(T_j)^{\mu_j-l_j}\times$$

$$\times\prod_{j\in e}(\varphi_j(|\gamma_j|))^{\mu_j-1}. \tag{27}$$

Now suppose that $|\gamma_j|\ge\min(\omega_j,T_j)\,,(j\in e_n)$, then

$$\|\Delta(\gamma,G)\,D^{\nu}f\|_{q,G}\le 2\,\|D^{\nu}f\|_{q,G}\le C\,(\omega,T)\,\|D^{\nu}f\|\prod_{j\in e_n}(\sigma_j(|\gamma_j|)).$$

Estimating for $\|D^{\nu}f\|_{q,G}$ by means of inequality (17), in this case, we again get the required inequality. Theorem 3 is proved.

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