Caspian Journal of Applied Mathematics, Ecology and Economics
V. 11, No 1, 2023, July
ISSN 1560-4055

Basic Property of Eigenfunctions of the Eigenvalue Problem for Fourth-Order Ordinary Differential Equations with a Spectral Parameter Contained in ll Boundary Conditions

Ayna E. Fleydanli

Abstract. This paper considers the spectral problem for fourth-order ordinary differential equations, all boundary conditions of which contain a spectral parameter. This problem describes small bending vibrations of an Euler-Bernoulli beam in the cross sections of which a longitudinal force acts, at both ends of which follower forces act, and also loads are attached to these ends using weightless rods, which are kept in balance by elastic springs. The basis properties of the system of eigenfunctions of the problem under consideration in the space L_p, $1 < p < \infty$, are studied.

Key Words and Phrases: Euler-Bernoulli beam, spectral parameter, eigenvalue, eigenfunction, basis property

2010 Mathematics Subject Classifications: 34A30, 34B08, 34B09, 34C10, 34C23, 47A75, 74H45

1. Introduction

We consider the following eigenvalue problem

$$\ell(y)(x) \equiv y^{(4)}(x) - (q(x)y'(x))' = \lambda y(x),\ 0 < x < 1, \tag{1.1}$$

$$y''(0) - a\lambda y'(0) = 0, \tag{1.2}$$

$$Ty(0) - b\lambda y(0) = 0, \tag{1.3}$$

$$y''(1) - c\lambda y'(1) = 0, \tag{1.4}$$

$$Ty(1) - d\lambda y(1) = 0, \tag{1.5}$$

where $\lambda \in \mathbb{C}$ is a spectral parameter, $Ty \equiv y''' - qy'$, q is a positive absolutely continuous function on $[0, 1]$, $a,\ b,\ c,\ d$ are real constants such that $a > 0$, $b > 0$, $c > 0$ and $d < 0$.

Note that since the second half of the last century, boundary value problems for the Sturm-Liouville equations of the second and fourth orders with boundary conditions depending on the spectral parameter have been intensively studied (see [1-10, 13, 16-25]). This is due to the fact that these problems describe small longitudinal, torsional and transverse vibrations of a beam, at the ends of which either loads or inertial loads are concentrated, or tracking forces act (see [14-16, 18, 24]). For example, the spectral problem (1.1)-(1.5) we are studying arises when describing small bending vibrations of an elastic homogeneous cantilever beam, in the cross sections of which a longitudinal force acts, loads are attached to the ends using weightless rods, which are kept in balance by elastic springs, as well as both of them are subject to tracking forces [14]. It should be noted that to study this problem of mechanics, we need to study the convergence of expansions in the system of root functions of problem (1.1)-(1.5) in various function spaces.

The spectral properties of second-order Sturm-Liouville problems with a spectral parameter in boundary conditions and fourth-order Sturm-Liouville problems with a spectral parameter in boundary conditions (but not in all boundary conditions) were studied in works [1-10, 13, 16-25] (see also their bibliography).

The purpose of this article is to study the spectral properties, including the basic properties of the system of root functions in the space L_p, $1 < p < \infty$ of the spectral problem (1.1)-(1.5).

2. Some properties of solutions to the initial boundary value problem (1.1), (1.3)-(1.5)

In this section we consider the initial boundary value problem (1.1), (1.3)-(1.5). For the study of this problem we introduce the following boundary condition

$$y'(0)\cos\alpha - y''(0)\sin\alpha = 0,\ \alpha \in [0, \pi/2]. \tag{2.1}$$

Following the corresponding reasoning carried out in [8], we can prove the following oscillatory theorem for the eigenvalue problem (1.1), (2.1), (1.3)-(1.5).

Theorem 2.1. *For each α the spectrum of problem (1.1), (2.1), (1.3)-(1.5) consists real and simple eigenvalues forming an infinitely increasing sequence $\{\lambda_k(\alpha)\}_{k=1}^{\infty}$ such that $\lambda_1(\alpha) = 0$ and $\lambda_k(\alpha) > 0$ for $k \geq 2$. $\{\lambda_k(\alpha)\}_{k=1}^{\infty}$ such that $\lambda_1(\alpha) = 0$ and $\lambda_k(\alpha) > 0$ for $k \geq 2$. Moreover, for each $k \in \mathbb{N}$ the eigenfunction $y_{k,\alpha}(x)$ corresponding to the eigenvalue $\lambda_{k,\alpha}(x)$ and its derivative has the following oscillatory property:*

(i) the eigenfunction $y_{k,\alpha}(x)$for $k \geq 3$ has either $k-2$ or $k-1$ simple zeros in $(0,1)$, while the function $y_{1,\alpha}(x)$ has no zeros and $y_{2,\alpha}(x)$ has one simple zero in $(0,1)$;

(ii) the function $y'_{k,\alpha}(x)$ for $k \geq 2$ has exactly $k-2$ simple zeros in the interval $(0,1)$.

By the maximum-minimum property of eigenvalues (see [15, Ch. 6, § 1, p. 405]), it follows from Theorem 2.1 that the following relation holds:

$$\lambda_1(\pi/2) = \lambda_1(\alpha) = \lambda_1(0) = 0 < \lambda_2(\pi/2) < \lambda_2(\alpha) < \lambda_2(0) < \lambda_3(\pi/2) < \lambda_3(\alpha) < \lambda_3(0) < \ \ldots \ . \tag{2.2}$$

Theorem 2.2. *For each fixed $\lambda \in \mathbb{C}\backslash\{0\}$ there is a nontrivial solution $y(x,\lambda)$, unique up to a constant factor, of problem (1.1), (1.3)-(1.5). The function $y(x,\lambda)$ for each fixed $x \in [0,l]$ is an entire function of parameter λ, which has the following representation:*

$$y(x,\lambda) = -D_2(\lambda)\left\{\psi_1(x,\lambda) + d\lambda\psi_4(x,\lambda)\right\} + D_1(\lambda)\left\{\psi_2(x,\lambda) + c\lambda\psi_3(x,\lambda)\right\}, \tag{2.3}$$

where $\varphi_k(x,\lambda), k = \overline{1,4}$, is solutions of equation (1.1) satisfying the Cauchy conditions (normalized for $x = 1$)

$$\psi_k^{(s-1)}(1,\lambda) = \delta_{ks}, \ s = 1,\, 2,\, 3, \ T\psi_k(1,\lambda) = \delta_{k4}, \tag{2.4}$$

δ_{ks} is the Kronecker delta, and

$$D_1(\lambda) = T\psi_1(0,\lambda) \ + \ d\lambda T\psi_4(0,\lambda) - \ b\lambda\left\{\psi_1(0,\lambda) \ + \ d\lambda\psi_4(0,\lambda)\right\},$$

and

$$D_2(\lambda) = T\psi_2(0,\lambda) \ + \ c\lambda T\psi_3(0,\lambda) - \ b\lambda\left\{\psi_2(0,\lambda) \ + \ c\lambda\psi_3(0,\lambda)\right\},$$

.

The proof of this theorem is similar to the proof of Theorem 2.2 of [8] with regard to Theorem 2.1.

Remark 2.1. Let $y(x,\lambda)$, $\lambda \in \mathbb{R}\backslash\{0\}$, be the nontrivial solution of the spectral problem (1.1), (1.3)-(1.5). Then this function can be normalized using the condition

$$y'(1,\lambda) = 1, \tag{2.5}$$

if $\lambda > 0$, and using the condition

$$y'(0,\lambda) = 1, \tag{2.6}$$

if $\lambda < 0$, in view of [12, Theorems 2.1 and 2.2].

By the second part of [12, Lemma 2.1] we get $D_1(\lambda) < 0$. Therefore, by Remark 2.1 and formula (2.3), without loss of generality, we can represent the function $y(x,\lambda)$ for $\lambda > 0$ in the form

$$y(x,\lambda) = -\tfrac{D_2(\lambda)}{D_1(\lambda)}\left\{\psi_1(x,\lambda) + d\lambda\psi_4(x,\lambda)\right\} + \psi_2(x,\lambda) + c\lambda\psi_3(x,\lambda). \tag{2.7}$$

Remark 2.2. If $\lambda = 0$, then according to relations (2.4) problem (1.1), (1.3)-(1.5) has two linearly independent solutions $y_1(x,0) = \psi_1(x,0) = 1$ and $y_2(x,0) = \psi_2(x,0)$, $x \in [0,1]$.

Remark 2.3. By [8, formulas (2.10) and (2.11)] (with replacement $x = 0$ by $x = 1$) from (2.4) for $y(x,\,\lambda)$ we obtain the following representation

$$y(x,\lambda) = A(\lambda)\left\{\psi_1(x,\lambda) + d\lambda\psi_4(x,\lambda)\right\} + \psi_2(x,\lambda) + c\lambda\psi_3(x,\lambda), \tag{2.8}$$

where

$$A(\lambda) = -\frac{\int\limits_0^1 \psi_2(t,\lambda)\,dt + c\lambda\int\limits_0^1 \psi_3(t,\lambda)\,dt + b\,\{\psi_2(0,\lambda) + c\lambda\psi_3(0,\lambda)\}}{\int\limits_0^1 \psi_1(t,\lambda)\,dt - d + d\lambda\int\limits_0^1 \varphi_4(t,\lambda)\,dt + b\,\{\psi_1(0,\lambda) + d\lambda\psi_4(0,\lambda)\}}. \tag{2.9}$$

Then passing to the limit as $\lambda \to 0$ in (2.8) we get

$$\lim_{\lambda\to 0} y(x,\lambda) = \psi_2(x,0) - \frac{\int\limits_0^1 \psi_2(t,0)\,dt + b\psi_2(0,0)}{1+b-d}.$$

Therefore, if we put

$$y(x,0) = \psi_2(x,0) - \frac{\int\limits_0^1 \psi_2(t,0)\,dt + b\psi_2(0,0)}{1+b-d}. \tag{2.10}$$

then the solution $y(x,\lambda)$ to problem (1.1), (1.3)-(1.5) will be defined everywhere on $[0,1]\times\mathbb{C}$.

We consider the function

$$G(\lambda) = \frac{y''(0,\lambda)}{y'(0,\lambda)}$$

which is well defined on

$$\mathcal{M} \equiv (\mathbb{C}\backslash\mathbb{R}) \cup (-\infty,\, \lambda_2(0)) \cup \left(\bigcup_{k=3}^{\infty} (\lambda_{k-1}(0), \lambda_k(0))\right).$$

It follows from Theorems 2.1 and 2.2 that $G(\lambda)$ is a meromorphic function of finite order and the eigenvalues $\lambda_k\,(\pi/2)$ and $\lambda_k(0)$, $k = 2, 3, \ldots$, of problem (1.1), (2.1), (1.3)-(1.5) for $\alpha = \pi/2$ and $\alpha = 0$ are zeros and poles of this function, respectively.

Lemma 2.1 *One has the following relations:*

$$\frac{dG(\lambda)}{d\lambda} = \frac{1}{y'^2(0,\lambda)}\left\{\int\limits_0^1 y^2(x,\lambda)\,dx + by^2(0,\lambda) + cy'^2(1,\lambda) - dy^2(1,\lambda)\right\},\ \lambda \in \mathcal{M}. \tag{2.11}$$

$$\lim_{\lambda\to-\infty} G(\lambda) = -\infty \tag{2.12}$$

The proof of formulas (2.11) and (2.12) is similar to that of [8, formula (2.19)] and [6, formula (3.8)], respectively.

Remark 2.4 By conditions $b > 0$, $c > 0$ and $d < 0$ it follows from (2.11) that

$$\frac{dG(\lambda)}{d\lambda} > 0 \text{ for } \lambda \in \mathcal{M}. \tag{2.13}$$

Remark 2.5 By (2.3), it follows from the second part of [12, Lemma 2.1] that

$$\psi_2(x,\lambda) < 0,\ \psi_2'(x,\lambda) > 0,\ \psi_2''(x,\lambda) < 0 \text{ and } T\psi_2(x,\lambda) > 0 \text{ for } x \in [0,1) \text{ and } \lambda > 0.$$

Hence we have

$$\psi_2''(x,0) \leq 0 \text{ and } T\psi_2(x,0) \geq 0 \text{ for } x \in [0,1]. \tag{2.14}$$

In view of (2.14) we get

$$\psi_2'''(x,0) \geq q(0)\,\psi'(x,0) > 0 \text{ for } x \in [0,1].$$

By the relation $\psi_2''(1,0) = 0$ it follows from last relation that $\psi_2''(0,0) < 0$, and consequently, we have the following relation

$$G(0) = \frac{\psi_2''(0,0)}{\psi_2'(0,0)} < 0. \tag{2.15}$$

3. The properties of eigenvalues of the eigenvalue problem (1.1)-(1.5)

Lemma 3.1 *The non-zero eigenvalues of problem (1.1)-(1.5) are real and simple.*

Proof. It is obvious that the non-zero eigenvalues of problem (1.1)-(1.5) are the roots of the equation

$$y''(0,\lambda) - a\lambda y'(0,\lambda) = 0. \tag{3.1}$$

If λ is a non-real eigenvalue of problem (1.1)-(1.5), then, due to the realness of the coefficients q, a, b, c and d from (1.1)-(1.5) it follows that $\overline{\lambda}$ is also its eigenvalue. Note that in this case to the eigenvalue $\overline{\lambda}$ corresponds the eigenfunction $y(x,\overline{\lambda}) = \overline{y(x,\lambda)}$, therefore (3.1) also holds for $\overline{\lambda}$.

By (1.1) for any λ, $\mu \in \mathbb{C}$ we have

$$(Ty(x,\mu))'\,y(x,\lambda) - (Ty(x,\lambda))'\,y(x,\mu) = (\mu-\lambda)y(x,\mu)y(x,\lambda). \tag{3.2}$$

Integrating equality (3.2) from 0 to 1, using the formula integration by parts to this resulting equality, and taking into account boundary conditions (1.3)-(1.5) we get

$$\begin{aligned} y''(0,\mu)\,y'(0,\lambda) - y''(0,\lambda)\,y'(0,\mu) &= (\mu-\lambda)\left\{\int_0^1 y(x,\mu)\,y(x,\lambda)dx + \right. \\ & \left. by(0,\mu)\,y(0,\lambda) + cy'(1,\mu)\,y'(1,\lambda) - dy(1,\mu)\,y(1,\lambda)\right\}. \end{aligned} \tag{3.3}$$

Setting $\mu = \overline{\lambda}$ in (3.3), using (3.1) and the relation $\lambda \neq \overline{\lambda}$ we obtain

$$\int_0^1 |\,y(x,\lambda)|^2 dx\ - a|y'(0,\lambda)|^2 + b\,|y(0,\lambda)|^2 + c\,|y'(1,\lambda)|^2 - d\,|y(1,\lambda)|^2 = 0. \tag{3.4}$$

On the other hand multiplying (1.1) by $\overline{y(x,\lambda)}$, integrating resulting equality from 0 to 1, using the formula integration by parts, and taking into account boundary conditions (1.2)-(1.5) we get

$$\begin{gathered}\int\limits_0^1 \left\{ |\,|y''(x,\lambda)|^2 + q(x)|y'(x,\lambda)|^2 \right\} dx = \\ \lambda \left\{ \int\limits_0^1 |\,y(x,\lambda)|^2 dx\; - a|y'(0,\lambda)|^2 + b\,|y(0,\lambda)|^2 + c\,|y'(1,\lambda)|^2 - d\,|y(1,\lambda)|^2 \right\},\end{gathered} \tag{3.5}$$

which, by (3.4), implies that

$$\int\limits_0^1 \left\{ |\,|y''(x,\lambda)|^2 + q(x)|y'(x,\lambda)|^2 \right\} dx = 0.$$

Since q is a positive continuous function on $[0,1]$ it follows from last relation that $y'(x,\lambda) \equiv 0$, which contradicts equality (1.1). The proof of this lemma is complete.

Remark 3.1 Note that the function on the left side of the equation (3.1) is entire and, by Lemma 3.1, does not have zero values for non-real λ. Hence, this function does not vanish identically. Consequently, the zeros of this function form a countable set without a finite limit point.

Lemma 3.2 *The non-zero eigenvalues of problem (1.1)-(1.5) are simple.*

Proof. If $\lambda \neq 0$ is an eigenvalue of (1.1)-(1.5) such that $y'(0,\lambda) = 0$, then it follows from (3.1) that $y''(0,\lambda) = 0$ in contradiction with relation (2.2). Hence by (3.1) non-zero eigenvalues of problem (1.1)-(1.5) are the roots of the equation

$$G(\lambda) = a\lambda. \tag{3.6}$$

Let $\tilde{\lambda} \neq 0$ be the double eigenvalue of problem (1.1)-(1.5). Then we have

$$G(\tilde{\lambda}) = a\tilde{\lambda} \text{ and } G'(\tilde{\lambda}) = a. \tag{3.7}$$

By the second relation on (3.7) from (2.11) we obtain

$$\int\limits_0^1 y^2(x,\tilde{\lambda})dx\; - ay'^2(0,\tilde{\lambda}) + b\,y^2(0,\tilde{\lambda}) + c\,y'^2(1,\tilde{\lambda}) - d\,y^2(1,\tilde{\lambda}) = 0. \tag{3.8}$$

Since $\tilde{\lambda}$ is real by (3.8) it follows from (3.5) that

$$\int\limits_0^1 \left\{ y''^2(x,\tilde{\lambda}) + q(x)y'^2(x,\tilde{\lambda}) \right\} dx = 0. \tag{3.9}$$

which implies that $y'(x,\tilde{\lambda}) \equiv 0$, in contradiction with equality (1.1). The proof of this lemma is complete.

Remark 3.2 For $\lambda = 0$, the general solution to problem (1.1), (1.3)-(1.5) has the form

$$v(x) = \tau_1 + \tau_2 \psi_2(x, 0),\ x \in [0, 1].$$

Then it follows from (3.1) that $\tau_2 \psi_2''(0, 0) = 0$. Hence by Remark 2.5 we have $\psi_2''(0, 0) \neq 0$, and consequently, $\tau_2 = 0$. Therefore, $\lambda = 0$ is a simple eigenvalue of the spectral problem (1.1)-(1.5) and without loss of generality we can assume that this eigenvalue has an eigenfunction $v(x) \equiv 1$.

Lemma 3.3 *In each of the intervals* $(-\infty, 0)$*,* $(0, \lambda_2(0)$*,* $(\lambda_{k-1}(0), \lambda_k(0))$*,* $k = 3, 4, \ldots$ *, equation (3.6) cannot have more than one solution.*

Proof. Let $\lambda^* \in (-\infty, 0)$ be a solution of problem (3.6). Then by Lemma 3.2 we have $G'(\lambda) - a \neq 0$. Since $\lambda^* \in (-\infty, 0)$ it follows from (3.5) (with $\tilde{\lambda}$ replaced by λ^*) that

$$\int_0^1 y^2(x, \lambda^*)dx - ay'^2(0, \lambda^*) + b\, y^2(0, \lambda^*) + c\, y'^2(1, \lambda^*)^2 - d\, y^2(1, \lambda^*) < 0. \tag{3.10}$$

and consequently, $G'(\lambda^*) - a < 0$. Therefore, the function $G(\lambda) - a\lambda$ except λ^* cannot have another solution in the interval $(-\infty, 0)$.

The remaining cases are considered similarly. The proof of this lemma is complete.

Theorem 3.1 *The eigenvalues of problem (1.1)-(1.5) form an infinitely increasing sequence* $\{\lambda_k\}_{k=1}^{\infty}$ *such that*

$$\lambda_1 \in (-\infty, 0),\ \lambda_2 = 0,\ \lambda_3 \in (\lambda_2(\pi/2), \lambda_2(0)),\ \ldots,\ \lambda_k \in (\lambda_{k-1}(\pi/2), \lambda_{k-1}(0)),\ \ldots\ . \tag{3.11}$$

Proof. Following the corresponding reasoning carried out in the proof of Lemma 3.3 of [7], we can verify that for the function $G(\lambda)$ the following representation holds

$$G(\lambda) = G(0) + \sum_{k=2}^{\infty} \frac{c_k \lambda}{\lambda_k(0)(\lambda - \lambda_k(0))},\ \lambda \in \mathcal{M}, \tag{3.12}$$

where $c_k = \underset{\lambda=\lambda_k(0)}{res} G(\lambda) < 0,\ k = 2, 3, \ldots$. Hence it follows from (3.12) that

$$G''(\lambda) = 2 \sum_{k=2}^{\infty} \frac{c_k}{(\lambda - \lambda_k(0))^3},\ \lambda \in \mathcal{M}. \tag{3.13}$$

By (3.13) we have $G''(\lambda) > 0$ for $\lambda \in (-\infty, \lambda_2(0))$, i.e. the function $G(\lambda)$ is convex on $(-\infty, \lambda_2(0))$.

In view of Lemma 2.1 and representation (3.12) we get the following relations:

$$\lim_{\lambda \to \lambda_k(0) - 0} G(\lambda) = +\infty,\ \lim_{\lambda \to \lambda_k(0) + 0} G(\lambda) = -\infty,\ k = 2, 3, \ldots\ . \tag{3.14}$$

Since the function $G(\lambda)$ is increasing (see Remark 2.4) and convex in the interval $(-\infty, \lambda_2(0))$ and $G(0) < 0$, and the function $a\lambda$ is increasing in the same interval, the straight line $a\lambda$ intersects the graph of the function $G(\lambda)$ in the interval $(-\infty, \lambda_2(0))$ at two points, one of which lies in $(-\infty, 0)$, and the other lies in the interval $(\lambda_2(\pi/2), \lambda_2(0))$. Thus, by Remark 3.2, problem (1.1)-(1.5) in the interval $(-\infty, \lambda_2(0))$ has three simple eigenvalues λ_1, λ_2 and λ_3 such that

$$\lambda_1 \in (-\infty, 0), \;\; \lambda_2 = 0 \text{ and } \lambda_3 \in (\lambda_2(\pi/2), \lambda_2(0)).$$

Next, by relations (2.11), (3.14) and Lemma 3.3, for each $k \in \mathbb{N}$, $k \geq 3$, the straight line $a\lambda$ intersects the graph of the function $G(\lambda)$ in the interval $(\lambda_{k-1}(0), \lambda_k(0))$ at one point which lies in $(\lambda_k(\pi/2), \lambda_k(0))$. Therefore, problem (1.1)-(1.5) in the interval $(\lambda_{k-1}(0), \lambda_k(0))$, $k = 3, 4, \ldots$, has one simple eigenvalues λ_{k+1} such that

$$\lambda_{k+1} \in (\lambda_k(\pi/2), \lambda_k(0)).$$

The proof of this theorem is complete.

Theorem 3.1 *For the eigenvalues and eigenfunctions of problem the following asymptotic formulas hold:*

$$\sqrt[4]{\lambda_k} = (k - 7/2)\,\pi + O\,(1/k)\,, \tag{3.15}$$

$$\begin{gathered} y_k(x) = -\tfrac{c\sqrt{\lambda_k}}{2}\left\{\sin\,(k - 7/2)\,\pi(x-1) + \cos\,(k - 7/2)\,\pi(x-1) + \right. \\ \left. (-1)^k e^{-\,(k-7/2)\pi x} - e^{(k-7/2)\pi(x-1)} + O\,(1/k)\right\}, \end{gathered} \tag{3.16}$$

where relation (3.16) holds uniformly for $x \in [0, 1]$.

The proof of this theorem is similar to that of [21, Theorem 3.1] with the use of [8, Theorem 3.2] and (3.11).

4. Operator interpretation of the eigenvalue problem (1.1)-(1.5)

It is known (see, for example, [6, 8]) that the spectral problem (1.1)-(1.5) reduces to the eigenvalue problem for the linear operator L in the Hilbert space $H = L_2(0, 1) \oplus \mathbb{C}^4$, equipped with scalar product

$$\begin{gathered} (\hat{y}, \hat{v})_H = (\{y, m, n, \varrho, \sigma\}, \{v, s, t, \varsigma, \tau\})_H = \\ \int_0^1 y(x)\,\overline{v(x)}\,dx + |\,a|^{-1} m\bar{s} + |\,b|^{-1} n\bar{t} + |\,c|^{-1} \varrho\bar{\sigma} + |\,d|^{-1} \varsigma\bar{\tau}, \end{gathered} \tag{4.1}$$

where operator L define by

$$L\hat{y} = L\{y, m, n, \tau, \sigma\} = \{\ell(y),\, y''(0),\, Ty(0),\, y''(1),\, Ty(1)\}$$

on the domain

$$\begin{aligned} D(L) = \{\{y\,(x),\, m,\, n\} \in H \,:\, y \in W_2^4(0, 1),\, \ell(y) \in L_2(0, 1), \\ m = ay'(0), n = by(0),\, \tau = cy'(1),\, \sigma = dy(1)\}. \end{aligned}$$

which is dense everywhere in H. Then problem (1.1)-(1.5) is equivalent to the spectral problem

$$L\hat{y} = \lambda\hat{y}, \ \hat{y} \in D(L),$$

i.e., the eigenvalues λ_k, $k \in \mathbb{N}$, of problem (1.1)-(1.5) and the operator L coincide and between the eigenvectors, there is a one-to-one correspondence

$$y_k(x) \leftrightarrow \ \hat{y}_k = \{y_k(x), \, m_k, \, n_k, \tau_k, \sigma_k\}, \ m_k = ay_k'(0),$$

$$n_k = by_k\,(0), \ \varrho_k = cy_k'(1), \ \sigma_k = dy_k\,(1), \ k \in \mathbb{N}.$$

If $a < 0$, then L is a positive, self-adjoint and discrete operator in H, and consequently, the system of eigenvectors $\{y_k(x), m_k, n_k, \varrho_k, \sigma_k\}_{k=1}^{\infty}$ of this operator forms an orthogonal basis in H.

If $a > 0$, then L is a closed (nonself-adjoint) and discrete operator in H.

Let J be the linear operator defined in H by

$$J\{y, m, n, \tau, \sigma\} = \{y, -m, n, \tau, \sigma\}$$

Note that J is a unitary and symmetric operator in H spectrum of which consists of two eigenvalues: -1 with multiplicity 1 and $+1$ with infinite multiplicity (see [13, Lemma 2.1]). Hence this operator generates the Pontryagin space $\Pi_1 = L_2(0,1) \oplus \mathbb{C}^4$ equipped with inner product (or more precisely J-metric) [11]

$$\begin{gathered}(\hat{y}, \hat{v})_{\Pi_1} = (\hat{y}, J\hat{v})_H = (\{y, m, n, \varrho, \sigma\}, \{v, s, t, \varsigma, \tau\})_{\Pi_1} = \\ \int_0^1 y(x)\,\overline{v(x)}\,dx - \, a^{-1}m\bar{s} \, + \, b^{-1}n\bar{t} + \, c^{-1}\varrho\bar{\sigma} - \, d^{-1}\varsigma\bar{\tau}.\end{gathered} \tag{4.2}$$

Theorem 4.1 *L is a J-self-adjoint operator in Π_1.*

The proof of this Theorem is similar to that of [13, Theorem 2.2] with the use of [11, Propostions 1^o and 2^o].

Theorem 4.2 *If L is the adjoint operator of L in H, then $L = JLJ$. Moreover, the system of eigenvectors $\{\hat{y}_k\}_{k=1}^{\infty}$, $\hat{y}_k = \{y_k, m_k, n_k, \varrho_k, \sigma_k\}$, of the operator L forms an unconditional basis in H.*

The first statement of this theorem follows from [11, § 3, Propostion 5] and the second statement follows from [11, § 4, Theorem 4.2].

By Theorem 3.1 we get

$$L\hat{y}_k = \lambda_k\hat{y}_k, \ k \in \mathbb{N}. \tag{4.3}$$

Let $\{\hat{v}_k^*\}_{k=1}^{\infty}$, $\hat{v}_k^* = \{v_k, s_k, t_k, \varsigma_k, \tau_k\}$, be the system of eigenvectors of operator L^*. Then, view of Theorem 3.1 and (4.3), we have

$$L^*\hat{v}_k = \lambda_k\hat{v}_k, \ k \in \mathbb{N}. \tag{4.4}$$

On the base of first part of Theorem 4.2 it follows from (4.3) and (4.4) that

$$\hat{v}_k^* = J\hat{y}_k, \ k \in \mathbb{N}. \tag{4.5}$$

By (4.1), (4.2), (4.5) and Theorem 4.1 for any k, $l \in \mathbb{N}$, $k \neq l$, we get

$$(\hat{y}_k, \hat{v}_l)_H = (\hat{y}_k, \hat{y}_l)_{\Pi_1} = 0. \tag{4.6}$$

By Lemma 3.2 and Remark 3.2 we have $G'(\lambda_k) - a \neq 0$ which, by (2.1), implies that

$$(\hat{y}_k, \hat{v}_k)_H = (\hat{y}_k, \hat{y}_k)_{\Pi_1} = \int_0^1 y_k^2(x)dx \, - a y_k'^2(0) + b\, y_k^2(0) + c\, y_k'^2(1)^2 - d\, y_k^2(1) \neq 0. \tag{4.7}$$

Theorem 4.3 *Let $\delta_k = (\hat{y}_k, \hat{y}_k)_{\Pi_1}$. Then each element $\hat{v}_k = \{v_k, s_k, t_k, \varsigma_k, \tau_k\}$, $k \in \mathbb{N}$, of the system $\{\hat{v}_k\}_{k=1}^{\infty}$ adjoint to the system $\{\hat{y}_k\}_{k=1}^{\infty}$ is defined as follows:*

$$\hat{v}_k = \delta_k^{-1} \hat{y}_k. \tag{4.8}$$

The proof of this theorem follows from (4.6) and (4.7).

5. Basis property of subsystems of the system of eigenfunctions of the spectral problem (1.1)-(1.5)

Let i, j, r and l be different arbitrary fixed natural numbers and

$$\Delta_{i,j,r,l} = \begin{vmatrix} s_i & t_i & \varsigma_i & \tau_i \\ s_j & t_j & \varsigma_j & \tau_j \\ s_r & t_r & \varsigma_r & \tau_r \\ s_l & t_l & \varsigma_l & \tau_l \end{vmatrix}. \tag{5.1}$$

Theorem 4.3 *If $\Delta_{i,j,r,l} \neq 0$, then the system $\{y_k(x)\}_{k=1,\, k \neq i,j,r,l}^{\infty}$ is a basis in $L_p(0,1), 1 < p < \infty$ (and even an unconditional basis for $p = 2$). If $\Delta_{i,j,r,l} = 0$, then the system $\{y_k(x)\}_{k=1,\, k \neq i,j,r,l}^{\infty}$ is neither complete nor minimal in $L_p(0,1)$, $1 < p < \infty$.*

The proof of this theorem is similar to that of [5, Theorem 4.1] with the use of (3.15) and (3.16).

By (4.8) from (5.1) we obtain

$$\Delta_{i,j,r,l} = \delta_i^{-1} \delta_j^{-1} \delta_r^{-1} \delta_l^{-1} \tilde{\Delta}_{i,j,r,l}, \tag{5.2}$$

where

$$\tilde{\Delta}_{i,j,r,l} = \begin{vmatrix} y_i'(0) & y_i(0) & y_i'(1) & y_i(1) \\ y_j'(0) & y_j(0) & y_j'(1) & y_j(1) \\ y_r'(0) & y_r(0) & y_r'(1) & y_r(1) \\ y_l'(0) & y_l(0) & y_l'(1) & y_l(1) \end{vmatrix}.$$

Since $\delta_k \neq 0$ for any $k \in \mathbb{N}$ it follows from (5.1) and (5.2) that

$$\Delta_{i,j,r,l} \neq 0 \Longleftrightarrow \tilde{\Delta}_{i,j,r,l} \neq 0. \tag{5.3}$$

By refining the asymptotic formulas for eigenvalues and eigenfunctions of problem (1.1)-(1.5) and applying Theorems 5.1, it is possible to establish sufficient conditions for the system $\{y_k(x)\}_{k=1,\, k\neq i,\, j,\, r,\, l}^{\infty}$ to form a basis in $L_p(0,1), 1<p<\infty$.

Theorem 4.3 *Let $i=2$ and $j,\ r,\ l,\ j<r<l$, be arbitrary sufficiently large fixed natural numbers, two of which are even and the third odd. Then the system $\{y_k(x)\}_{k=1,\, k\neq i,\, j,\, r,\, l}^{\infty}$ is a basis in $L_p(0,1), 1<p<\infty$, which is an unconditional basis in $L_2(0,1)$.*

Proof. Following the corresponding reasoning carried out in the proof of formulas (5.15) and (5.17) of Theorem 5.4 in [7] we obtain the following asymptotic formulas

$$\begin{gathered}\varrho_k=\left(k-\tfrac{7}{2}\right)\pi+O\left(\tfrac{1}{k}\right),\ y_k'(0)=(-1)^k\tfrac{c}{a}\left(1+O\left(\tfrac{1}{\varrho_k^2}\right)\right),\\ y_k(0)=(-1)^k\ \tfrac{c}{b}\,\varrho_k\left(1+O\left(\tfrac{1}{\varrho_k^2}\right)\right),\ y_k(1)=\tfrac{c}{d}\,\varrho_k\left(1+\tfrac{1}{d\varrho_k}+O\left(\tfrac{1}{\varrho_k^2}\right)\right),\end{gathered}\tag{5.4}$$

where $\varrho_k=\sqrt[4]{\lambda_k}$.

Let $i=2,\ j,\ l,\ r,\ j<r<l$, be arbitrary fixed sufficiently large natural numbers such that j and r be even, and l be odd. Then, by (5.4) we have

$$\tilde{\Delta}_{1,\,j,\,r,\,l}=\frac{c}{a}\begin{vmatrix}0 & 1 & 0 & 1\\ 1 & \frac{c}{b}\varrho_j & 1 & \frac{c}{d}\varrho_j+\frac{c}{d^2}\\ 1 & \frac{c}{b}\varrho_r & 1 & \frac{c}{d}\varrho_r+\frac{c}{d^2}\\ -1 & -\frac{c}{b}\varrho_l & 1 & \frac{c}{d}\varrho_l+\frac{c}{d^2}\end{vmatrix}+O\left(\frac{1}{\varrho_j}\right)=$$

$$\frac{2c}{a}\begin{vmatrix}0 & 1 & 1\\ 1 & \frac{c}{b}\varrho_j & \frac{c}{d}\varrho_j+\frac{c}{d^2}\\ 1 & \frac{c}{b}\varrho_r & \frac{c}{d}\varrho_r+\frac{c}{d^2}\end{vmatrix}+O\left(\frac{1}{\varrho_j}\right)=\frac{2c}{a}\begin{vmatrix}0 & 1 & 1\\ 0 & \frac{c}{b}(\varrho_j-\varrho_r) & \frac{c}{d}(\varrho_j-\varrho_r)\\ 1 & \frac{c}{b}\varrho_r & \frac{c}{d}\varrho_r+\frac{c}{d^2}\end{vmatrix}+$$

$$O\left(\frac{1}{\varrho_j}\right)=\frac{2c^2}{a}\left(\frac{1}{d}-\frac{1}{b}\right)(\varrho_j-\varrho_r)+O\left(\frac{1}{\varrho_j}\right)>0.$$

Hence bin view of (5.2) and (5.3) it follows from Theorem 5.1 that the system $\{y_k(x)\}_{k=1,\, k\neq i,\, j,\, r,\, l}^{\infty}$ is a basis in $L_p(0,1), 1<p<\infty$, and for $p=2$ this basis is an unconditional basis.

Other cases are considered similarly. The proof of this theorem is complete.

References

[1] Z.S. Aliyev, Basis properties of a fourth order differential operator with spectral parameter in the boundary condition, *Cent. Eur. J. Math.* **8**(2) (2010), 378–388.

[2] Z.S. Aliyev, Basis properties in L_p of systems of root functions of a spectral problem with spectral parameter in a boundary condition, *Diff. Equat.* **47**(6) (2011), 766–777.

[3] Z.S. Aliyev, Basis properties in L_p of systems of root functions of a spectral problem with spectral parameter in a boundary condition, *Diff. Equat.* **47**(6) (2011), 766–777.

[4] Z.S. Aliyev, On basis properties of root functions of a boundary value problem containing a spectral parameter in the boundary conditions. *Dokl. Math.* **87**(2) (2013), 137–139.

[5] Z.S. Aliev, A.A. Dunyamalieva, Defect basis property of a system of root functions of a Sturm-Liouville problem with spectral parameter in the boundary conditions, *Diff. Equat.* **51**(10) (2015), 1249–1266.

[6] Z.S. Aliyev, S.B. Guliyeva, Properties of natural frequencies and harmonic bending vibrations of a rod at one end of which is concentrated inertial load, *J. Differential Equations* **263**(9) (2017), 5830–5845.

[7] Z.S. Aliyev, N.B. Kerimov, V.A. Mehrabov, Convergence of eigenfunction expansions for a boundary value problem with spectral parameter in the boundary conditions. I, *Diff. Equat.* 56(2), (2020), 143–157.

[8] Z.S. Aliyev, G.T. Mamedova, Some properties of eigenfunctions for the equation of vibrating beam with a spectral parameter in the boundary conditions, *J. Differential Equations* **269**(2) (2020), 1383–1400.

[9] Z.S. Aliyev, F.M. Namazov, On the spectral problem arising in the mathematical model of bending vibrations of a homogeneous rod, *Complex Anal. Oper. Theory* **13**(8) (2019), 3675–3693.

[10] J. Ben Amara, A.A. Vladimirov, On oscillation of eigenfunctions of a fourth-order problem with spectral parameters in the boundary conditions, *J. Math. Sci.* 150(5) (2008), 2317–2325.

[11] T.Ya. Azizov and S.I. Iokhvidov, Linear operators in Hilbert spaces with G-metric, *Russ. Math. Surv.* **26**(4) (1971), 45–97.

[12] D.O. Banks, G.J. Kurowski, A Prufer transformation for the equation of a vibrating beam subject to axial forces, *J.Differential Equations* **24**(1) (1977), 57–74.

[13] P.A. Binding, P.J. Browne, Application of two parameter eigencurves to Sturm-Liouville problems with eigenparameter-dependent boundary conditions, *Proc. R. Soc. Edinb., Sect. A* **125**(6) (1995), 1205–1218.

[14] B.B. Bolotin, *Vibrations in technique: Handbook in 6 volumes, The vibrations of linear systems, I*, Engineering Industry, Moscow, 1978.

[15] R. Courant, D. Hilbert, *Methods of mathematical phusics, I*, Interscience, New York, 1953.

[16] C.T. Fulton, Two-point boundary value problems with eigenvalue parameter in the boundary conditions, *Proc. Roy. Soc. Edinburgh, Sect.**A*** **77**(3-4) (1977), 293–308.

[17] C. Gao, M. Ran, Spectral properties of a fourth-order eigenvalue problem with quadratic spectral parameters in a boundary condition, *AIMS Math.* **5**(2) (2020), 904–922.

[18] N.Yu. Kapustin, On a spectral problem arising in a mathematical model of torsional vibrations of a rod with pulleys at the ends, *Diff. Equat.* **41**(10) (2005), 1490–1492.

[19] N.Yu. Kapustin, E.I. Moiseev, On the basis property in the space L_p of systems of eigenfunctions corresponding to two problems with spectral parameter in the boundary condition, *Diff. Equ.* **36**(10) (2000), 1357–1360.

[20] N.B. Kerimov, Z.S. Aliyev, The oscillation properties of the boundary value problem with spectral parameter in the boundary condition, *Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci.* **25**(7) (2005), 61–68.

[21] N.B. Kerimov, Z.S. Aliev, On the basis property of the system of eigenfunctions of a spectral problem with spectral parameter in a boundary condition, *Diff. Equat.* **43**(7) (2007), 905–915.

[22] N.B. Kerimov, R.G. Poladov, Basis properties of the system of eigenfunctions in the Sturm-Liouville problem with a spectral parameter in the boundary conditions, *Doklady Math.* **85**(1) (2012), 8–13.

[23] J. Qin, K. Li, Z. Zheng, J. Cai, Eigenvalues of fourth-order differential operators with eigenparameter dependent boundary conditions, *AIMS Mathematics* **7**(5) (2022), 9247–9260.

[24] A.N. Tikhonov and A.A. Samarskii, *Equations of mathematical physics*, Macmillan, New York, 1963

[25] J. Walter, Regular eigenvalue problems with eigenvalue parameter in the boundary condition, *Math. Z.* **133**(4) (1973), 301–312.

Ayna E. Fleydanli
Sumgait State University, Sumgait, Azerbaijan
E-mail: aynafleydanli@gmail.com

Received 02 February 2023
Accepted 12 March 2023