Caspian Journal of Applied Mathematics, Ecology and Economics
V. 11, No 1, 2023, July
ISSN 1560-4055

Transformation Operators for one Second-Order Differential Equation with Increasing Coefficient

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Abstract. The Stark equation with a step-like perturbed potential is considered. Using transformation operators, we obtain representations of solutions of this equation with conditions at infinity. Estimates for the kernels of the transformation operators are obtained.
Key Words and Phrases: Stark equation, transformation operator, Airy functions, triangular representation.
2010 Mathematics Subject Classifications: Primary 34A55, 34B20, 34L05

1. Introduction

The Stark effect is the shifting and splitting of spectral lines of atoms and molecules due to presence of an external electric field. The effect is named after Stark, who discovered it in 1913. The Stark effect has been of marginal benefit in the analysis of atomic spectra, but has been a major tool for molecular rotational spectra. The perturbation theory of the Stark effect is of particular interest. The application of transformation operators to the perturbation theory of linear operators is well known (see [1], [2] and the references therein). These operators arose from the general ideas of the theory of generalized shift operators created by Delsarte [3].

For arbitrary Sturm–Liouville equations, transformation operators were constructed by Povzner [4]. Marchenko [5] used transformation operators for studying inverse spectral problems and the asymptotic behavior of the spectral function of the singular Sturm–Liouville operator. Levin [6] introduced transformation operators of a new form that preserve the asymptotic expansions of solutions at infinity. Marchenko [5] used them to solve the inverse problem of scattering theory. Similar problems for the Schrodinger equation with unbounded potentials were considered in [7]- [9].

We consider the differential equation

$$-y'' + xy + p(x)\,y + q(x)\,y = \lambda y,\ -\infty < x < \infty, \quad \lambda \in C. \tag{1}$$

where real potentials $p(x)$ and $q(x)$ satisfy the conditions

$$p(x) = \begin{cases} \alpha_+, x \geq 0, \\ \alpha_-, \, x < 0, \end{cases} \tag{2}$$

$$q(x) \in C(-\infty, +\infty), \int_{-\infty}^{\infty} |xq(x)|\, dx < \infty. \tag{3}$$

In the present paper, using transformation operators, we obtain representations of solutions of this equation with conditions at infinity. The results obtained can be used to solve inverse spectral problems for an equation (1). Some questions of the spectral theory of the one-dimensional Stark equation were studied in [10]-[13].

2. The transformation operators

In what follows, we deal with special functions satisfying the Airy equation

$$-y'' + zy = 0.$$

It is well known (e.g., see [14]) that this equation has two linearly independent solutions $Ai(z)$ and $Bi(z)$ with the initial conditions

$$Ai(0) = \frac{1}{3^{\frac{2}{3}}\Gamma\left(\frac{2}{3}\right)},\ Ai'(0) = \frac{1}{3^{\frac{1}{3}}\Gamma\left(\frac{1}{3}\right)},$$
$$Bi(0) = \frac{1}{3^{\frac{1}{6}}\Gamma\left(\frac{2}{3}\right)},\ Bi'(0) = \frac{3^{\frac{1}{6}}}{\Gamma\left(\frac{1}{3}\right)}.$$

The Wronskian $\{Ai(z), Bi(z)\}$ of these functions satisfies

$$\{Ai(z), Bi(z)\} = Ai(z)\, Bi'(z) - Ai'(z)\, Bi(z) = \pi^{-1}.$$

Both functions are entire functions of order $\frac{3}{2}$ and type $\frac{2}{3}$. Note that the functions $Ai(x-\lambda)$, $Ai(x-\lambda) - iBi(x-\lambda)$ satisfy the relations (see [2]) $Ai(x-\lambda) \in L_2(0, +\infty)$, $Ai(x-\lambda) - iBi(x-\lambda) \in L_2(-\infty, 0)$ for $Im\lambda \geq 0$.
In what follows we will need special solutions of the unperturbed equation

$$-y'' + xy + p(x)\, y = \lambda y,\ -\infty < x < \infty, \quad \lambda \in C. \tag{4}$$

Lemma 1. *For any λ from the complex plane, equation (4) has solutions $\psi_{\pm}(x, \lambda)$ in the form*

$$\psi_{+}(x, \lambda) =$$

$$= \begin{cases} Ai(x + \alpha_{+} - \lambda),\ x \geq 0, \\ \pi\left[Ai(\alpha_{+} - \lambda)\, Bi'(\alpha_{-} - \lambda) - Ai'(\alpha_{+} - \lambda)\, Bi(\alpha_{-} - \lambda)\right] Ai(x + \alpha_{-} - \lambda) + \\ +\pi\left[Ai(\alpha_{-} - \lambda)\, Ai'(\alpha_{+} - \lambda) - Ai(\alpha_{+} - \lambda)\, Ai'(\alpha_{-} - \lambda)\right] Bi(x + \alpha_{-} - \lambda),\ x < 0, \end{cases} \tag{5}$$

$$\psi_{-}(x,\lambda)=\begin{cases}\pi\left\{Bi'(\alpha_{+}-\lambda)\left[Ai(\alpha_{-}-\lambda)-iBi(\alpha_{-}-\lambda)\right]-\right.\\ \left.Bi(\alpha_{+}-\lambda)\left[Ai'(\alpha_{-}-\lambda)-iBi'(\alpha_{-}-\lambda)\right]\right\}Ai(x+\alpha_{+}-\lambda)+\\ \pi\left\{Ai(\alpha_{+}-\lambda)\left[Ai(\alpha_{-}-\lambda)-iBi(\alpha_{-}-\lambda)\right]-\right.\\ \left.Ai'(\alpha_{+}-\lambda)\left[Ai'(\alpha_{-}-\lambda)-iBi'(\alpha_{-}-\lambda)\right]\right\}Bi(x+\alpha_{+}-\lambda),\ x\geq 0\\ Ai(x+\alpha_{-}-\lambda)-iBi(x+\alpha_{-}-\lambda),\ x<0.\end{cases} \tag{6}$$

Proof. Obviously, when $x \geq 0$ one of the solutions of equation (4) is function $Ai(x+\alpha_{+}-\lambda)$. On the other hand, for $x \leq 0$ any solution of equation (4) can be represented as

$$CAi(x+\alpha_{-}-\lambda)+DBi(x+\alpha_{-}-\lambda).$$

If we glue these solutions at a point $x=0$, we get

$$C=\pi\left[Ai(\alpha_{+}-\lambda)Bi'(\alpha_{-}-\lambda)-Ai'(\alpha_{+}-\lambda)Bi(\alpha_{-}-\lambda)\right],$$

$$D=\pi\left[Ai(\alpha_{-}-\lambda)Ai'(\alpha_{+}-\lambda)-Ai(\alpha_{+}-\lambda)Ai'(\alpha_{-}-\lambda)\right].$$

Thus, formula (5) is established. Formula (6) is derived similarly.

The lemma is proved.

We shall use the following notation

$$\sigma_{\pm}(x)=\pm\int_{x}^{\pm\infty}\left|p(t)-\alpha_{\pm}+q(t)\right|dt.$$

In the following theorem the representation of solution from the equation (1) is found by means of transformation operator.

Theorem 1. *If the potentials $p(x)$ and $q(x)$ satisfy the conditions (2),* (3) *then for any λ from the closed upper half-plane equation* (1) *has a solution $f_{+}(x,\lambda)$ that can be represented in the form*

$$f_{+}(x,\lambda)=\psi_{+}(x,\lambda)+\int_{x}^{\infty}K_{+}(x,t)\ \psi_{+}(t,\lambda)\,dt, \tag{7}$$

where kernel $K_{+}(x,t)$ is continuous function and satisfies relations

$$K_{+}(x,t)=O\left(\sigma_{+}\left(\frac{x+t}{2}\right)\right),x+t\to\infty, K_{+}(x,x)=\frac{1}{2}\int_{x}^{\infty}\left[p(t)-\alpha_{+}+q\ (t)\right]\,dt. \tag{8}$$

Proof. We rewrite the perturbed equation (1) in the form

$$-y''+xy+Q(x)\,y=(\lambda-\alpha_{+})\,y,\ -\infty<x<\infty. \tag{9}$$

where $Q(x) = p(x) - \alpha_+ + q(x)$. Obviously, the $Q(x)$ function for all $x > a, a > -\infty$ satisfies the condition

$$Q(x) \in C(-\infty, +\infty), \int_a^\infty |xQ(x)|\, dx < \infty. \tag{10}$$

Let $f_+(x, \lambda)$ be solution of equation (10) with the asymptotic behavior $f_+(x, \lambda) = f_0(x, \lambda)(1 + o(1)), x \to +\infty$, where $f_0(x, \lambda) = Ai(x + \alpha_+ - \lambda)$. Subject to the conditions (11), such solution exist, is determined uniquely by its asymptotic behavior. With the aid of operator transformations, we have the representation

$$f_+(x, \lambda) = f_0(x, \lambda) + \int_x^\infty K(x, t)\ f_0(t, \lambda)\, dt, \tag{11}$$

Moreover, the kernel $K(x, t)$ is a continuous function and satisfies the following relations

$$K(x, t) = O\left(\sigma_+\left(\frac{x+t}{2}\right)\right), x + t \to \infty, \tag{12}$$

$$K(x, x) = \frac{1}{2}\int_x^\infty Q\ (t)\ dt. \tag{13}$$

In addition, rewriting the unperturbed equation (4) in the form

$$-y'' + xy + Q_0(x)\, y = (\lambda - \alpha_+)\, y,\ -\infty < x < \infty.$$

where $Q_0(x) = p(x) - \alpha_+$, we obtain

$$\psi_+(x, \lambda) = f_0(x, \lambda) + \int_x^\infty K_0(x, t)\ f_0(t, \lambda)\, dt\,. \tag{14}$$

Moreover, in this case, $K_0(x, t)$ satisfies the identity$K_0(x, t) \equiv, x \geq 0$. From the well-known properties of the transformation operators it follows that (see [5]) the function $f_0(x, \lambda)$ also admits the representation

$$f_0(x, \lambda) = \psi_+(x, \lambda) + \int_x^\infty \tilde{K}_0(x, t)\ \ \psi_+(t, \lambda)\, dt\,, \tag{15}$$

where the kernels$K_0(x, t), \tilde{K}_0(x, t)$ are connected by the equality

$$K_0(x, t) + \tilde{K}_0(x, t) + \int_x^t \tilde{K}_0(x, u)\ \ K_0(u, t)\, du = 0. \tag{16}$$

Substituting the expression (16) from the $f_0(x, \lambda)$ in (12), we get

$$\begin{aligned} f_+(x, \lambda) &= \psi_+(x, \lambda) + \textstyle\int_x^\infty K(x, t)\left[\psi_+(t, \lambda) + \int_t^\infty \tilde{K}_0(t, u)\, \psi_+(u, \lambda)\, du\right]\ dt = \\ &= \psi_+(x, \lambda) + \textstyle\int_x^\infty K(x, t)\, \psi_+(t, \lambda)\, dt + \int_x^\infty K(x, t) \int_t^\infty \tilde{K}_0(t, u)\, \psi_+(u, \lambda)\, dudt = \\ &= \psi_+(x, \lambda) + \textstyle\int_x^\infty K(x, t)\, \psi_+(t, \lambda)\, dt + \int_x^\infty \left(\int_x^t K(x, u)\, \tilde{K}_0(u, t)\, du\right) \psi_+(t, \lambda)\, dt. \end{aligned}$$

Setting

$$K_+(x,t) = K(x,t) + \int_x^t K(x,u) \quad \tilde{K}_0(u,t)\,du, \tag{17}$$

one can recast the last relation in the form

$$f_+(x,\lambda) = \psi_+(x,\lambda) + \int_x^{\infty} K_+(x,t) \ \ \psi_+(t,\lambda)\,dt.$$

Formula (8) is a straightforward consequence of (13), (17) . Taking $t = x$ in the equality (17), we find that$K_+(x,t) = K(x,t)$. Whence, by virtue of (15), formula (9) follows.

The theorem is proved.

The following theorem is proved in a similar way.

Theorem 2. *If the potentials $p(x)$ and $q(x)$ satisfy the conditions (2), (3), then, for any λ from the closed upper half-plane, equation* (1) *has a solution $f_-(x,\lambda)$ representable as*

$$f_-(x,\lambda) = \psi_-(x,\lambda) + \int_{-\infty}^{x} K_-(x,t) \ \ \psi_-(t,\lambda)\,dt.$$

where the kernel $K_-(x,t)$ is continuous function and satisfy the following conditions

$$K_-(x,t) = O\left(\sigma_-\left(\frac{x+t}{2}\right)\right), x+t \to -\infty, K_-(x,x) = \frac{1}{2}\int_{-\infty}^{x} \left[p(t) - \alpha_- + q\ (t)\right]\,dt.$$

References

[1] J. Avron, I. Herbst, *Spectral and scattering theory of Schrodinger operators related to the Stark effect,* Commun. Math. Phys., 52, 1977, 239–254.

[2] F. Calogero, A. Degasperis, *Inverse spectral problem for the one-dimensional Schrodinger equation with an additional linear potential,* Lett. Nuovo Cimento, 23, 1978, 143–149 .

[3] J. Delsarte, *Sur une extension de la formule de Taylor,* J.Math. Pures Appl. **17**, 1938, 213–231.

[4] A. Ya.Povzner, *On differential equations of Sturm–Liouville type on a half-axis,* Mat. Sb. **23**(1), 1948, 3–52.

[5] V. A.Marchenko, *Sturm–Liouville Operators and Their Applications,* Naukova Dumka, Kiev, (1977) [in Russian].

[6] B. Ya. Levin, *Transformation of Fourier and Laplace type by means of solutions of a differential equation of second order,* Dokl. Akad. Nauk SSSR, **106** (2), 1956, 187–190.

[7] M. G. Gasymov, B. A.Mustafaev, *Inverse scattering problem for an anharmonic equation on the semiaxis,* Dokl. Akad. Nauk SSSR **228**(1), 1976, 11–14 .

[8] Li.Yishen, *One special inverse problem of the second-order differential equation for the whole real axis,* Chinese Ann. Math. **2**(2), 1981, 147–155.

[9] A.Kh.Khanmamedov, D.H.Orudjov, *Triangular Representation of the Solution to the Schrodinger Equation with an Additional Linear Potential,* Azerbaijan Journal of Mathematics, vol 13, no 2,2023, 100-109.

[10] E.L.Korotyaev, Resonances for 1D Stark operators, Journal Spectral Theory, **7**(3), 2017, 633-658.

[11] E.L. Korotyaev *Asymptotics of resonances for 1D Stark operators*, Lett. Math. Phys., **118**(5), 2018, 1307-1322.

[12] A. M.Savchuk, A. A.Shkalikov, *Spectral Properties of the Complex Airy Operator on the Half-Line,* Functional Analysis and Its Applications, **51**(1), 2017, 66–79.

[13] A. Kh. Khanmamedov, M. G. Makhmudova, *Inverse spectral problem for the Schrodinger equation with an additional linear potential,* Theoretical and Mathematical Physics, 202(1),2020, 58–71.

[14] M.Abramowitz, I.Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables,* Dover Publications, New York, 1972.

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Received 20 February 2023
Accepted 01 April 2023