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Inverse Boundary Value Problem for Benney-Luke Linearized Equation with Nonlocal Time-integral Conditions of Second Kind

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Abstract. In the paper we study an inverse boundary value problem with a time-dependent unknown coefficient for a linearized Benney-Luke equation with nonlocal time- integral conditions of second kind. The essence of the problem is to determine the unknown coefficient together with the solution. The problem is considered in a rectangular domain. When solving the original inverse problem, a transition is made from the original inverse problem to some auxiliary inverse problem. The existence and uniqueness of the solution of the auxiliary problem is proved by means of compressed mappings. Then the transition to the original inverse problem is made again, as a result a conclusion is made about the solvability of the original inverse problem.

Key Words and Phrases: inverse problem, Benney-Luke equation, existence and uniqueness of the classic solution.

2010 Mathematics Subject Classifications: 35R30

1. Introduction

Many problems of mathematical physics, mechanics of continuum are boundary value problems reduced to the integration of a differential equation or a system of partial equations under given boundary and initial conditions. Many problems of gas dynamics, theory of elasticity, theory of plates and shells are reduced to the consideration of higher order partial differential equations [1]. Differential equations of fourth order are of great interest from the point of view of applications (see, e.i. [2, 3]). Benney-Luke type partial differential equations have applications in mathematical physics (see [3]).

The problems in which together with the solution of one or another differential equation, it is necessary to determine the coefficient (coefficients) of the equation itself, or the right hand side of the equation, in mathematics and in mathematical modeling are called inverse problems. Theory of inverse problems for differential equations is a dynamically developing section of modern science. Last time, the inverse problems arise in very different fields of human activity as seismology, mineral exploration, biology, medicine, quality control of industrial products, etc. that places them among the pressing problems of modern mathematics. Different inverse problems for certain types of partial differential equations have been studied in many works. Here first of all we note the works of A.N.Tikhonov [4], M.M.Lavrent'ev [5, 6], V.K.Ivanov [7] and their followers. For more information you can read the monograph of A.M.Denisov [9].

Theory of inverse boundary value problems for fourth order equations still remain small studied. The works [**?** 10, 11, 12] have been devoted to the inverse boundary value problems for the Benney–Luke equation.

The goal of the present paper is to prove the existence and uniqueness of the solution of the inverse boundary value problem for the Benney-Luke equation with time non-local integral conditions of second kind.

2. Statement of the problem and its reduction to the equivalent problem

Let us consider for the equation

$$u_{tt}(x,t) - u_{xx}(x,t) + \alpha u_{xxxx}(x,t) - \beta u_{xxtt}(x,t) = a(t)u(x,t) + f(x,t), (x,t) \in D_T \quad (1)$$

in the domain $D_T = \{(x,t) : 0 \le x \le 1,\ 0 \le t \le T\}$, the inverse boundary value problem with the nonlocal initial conditions

$$u(x,0) = \int_0^T p_1(t)u(x,t)dt + \varphi(x), u_t(x,0) = \int_0^T p_2(t)u(x,t)dt + \psi(x)(0 \le x \le 1), \quad (2)$$

the boundary conditions

$$u_x(0,t) = 0, u(1,t) = 0, u_{xxxx}(0,t) = 0, u_{xx}(1,t) = 0(0 \le t \le T) \quad (3)$$

and with the additional condition

$$u(0,t) = h(t)\,(0 \le t \le T), \quad (4)$$

where $\alpha > 0$, $\beta > 0$ are fixed numbers, $f(x,t)$, $p_1(t)$, $p_2(t)$, $\varphi(x)$, $\psi(x)$, $h(t)$ are the given functions $u(x,t)$, $a(t)$ are the desired functions

We introduce the denotation

$$\tilde{C}^{4,2}(D_T) = \left\{u(x,t) :\ u(x,t) \in C^2(D_T)\,, u_{ttx}(x,t),\right.$$
$$\left. u_{ttxx}(x,t), u_{xxx}(x,t), u_{xxxx}(x,t) \in C(D_T)\right\}.$$

Definition 1. *Under the classical solution of the inverse boundary value problem* (1)-(4) *we understand the pair* $\{u(x,t)\,,\ a(t)\}$ *of functions* $u(x,t) \in \tilde{C}^{4,2}(D_T)$, $a(t) \in C[0,T]$ *satisfying the equation* (1) *and conditions* (2)-(4) *in the usual sense.*

To study the problem (1)-(4) at first we consider the following problem:

$$y''(t) = a(t)\,y(t) \quad (0 \le t \le T), \quad (5)$$

$$y(0) = \int_0^T p_1(t)y(t)dt, \ y'(0) = \int_0^T p_2(t)y(t)dt, \tag{6}$$

where $a(t) \in C[0,T]$, $p_1(t)$, $p_2(t) \in C[0,T]$ are the given functions, $y = y(t)$ is an unknown function and under the solution of problem (5), (6) we understand the function $y(t)$ belonging to $C^2[0,T]$ and satisfying condition of (5), (6) in the usual sense.

The following lemma is valid

Lemma 1 ([13]). *Let us functions $p_1(t) \in C[0,T], p_2(t) \in C[0,T]$, $a(t) \in C[0,T]$ and*

$$\|a(t)\|_{C[0,T]} \leq R = const.$$

Furthermore

$$\left(T \|p_2(t)\|_{C[0,T]} + \|p_1(t)\|_{C[0,T]} + \frac{T}{2}R\right) T < 1.$$

Then problem (5), (6) *has only a trivial solution.*

Along with the inverse boundary value problem (1)- (4) we consider the following auxiliary inverse boundary value problem:

It is required to determine the pair $\{u(x,t), a(t)\}$ of functions $u(x,t) \in \tilde{C}^{4,2}(D_T)$, $a(t) \in C[0,T]$ from the relations (1)-(4) ,

$$h''(t) - u_{xx}(0,t) + \alpha u_{xxxx}(0,t) - \beta u_{xxtt}(0,t) = a(t)h(t) + f(0,t) \ (0 \leq t \leq T) \tag{7}$$

The following theorem is valid

Theorem 1. *Let $\varphi(x)$, $\psi(x) \in C[0,1]$, $p_i(t) \in C[0,T]$ $(i = 1,2)$, $f(x,t) \in C(D_T)$, $h(t) \in C^2[0,T]$, $h(t) \neq 0$ $(0 \leq t \leq T)$ and the following agreement conditions be fulfilled:*

$$\varphi(0) = h(0) - \int_0^T p_1(t)h(t)dt, \quad \psi(0) = h''(0) - \int_0^T p_2(t)h(t)dt.$$

Then the following statements are valid:

A. *Each classic solution $\{u(x,t), a(t)\}$ of problem* (1)-(4) *is the solution of problem* (1)-(3), (7) *as well;*

B. *Each solution of $\{u(x,t), a(t)\}$ of problem* (1)-(3), (7), *such that*

$$\left(T \|p_2(t)\|_{C[0,T]} + \|p_1(t)\|_{C[0,T]} + \frac{T}{2} \|a(t)\|_{C[0,T]}\right) T < 1 \tag{8}$$

is the classic solution (1)-(4).

Proof. Let $\{u(x,t), a(t)\}$ be a classic solution of problem (1)-(4). Substituting $x = 0$ in equation (1), we find:

$$u_{tt}(0,t) - u_{xx}(0,t) + \alpha u_{xxxx}(0,t) - \beta u_{xxtt}(1,t) =$$

$$= a(t)u(0,t) + f(0,t) \ \ (0 \le t \le T)\,, \tag{9}$$

In what follows, assuming $h(t) \in C^2[0,T]$ and differentiating twice (4), we have:

$$u_t(0,t) = h'(t)\,,\ u_{tt}(0,t) = h''(t)\,(0 \le t \le T).$$

Taking into account these relations, from (9) allowing for (4) we obtain the fulfillment of (7). Now, we assume that $\{u(x,t), a(t)\}$ is the solution of problem (1)- (3), (7). Then from (7) and (9) we obtain:

$$\frac{d^2}{dt^2}(u(0,t) - h(t)) = a(t)(u(0,t) - h(t)) \ \ (0 \le t \le T). \tag{10}$$

By virtue of (2) and agreement conditions $\varphi(0) = h(0) \ - \int_0^T p_1(t)h(t)dt$,
$\psi(0) = h''(0) - \int_0^T p_2(t)h(t)dt$ we have:

$$u(0,0) - h(0) - \int_0^T p_1(t)(u(0,t) - h(t))dt = u(0,0) - \int_0^T p_1(t)u(0,t)dt -$$

$$= \varphi(0) - \left(h(0) - \int_0^T p_1(t)h(t)dt\right) = 0,$$

$$u_t(0,0) - h'(0) - \int_0^T p_2(t)(u(0,t) - h(t))dt = u_t(0,0) - \int_0^T p_2(t)u(0,t)dt -$$

$$= \varphi(0) - \left(h'(0) - \int_0^T p_2(t)h(t)dt\right) = 0. \tag{11}$$

From (10), (11) by lemma 1 we conclude that condition (4) is fulfilled. The theorem is proved.

3. Studying the existence and uniqueness of the classic solution of the inverse boundary value problem

We will look for the first component $u(x,t)$ of the solution $\{u(x,t), a(t)\}$ of problem (1)-(3), (7) in the form:

$$u\,(x,t) = \sum_{k=1}^{\infty} u_k\,(t) \cos \lambda_k x \ \ (\lambda_k = \frac{\pi}{2}(2k = 1)), \tag{12}$$

where

$$u_k(t) = 2\int_0^1 u(x,t) \cos \lambda_k x\, dx.$$

Applying the formal scheme of the Fourier method, from (1), (2) we obtain:

$$(1 + \alpha\lambda_k^2)u_k''(t) + \lambda_k^2(1 + \alpha\lambda_k^2)u_k(t) = F_k(t; u, a, b) \ \ (0 \le t \le T;\ k = 1, 2, ...), \tag{13}$$

$$u_k(0) = \int_0^T p_1(t)u_k(t)dt + \varphi_k, \;\; u_k'(0) = \int_0^T p_2(t)u_k(t)dt + \psi_k \;\; (k = 1, 2, ...), \tag{14}$$

where

$$F_k(t; u, a) = f_k(t) + a(t)u_k(t), f_k(t) = 2\int_0^1 f(x,t)\cos\lambda_k x dx,$$

$$\varphi_k = 2\int_0^1 \varphi(x)\cos\lambda_k x dx, \psi_k = 2\int_0^1 \psi(x)\cos\,\lambda_k x dx \;\; (k = 1, 2, ...).$$

Solving the problem (13), (14), we find:

$$u_k(t) = \left(\int_0^T p_1(t)u_k(t)dt + \varphi_k\right)\cos\beta_k t + \frac{1}{\beta_k}\left(\int_0^T p_2(t)u_k(t)dt + \psi_k\right)\sin\beta_k t +$$

$$+\frac{1}{\beta_k(1+\beta\lambda_k^2)}\int_0^t F_k(\tau\,; u, a)\,\sin\beta_k\,(t-\tau)\;d\tau\,. \tag{15}$$

where

$$\beta_k = \lambda_k\sqrt{\frac{1+\alpha\lambda_k^2}{1+\beta\lambda_k^2}}(k = 1, 2, ...).$$

After substituting the expression $u_k(t)$ $(k = 1, 2, ...)$ from (15) to (12), for determining the component $u\,(x, t)$ of the solution of problem (1)-(3), (7) we obtain:

$$u(x,t) = \sum_{k=1}^{\infty}\left\{\left(\int_0^T p_1(t)u_k(t)dt + \varphi_k\right)\cos\beta_k t + \frac{1}{\beta_k}\left(\int_0^T p_2(t)u_k(t)dt + \psi_k\right)\sin\beta_k t +\right.$$

$$\left. + \frac{1}{\beta_k(1+\beta\lambda_k^2)}\int_0^t F_{2k-1}(\tau\,; u, a, b)\,\sin\beta_k\,(t-\tau)\;d\tau\right\}\cos\,\lambda_k x. \tag{16}$$

Now, from (7) allowing for (15), we obtain:

$$a(t)h(t)\; = h''(t)\; - f(0,t) +$$

$$+\sum_{k=1}^{\infty}(\lambda_k^2(1+\alpha\lambda_k^2)u_k(t) + \beta\lambda_k^2 u_k''(t)).$$

or taking into account

$$u_k''(t) = -\frac{\lambda_k^2(1+\alpha\lambda_k^2)}{1+\beta\lambda_k^2}u_k(t) + \frac{1}{1+\beta\lambda_k^2}F_k\,(t; u, a)$$

we have:

$$a(t)h(t)\; = h''(t)\; - f(0,t) +$$

$$+\sum_{k=1}^{\infty}\left(\beta_k^2 u_k(t) + \frac{\beta\lambda_k^2}{1+\beta\lambda_k^2}F_k\,(t; u, a)\right)\,. \tag{17}$$

Now from (17) we find:

$$a(t) = [h(t)]^{-1}\left\{h''(t)\ - f(0,t) + \right.$$

$$\left. + + \sum_{k=1}^{\infty}\left(\beta_k^2 u_k(t) + \frac{\beta\lambda_k^2}{1+\beta\lambda_k^2}F_k\left(t;u,a\right)\right)\right\}. \tag{18}$$

Substituting the expression $u_k\left(t\right)$ $(k=1,2,...)$ from (15) in (18), we obtain:

$$a(t) = [h(t)]^{-1}\left\{h''(t)\ - f(0,t) + \right.$$

$$+\sum_{k=1}^{\infty}\left(\beta_k^2\left[\left(\int_0^T p_1(t)u_k(t)dt + \varphi_k\right)\cos\beta_k t + \frac{1}{\beta_k}\left(\int_0^T p_2(t)u_k(t)dt + \psi_k\right)\sin\beta_k\ + \right.\right.$$

$$\left.\left. + \frac{1}{\beta_k(1+\beta\lambda_k^2)}\int_0^t F_k(\tau\,;u,a)\,\sin\beta_k\left(t-\tau\right)\,d\tau\right] + \frac{\beta\lambda_k^2}{1+\beta\lambda_k^2}F_k(t\,;u,a)\right\}. \tag{19}$$

Thus, the solution of the problem (1)-(3), (7) is reduced to the solution of the system (16),(19) with respect to the unknown functions $u(x,t)$ and $a(t)$.

The following lemma is very important for studying the uniqueness of the solution of problem (1)-(3), (7)

Lemma 2. *If* $\{u(x,t),a(t)\}$ *is any classic solution of problem* (1)-(3), (7), *the function* $u_k(t)$ $(k=1,2,\ldots)$ *determined by the relation*

$$u_k(t) = 2\int_0^1 u(x,t)\cos\lambda_k x\,dx.(k=1,2,....),$$

$[0,T]$ *satisfy on the countable system* (15).

Obviously, if $u_k(t) = 2\int_0^1 u(x,t)\cos\lambda_k x dx (k=1,2,...)$ is the solution of the system (15), then the pair $\{u(x,t),a(t)\}$ of functions $u(x,t) = \sum_{k=1}^{\infty} u_k(t)\cos\lambda_k x$ and $a(t)$ is the solution of the system (16), (19) .

The following corollary follows from lemma 2

Corollary 1. *Let the system* (16), (19) *have a unique solution. Then problem* (1)-(3), (7) *can have at most one solution, i.e. if problem* (1)-(3), (7) *has a solution, this solution is unique.*

1. Denote by $B_{2,T}^5$ [14], the totality of all the functions $u(x,t)$ of the form

$$u(x,t) = \sum_{k=1}^{\infty} u_k(t)\cos\lambda_k x\ ,$$

considered in D_T, where each of the functions $u_k(t)$ is continuous on $[0,T]$ and

$$I(u) \equiv \left\{ \sum_{k=1}^{\infty} (\lambda_k^5 \left\| u_k(t) \right\|_{C[0,T]})^2 \right\}^{\frac{1}{2}} < +\infty..$$

We determine the norm in this set as follows:

$$\left\| u(x,t) \right\|_{B_{2,T}^5} = I(u).$$

2. Denote by E_T^5 a space consisting of the topological product

$$B_{2,T}^5 \times C[0,T].$$

The norm of the element $z = \{u, a\}$ is determined by the formula

$$\left\| z \right\|_{E_T^5} = \left\| u(x,t) \right\|_{B_{2,T}^5} + \left\| a(t) \right\|_{C[0,T]} .$$

It is known that $B_{2,T}^5$ and E_T^5 are Banach spaces.

Now in the space E_T^5 we consider the operator

$$\Phi(u,a) = \{\Phi_1(u,a), \Phi_2(u,a))\} ,$$

where

$$\Phi_1(u,a) = \tilde{u}(x,t) = \sum_{k=1}^{\infty} \tilde{u}_k(t) \cos \lambda_k x, \Phi_2(u,a) = \tilde{a}(t),$$

while $\tilde{u}_k(t) (k = 1, 2, ...)$ and $\tilde{a}(t)$ are equal to the right hand sides of (15) and (19) respectively.

It is easy to see that

$$1 + \beta\lambda_k^2 > \beta\lambda_k^2, \quad \frac{1}{1+\beta\lambda_k^2} < \frac{1}{\beta\lambda_k^2},$$

$$\sqrt{\frac{\alpha}{1+\beta}}\, \lambda_k \leq \beta_k \leq \sqrt{\frac{1+\alpha}{\beta}}\, \lambda_k, \quad \sqrt{\frac{\beta}{1+\alpha}}\, \frac{1}{\lambda_k} \leq \frac{1}{\beta_k} \leq \sqrt{\frac{1+\beta}{\alpha}}\, \frac{1}{\lambda_k},$$

Taking into account this relation, we find:

$$\left(\sum_{k=1}^{\infty} (\lambda_k^5 \left\| \tilde{u}_k(t) \right\|_{C[0,T]})^2 \right)^{\frac{1}{2}} \leq \sqrt{6} \left(\sum_{k=1}^{\infty} (\lambda_k^5 \left| \varphi_k \right|)^2 \right)^{\frac{1}{2}} +$$

$$+\sqrt{6}T \left\| p_1(t) \right\|_{C[0,T]} \left(\sum_{k=1}^{\infty} \left(\lambda_k^5 \left\| u_k(t) \right\|_{C[0,T]} \right)^2 \right)^{\frac{1}{2}} + \sqrt{\frac{6(1+\beta)}{\alpha}} \left(\sum_{k=1}^{\infty} (\lambda_k^4 \left| \psi_k \right|)^2 \right)^{\frac{1}{2}} +$$

$$+\sqrt{\frac{6(1+\beta)}{\alpha}}\,T\,\|p_2(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k^5\,\|u_k(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}+$$

$$+\frac{1}{\beta}\sqrt{\frac{6(1+\beta)T}{\alpha}}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k^2\,|\,f_k(\tau)\,|)^2d\tau\right)^{\frac{1}{2}}+$$

$$\frac{1}{\beta}\sqrt{\frac{6(1+\beta)}{\alpha}}T\,\|a(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^5\,\|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}, \tag{20}$$

$$\|\tilde{a}(t)\|_{C[0,T]}\le\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left\{\left\|h''(t)-f(0,t)\right\|_{C[0,T]}+\right.$$

$$+\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\left[\sqrt{\frac{1+\alpha}{\beta}}\left[\left(\sum_{k=1}^{\infty}(\lambda_k^5\,|\varphi_k|)^2\right)^{\frac{1}{2}}+\right.\right.$$

$$+T\,\|p_1(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k^5\,\|u_k(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}+\sqrt{\frac{1+\beta}{\alpha}}\left(\sum_{k=1}^{\infty}(\lambda_k^4\,|\,\psi_k\,|)^2\right)^{\frac{1}{2}}+$$

$$+\sqrt{\frac{1+\beta}{\alpha}}T\,\|p_2(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k^5\,\|u_k(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}+$$

$$+\frac{1}{\beta}\sqrt{\frac{T(1+\beta)}{\alpha}}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k^2\,|\,f_k(\tau)\,|)^2d\tau\right)^{\frac{1}{2}}+$$

$$\left.+\frac{1}{\beta}\sqrt{\frac{1+\beta}{\alpha}}T\,\|a(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^5\,\|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}\right]+$$

$$\left.\left.+\left(\sum_{k=1}^{\infty}(\lambda_k^2\,\|f_k(t)\|_{C[0,T]}\,|)^2d\tau\right)^{\frac{1}{2}}+\|a(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^5\,\|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}\right]\right\}. \tag{21}$$

Assume that the data of problem (1)-(3), (7) satisfy the following conditions:

$$1.\alpha>0,\beta>0,p_1(t)\in C[0,T]\,,p_2(t)\in C[0,T]\,.$$

$$2.\varphi(x)\in C^4[0,1],\ \varphi^{(5)}(x)\in L_2(0,1),\ \ \varphi'(0)=\ \varphi(1)=\varphi'''(0)=\varphi''(1)=\varphi^{(4)}(1)=0.$$

$$3.\psi(x)\in C^3[0,1],\ \psi^{(4)}(x)\in L_2(0,1),\ \ \psi'(0)=\psi(1)=\psi'''(0)=\psi''(1)=0.$$

$$4.f(x,t),f_x(x,t)\in C(D_T),\ f_{xx}(x,t)\in L_2(D_T),\ f_x(0,t)=f(1,t)=0\ (0\le t\le T)\,.$$

$$5.h(t)\in C^2[0,T]\,,\ h(t)\ne 0\ \ (0\le t\le T)$$

Then from (20) and (21) we find:

$$\left\{\sum_{k=1}^{\infty}\left(\lambda_k^5 \parallel \tilde{u}_{2k}(t) \parallel_{C[0\,,T]}\right)^2\right\}^{\frac{1}{2}} \le A_1(T)+$$

$$+B_1(T)\left\|a(t)\right\|_{C[0,T]}\left\|u(x,t)\right\|_{B^5_{2,T}}+C_1(T)\left\|u(x,t)\right\|_{B^5_{2,T}}, \tag{22}$$

$$\left\|\tilde{a}(t)\right\|_{C[0,T]} \le A_2(T)+$$

$$+B_2(T)\left\|a(t)\right\|_{C[0,T]}\left\|u(x,t)\right\|_{B^5_{2,T}}+C_2(T)\left\|u(x,t)\right\|_{B^5_{2,T}}, \tag{23}$$

where

$$A_1(T)=\sqrt{6}\left\|\varphi^{(5)}(x)\right\|_{L_2(0,1)}+\sqrt{\frac{6(1+\beta)}{\alpha}}\left\|\psi^{(4)}(x)\right\|_{L_2(0,1)}+$$

$$+\frac{1}{\beta}\sqrt{\frac{6T(1+\beta)}{\alpha}}\left\|f_{xx}(x,t)\right\|_{L_2(D_T)},$$

$$B_1(T)=\frac{1}{\beta}\sqrt{\frac{6(1+\beta)}{\alpha}}T, C_1(T)=\sqrt{6}\,T\left(\left\|p_1\,(t)\right\|_{C[0,T]}+\sqrt{\frac{1+\beta}{\alpha}}\left\|p_2\,(t)\right\|_{C[0,T]}\right),$$

$$A_2(T)=\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left\{\left\|h''(t)-f(0,t)\right\|_{C[0,T]}+\right.$$

$$+\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\left[\sqrt{\frac{1+\alpha}{\beta}}\left[\left\|\varphi^{(5)}(x)\right\|_{L_2(0,1)}+\sqrt{\frac{1+\beta}{\alpha}}\left\|\psi^{(4)}(x)\right\|_{L_2(0,1)}+\right.\right.$$

$$\left.\left.+\frac{1}{\beta}\sqrt{\frac{T(1+\beta)}{\alpha}}\left\|f_{xx}(x,t)\right\|_{L_2(D_T)}\right]+\left\|\left\|f_{xx}(x,t)\right\|_{C[0,T]}\right\|_{L_2(0,1)}\right]\right\},$$

$$B_2(T)=\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\sqrt{\frac{1+\alpha}{\beta}}\left(\frac{1}{\beta}\sqrt{\frac{1+\beta}{\alpha}}T+1\right)$$

$$C_2(T)=\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\sqrt{\frac{1+\alpha}{\beta}}\left[T\left\|p_1\,(t)\right\|_{C[0,T]}+\right.$$

$$\left.+\sqrt{\frac{1+\beta}{\alpha}}T\left\|p_2\,(t)\right\|_{C[0,T]}\right].$$

Now, from (22)-(23) we obtain:

$$\left\|\tilde{u}(x,t)\right\|_{B^6_{2,T}}+\left\|\tilde{a}(t)\right\|_{C[0,T]}\le A(T)+$$

$$+B(T)\left\|a(t)\right\|_{C[0,T]}\left\|u(x,t)\right\|_{B^5_{2,T}}+C(T)\left\|u(x,t)\right\|_{B^5_{2,T}}, \tag{24}$$

where

$$A(T) = A_1(T) + A_2(T), B(T) = B_1(T) + B_2(T)\,,$$

$$C(T) = C_1(T) + C_2(T)\,,$$

We now can prove the following theorem

Theorem 2. *If conditions 1- 5 are fulfilled and the inequality*

$$(B(T)(A(T)+2) + C(T))(A(T)+2) < 1\ , \tag{25}$$

is valid, the problem (1)-(3), (7)) *has a unique solution in the sphere* $K = K_R(\|z\|_{E_T^5} \leq R \leq A(T)+2)$ *of the space* E_T^5 .

Proof. In the space E_T^5 we consider the equation

$$z = \Phi z\,, \tag{26}$$

where $z = \{u,\, a\}$, the components $\Phi_i(u,a)$ $(i = 1,2)$ of the operator $\Phi(u,a)$ are determined by the right hand sides of equations (16) and (19).

Let us consider the operator $\Phi(u,a)$ in the sphere $K = K_R$ from E_T^5. Similar to (24) we obtain that for all $z, z_1, z_2 \in K_R$ the following estimations are valid:

$$\|\Phi z\|_{E_T^5} \leq A(T) + B(T)\,\|a(t)\|_{C[0,T]}\,\|u(x,t)\|_{B_{2,T}^5} + C(T)\,\|u(x,t)\|_{B_{2,T}^5} \leq$$

$$\leq A(T) + B(T)(A(T)+2)^2 + C(T)(A(T)+2)\ , \tag{27}$$

$$\|\Phi z_1 - \Phi z_2\|_{E_T^5} \leq B(T)R(\|a_1(t) - a_2(t)\|_{C[0,T]} + \|u_1(x,t) - u_2(x,t)\|_{B_{2,T}^5}) +$$

$$+ C(T)\,\|u_1(x,t) - u_2(x,t)\|_{B_{2,T}^5}\ . \tag{28}$$

Now, by virtue of (25), from (27) and (28) it is clear that the operator Φ satisfies the principle of compressed mappings on the set $K = K_R$. Therefore the operator Φ in the sphere $K = K_R$ has a unique fixed point $\{z\} = \{u\,,\, a\}$, that is the solution of equation (26), i.e. it is a unique solution of the system (16), (19) in the sphere $K = K_R$. Then the function $u(x,t)$, as an element of the space $B_{2,T}^5$, is continuous and has continuous derivatives $u_x(x,t)$,$u_{xx}(x,t)$,$u_{xxx}(x,t)$ and $u_{xxxx}(x,t)$ in D_T.

Similar to [13] we can show that $u_t(x,t), u_{tt}(x,t), u_{ttx}(x,t),\ u_{ttxx}(x,t) \in C(D_T)$.

In what follows, we can check that equation (1) and conditions (2), (3), (7) are satisfied in a usual sense. Consequently, $\{u(x,t), a(t)\}$ is the solution of the problem (1)-(3), (7), and by virtue of the Corollary of lemma 2 , it is unique in the sphere $K = K_R$. The theorem is proved.

The validity of the following statement follows directly from theorem 2 and by virtue of theorem 1.

Theorem 3. *Let all the conditions of theorem 2 , and the following agreement conditions be fulfilled*

$$\varphi(0) = h(0) \ - \int_0^T p_1(t)h(t)dt, \quad \psi(0) = h''(0) - \int_0^T p_2(t)h(t)dt$$

and

$$\left(T \left\| p_2(t) \right\|_{\ C[0,T]} + \left\| p_1(t) \right\|_{\ C[0,T]} + \frac{T}{2}(A(T) + 2) \right) T < 1$$

Then problem (1)-(4) *has a classic unique solution in the sphere* $K = K_R(\|z\|_{E_T^5} \leq R = A(T) + 2)$ *of the space* E_T^5.

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