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Global Bifurcation from Infinity in Nonlinear Dirac Problems with Eigenvalue Parameter in the Boundary Conditions

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Abstract. In this paper we consider global bifurcation from infinity for nonlinear Dirac system with a spectral parameter in boundary conditions. We show the existence of two families of global continua of the set of solutions to this problem, emanating from the points of the line $\mathbb{R} \times \{\infty\}$, corresponding to the eigenvalues of the linear problem, which is obtained from the nonlinear problem by setting the nonlinear term to zero and contained in classes of vector-functions with a fixed oscillation count in the neighborhood of these points.

Key Words and Phrases: nonlinear Dirac problem, eigenvalue, eigenvector-function, bifurcation from infinity, asymptotic bifurcation point, oscillation count

2010 Mathematics Subject Classifications: 34A30, 34B05, 34B15, 34C10, 34C23, 34K29, 47J10, 47J15

1. Introduction

We consider the following nonlinear Dirac equation

$$(\ell w)(x) \equiv Bw'(x) - P(x)w(x) = \lambda w(x) + g(x, w(x), \lambda), \; 0 < x < \pi, \tag{1.1}$$

subject to the boundary conditions

$$\begin{aligned} U_1(\lambda, w) := (\lambda \cos\alpha + a_0, \lambda \sin\alpha + b_0)w(0) = \\ (\lambda \cos\alpha + a_0)\, v(0) + (\lambda \sin\alpha + b_0)\, u\,(0) = 0, \end{aligned} \tag{1.2}$$

$$\begin{aligned} U_2(\lambda, w) := (\lambda \cos\beta + a_1, \lambda \sin\beta + b_1)w(\pi) = \\ (\lambda \cos\beta + a_\pi)\, v(0) + (\lambda \sin\beta + b_0)\, u\,(\pi) = 0, \end{aligned} \tag{1.3}$$

where

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \; P(x) = \begin{pmatrix} p\,(x) & 0 \\ 0 & r(x) \end{pmatrix}, \; w(x) = \begin{pmatrix} u(x) \\ v(x) \end{pmatrix},$$

$\lambda \in \mathbb{C}$ is an eigenvalue parameter, $p(x)$, $r(x) \in C([0,\pi];\mathbb{R})$, $[0,\ \pi]$, α, β, a_0, b_0, a_1 and b_1 are real constants such that $0 \leq \alpha$, $\beta < \pi$ and

$$\sigma_0 = a_0 \sin\alpha - b_0 \cos\alpha < 0,\ \ \sigma_1 = a_1 \sin\beta - b_1 \cos\beta > 0. \tag{1.4}$$

Here the function $g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in C\left([0,\pi] \times \mathbb{R}^2 \times \mathbb{R};\mathbb{R}^2\right)$ satisfies the following condition:

$$g(x,w,\lambda) = o(|w|) \text{ as } |w| \to \infty, \tag{1.5}$$

uniformly in $(x,\lambda) \in [0,\pi] \times \Lambda$, for any bounded interval $\Lambda \subset \mathbb{R}$ (we denote by $|\cdot|$ the Euclidean norm in $\mathbb{R}^2$).

In relativistic quantum mechanics and particle physics, the Dirac equation is a relativistic wave equation for describing spin-1/2 particles (i.e., fermions) [13, 19]. This equation is consistent with both the principles of quantum mechanics and the special theory of relativity, and was the first theory to fully explain the theory of relativity in the context of quantum mechanics.

The nonlinear Dirac equation since the first nonlinear generalization of the linear Dirac equation by the Russian physicist D.D. Ivanenko [11] naturally emerged as a practical model in many physical systems. Note that the nonlinear Dirac equations model the state of relativistic electrons (or other 1/2 particles) in external fields and during self-interaction. For example, equation (1.1) in the case of $p(x) = V(x) + m$, $r(x) = V(x) - m$ and $g(x,w,\lambda) = \tilde{g}(x,|w|)w$, where $V(x)$ is the potential function, m is the particle mass and $\tilde{g}$ is a nonlinear self-coupling, describes the dynamics of self-acting massive Dirac fermions with spin 1/2 inside external non-stationary electromagnetic potentials [19].

The global bifurcation from infinity in nonlinear Sturm-Liouville problems, fourth-order nonlinear Sturmian systems and nonlinear Dirac systems in cases where the spectral parameter is not contained in the boundary conditions and is contained in the boundary condition has been well studied in [3-5, 9, 14-18, 20]. In these papers, the authors show the existence of global connected components of the set of solutions to the nonlinear problems under consideration, which branch from the bifurcation points and intervals of the line $\mathbb{R} \times \{\infty\}$. Moreover, these components are contained in classes of functions that have oscillating properties of eigenvector-functions of linear problems, obtained from nonlinear problems by equating nonlinear terms to zero, in the neighborhoods of bifurcation points and intervals.

Note that global bifurcation from zero of nontrivial solutions to the nonlinear Dirac problem (1.1)-(1.3) was studied in [2].

In this paper, we consider the structure and behavior of global continua bifurcating from infinity to the nonlinear Dirac problem (1.1)-(1.3).

The rest of the article is organized as follows. Section 2 first provides information about the oscillatory properties of the eigenvector functions of the linear problem (1.1)-(1.3) for $g \equiv 0$ in terms of the Prüfer angular function (see [8]). Then we present the construction of classes of vector-functions in $C([0,\pi];\mathbb{R}^2)$ and $C([0,\pi];\mathbb{R}^2) \oplus \mathbb{R}^2$, that have these oscillatory properties of these eigenvector-functions. Moreover, the problem is reduced to equivalent operator equations in the space $C([0,\pi];\mathbb{R}^2\oplus\mathbb{R}^2)$. Section 3 shows the existence of global continua of nontrivial solutions to problem (1.1)-(1.3), emanating from asymptotic bifurcation points and contained in the above-mentioned classes of vector-functions in the neighborhoods of these points.

2. Preliminary

Let the continuous functions $\gamma(\lambda)$ and $\delta(\lambda)$ on $\mathbb{R}$ be defined by

$$\cot\gamma(\lambda) = -\frac{\lambda\cos\alpha + a_0}{\lambda\sin\alpha + b_0},\ \gamma\left(-\frac{b_0}{\sin\alpha}\right) = 0,\ \text{for } \alpha\neq 0; \tag{2.1}$$

$$\cot\delta(\lambda) = -\frac{\lambda\cos\beta + a_1}{\lambda\sin\beta + b_1},\ \delta\left(-\frac{b_0}{\sin\beta}\right) = 0,\ \text{for } \beta\neq 0. \tag{2.2}$$

In view of (1.4) the function $\gamma(\lambda)$ strictly increases and the function $\delta(\lambda)$ strictly decreases in the domains of their definition. Moreover, it follows from (2.1) and (2.2) that

$$\gamma(\lambda)\in(-\alpha,\pi-\alpha),\ \lim_{\lambda\to-\infty}\gamma(\lambda) = -\alpha \ \text{ and } \lim_{\lambda\to+\infty}\gamma(\lambda) = \pi-\alpha, \tag{2.3}$$

$$\delta(\lambda)\in(-\beta,\pi-\beta),\ \lim_{\lambda\to-\infty}\delta(\lambda) = \pi-\beta \ \text{ and } \lim_{\lambda\to+\infty}\delta(\lambda) = -\beta. \tag{2.4}$$

Let $E = C\left([0,\pi];\,\mathbb{R}^2\right)$ be the Banach space with the usual norm

$$||w|| = \max_{x\in[0,\,\pi]}|u(x)| + \max_{x\in[0,\,\pi]}|v(x)|.$$

Let S be the subset of E defined as follows:

$$S = \{w\in E:\ |u(x)+|v(x)|\ >0,\ x\in[0,\pi]\}.$$

For each fixed $(\lambda,w) = \left(\lambda,\begin{pmatrix} u \\ v\end{pmatrix}\right)\in\mathbb{R}\times S$ we define a continuous function $\theta(\lambda,w,x)$ on $[0,\pi]$ by

$$\cot\theta(\lambda,w,x) = \frac{u(x)}{v(x)},\ \ \theta(\lambda,w,0) = \gamma(\lambda). \tag{2.5}$$

Now we consider the linear Dirac problem

$$\begin{cases} (\ell w)(x) = \lambda w(x),\ x\in(0,\pi), \\ U_1(\lambda,w) = 0, \\ U_2(\lambda,w) = 0, \end{cases} \tag{2.6}$$

that obtained from (1.1)-(1.3) by setting $g\equiv 0$.

Remark 2.1. Problem (2.6) was studied in [1], where it was shown that the eigenvalues λ_k, $k \in \mathbb{Z}$, of this problem are real and simple, and they can be numbered in ascending order on the real axis

$$\ldots < \lambda_{-k} < \ldots < \lambda_{-2} < \lambda_{-1} < \lambda_0 < \lambda_1 < \lambda_2 < \ldots < \lambda_k < \ldots ,$$

so that the corresponding angular function $\theta(\lambda_k, w_k, x)$ at $x = 0$ and $x = \pi$ satisfies the conditions

$$\theta(\lambda_k, w_k, 0) = \gamma(\lambda_k), \quad \theta(\lambda_k, w_k, \pi) = \delta(\lambda_k) + \pi k, \tag{2.7}$$

where $w_k(x)$, $k \in \mathbb{Z}$, is an eigenvector-function corresponding to the kth eigenvalue λ_k.

Note that the spectral properties of problem (2.6) was studied in [12] in the case of $\alpha, \beta \in [-\pi/2, \pi/2]$.

We define the integers m_{-1} and m_1 as follows:

$$\begin{cases} m_{-1} = \max\left\{k \in \mathbb{Z} : \lambda_k + p(x) < 0,\ \lambda_k + r(x) < 0,\ x \in [0, \pi]\right\}, \\ \\ m_1 = \min\left\{k \in \mathbb{Z} : \lambda_k + p(x) > 0,\ \lambda_k + r(x) > 0,\ x \in [0, \pi]\right\}. \end{cases} \tag{2.8}$$

Remark 2.2. From (2.8) it follows that the function $\theta(\lambda_k, w_k, x)$ satisfies the first and second parts of statement (ii) of [6, Theorem 2.1] for $k \geq m_1$ and $k \leq m_{-1}$, respectively.

From now on ν will denote an element of $\{+, -\}$ that is, either $\nu = +$ or $\nu = -$.

For each $k \in \mathbb{Z}$, $k \geq m_1$ or $k \leq m_{-1}$, each $\lambda \in \mathbb{R}$ and each ν we define the set $S^\nu_{k,\lambda}$ of functions $w = \begin{pmatrix} u \\ v \end{pmatrix} \in S$ which satisfy the following conditions:

(i) $\theta(\lambda, w, \pi) = \delta(\lambda) + k\pi$;

(ii) if $k \geq m_1$, then for fixed (λ, w), as x increases, the function θ cannot tend to a multiple of $\pi/2$ from above, and as x decreases, the function θ cannot tend to a multiple of $\pi/2$ from below; if $k \leq m_{-1}$, then for fixed (λ, w), as x increases, the function θ cannot tend to a multiple of $\pi/2$ from below, and as x decreases, the function θ cannot tend to a multiple of $\pi/2$ from above.

(iii) the function $\nu u(x)$ is positive in a deleted neighborhood of the point $x = 0$.

Let $S_{k,\lambda} = S^+_{k,\lambda} \cup S^-_{k,\lambda}$. By Remarks 2.1 and 2.2 for each $\lambda \in \mathbb{R}$ the sets $S^+_{k,\lambda}$, $S^-_{k,\lambda}$ and $S_{k,\lambda}$, $k \geq m_1$ and $k \leq m_{-1}$, are nonempty. It follows from the definition of these sets that for each fixed $\lambda \in \mathbb{R}$ they are disjoint and open in E. Moreover, if $w = \begin{pmatrix} u \\ v \end{pmatrix} \in \partial S^\nu_{k,\lambda}$ for some $\lambda \in \mathbb{R}$, then there exists $\xi \in [0, \pi]$ such that $|w(\xi)| = 0$, i.e., $u(\xi) = v(\xi) = 0$ [7, Remark 2.7].

For each $k \in \mathbb{Z}$, $k \geq m_1$ or $k \leq m_{-1}$, and each ν let S_k and S^ν_k be the subsets of S defined by

$$S_k = \bigcup_{\lambda \in \mathbb{R}} S_{k,\lambda} \ \text{ and } \ S^\nu_k = \bigcup_{\lambda \in \mathbb{R}} S^\nu_{k,\lambda},$$

respectively. Note that the sets S_k and S^ν_k are disjoint and open in E. Moreover, if $w = \begin{pmatrix} u \\ v \end{pmatrix} \in \partial S^\nu_k$, then there exists $\xi \in [0, \pi]$ such that $u(\xi) = v(\xi) = 0$.

Let $\hat{E}$ be the Banach space $E \oplus \mathbb{R}^2$ with the norm

$$||\hat{w}||_0 = \left\| \begin{pmatrix} w \\ l \\ s \end{pmatrix} \right\|_0 = ||w|| + |l| + |s|,$$

and let

$$\hat{S} = \{\hat{w} = (w, l, s)^t \in \hat{E} : w \in S\}.$$

For each $k \in \mathbb{Z}$, $k \geq m_1$ or $k \leq m_{-1}$, and each ν let $\hat{S}_k^\nu$ the subset of $\hat{S}$ given by

$$\hat{S}_k^\nu = \left\{ \hat{w} = \begin{pmatrix} w \\ l \\ s \end{pmatrix} : w \in S_k^\nu \right\} \text{ and } \hat{S}_k = \hat{S}_k^+ \cup \hat{S}_k^-.$$

Let A is a linear operator defined in $\hat{E}$ by

$$Aw = A \begin{pmatrix} w \\ l \\ s \end{pmatrix} = \begin{pmatrix} \ell w \\ a_0 v(0) + b_0 u(0) \\ a_1 v(\pi) + b_1 u(\pi) \end{pmatrix} \tag{2.9}$$

on the domain

$$D(L) = \left\{ \hat{w} = \left(\begin{pmatrix} w \\ l \\ s \end{pmatrix} \right) \in \hat{E} : w \in C^1([0,\pi]; \mathbb{R}^2),\ l = -\left(v(0)\cos\alpha + u(0)\sin\alpha\right), \right.$$

$$\left. s = -\left(v(\pi)\cos\beta + u(\pi)\sin\beta\right) \right\}.$$

Here in the space $C^1([0,\pi] \oplus \mathbb{R}^2)$ the norm is determined by

$$||\hat{w}||_1 = \left\| \begin{pmatrix} w \\ l \\ s \end{pmatrix} \right\|_1 = ||w|| + ||w'|| + |l| + |s|.$$

It is obvious that the operator L is well defined. Then the problem (2.1) is recast in the equivalent operator form

$$A\hat{w} = \lambda\hat{w},\ \hat{w} \in D(A). \tag{2.10}$$

The operator A is closed in $\hat{E}$ with compact resolvent. Therefore, if $\lambda = 0$ is not an eigenvalue of the operator A, then there exists $A^{-1} : \hat{E} \to \hat{D}(A)$ which is completely continuous.

As the norms in $\mathbb{R} \times E$ and $\mathbb{R} \times \hat{E}$, we take

$$||(\lambda, w)|| = \left\{|\lambda|^2 + ||w||^2\right\}^{1/2} \text{ and } ||(\lambda, \hat{w})||_0 = \left\{|\lambda|^2 + ||\hat{w}||_0^2\right\}^{1/2},$$

respectively. By $\mathcal{B}_{\lambda,\varepsilon}$, $\varepsilon > 0$, we denote the open ball in $\mathbb{R} \times \hat{E}$ of radius ε with center at $(\lambda, \hat{0})$.

Now let the nonlinear operators $F : \mathbb{R} \times \hat{E} \to \hat{E}$ and $G : \mathbb{R} \times \hat{E} \to \hat{E}$ are defined by

$$G(\lambda, \hat{w}) = G\left(\lambda, \begin{pmatrix} w \\ l \\ s \end{pmatrix}\right) = \begin{pmatrix} g(x, w, \lambda) \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} g_1(x, w, \lambda) \\ g_2(x, w, \lambda) \\ 0 \\ 0 \end{pmatrix}. \tag{2.11}$$

Then problem (1.1)-(1.3) we can rewrite in the following equivalent operator form

$$A\hat{w} = \lambda\hat{w} + G(\lambda, \hat{w}). \tag{2.12}$$

It is obvious that in this case there is a one-to-one correspondence between the solutions of problems (1.1)-(1.3) and (2.12):

$$(\lambda, w) \leftrightarrow (\lambda, \hat{w}). \tag{2.13}$$

Let

$$\hat{A} = A^{-1} \text{ and } \hat{G} = A^{-1}G. \tag{2.14}$$

Then problem (2.10) reduces to the following equivalent form

$$\hat{w} = \lambda\hat{A}\hat{w} + \hat{G}(\lambda, \hat{w}). \tag{2.15}$$

Moreover, the linear problem (2.10) reduces to the linear problem

$$\hat{w} = \lambda\hat{A}\hat{w}. \tag{2.16}$$

Remark 2.3. The eigenvector function w_k corresponding to the eigenvalue λ_k of problem (2.16) will be unique using the requirement

$$\hat{w}_k \in \hat{S}_k^+ \text{ and } ||\hat{w}_k||_0 = 1.$$

3. Global bifurcation of solutions from infinity to problem (1.1)-(1.3)

Let $\tilde{\mathcal{C}}$ be the set of nontrivial solutions of the nonlinear Dirac problem (2.12) (also to problem (2.10)).

Remark 3.1. We add the points $\{(\lambda, \infty) : \lambda \in \mathbb{R}\}$ to our space $\mathbb{R} \times \hat{E}$ and defining an appropriate topology on the resulting set, we include that (λ, ∞) is an element of $\mathbb{R} \times \hat{E}$.

Theorem 3.1 *For each $k \in \mathbb{Z}$ and each ν there exists a component $\hat{\mathcal{C}}_k^\nu$ of the set $\hat{\mathcal{C}}$ that meet (λ_k, ∞) with respect to the set $\mathbb{R} \times \hat{S}_k^\nu$ and for this set at least one of the following statements holds:*

(i) $\hat{\mathcal{C}}_k^\nu$ meets (λ'_k, ∞) with respect to the set $\mathbb{R} \times \hat{S}_{k'}^{\nu'}$ for some $(k', \nu') \neq (k, \nu)$;

(ii) $\hat{\mathcal{C}}_k^\nu$ meets $\hat{\mathcal{R}} = \{(\lambda, \hat{0}) : \lambda \in \mathbb{R}\}$ for some $\lambda \in \mathbb{R}$;

(iii) the natural projection $P_{\hat{\mathcal{R}}}(\hat{\mathcal{C}}_k^\nu)$ onto $\hat{\mathcal{R}}$ is unbounded.

Proof. The proof will be carried out in three steps.

Step 1. Recall that the operator $\hat{A} : \hat{E} \to D(A)$ is completely continuous. Consequently, there exists positive constants C_1 such that for any $\hat{w} \in \hat{E}$ the following relation holds:

$$||\hat{A}\hat{w}||_1 \leq C_1 ||\hat{w}||_0. \tag{3.1}$$

By (1.5) it follows from [5, Lemma 2] that for any sufficiently small $\epsilon \in (0,1)$ there is a sufficiently large $\varrho_\epsilon > 0$ such that for any $x \in [0,\pi]$, $w \in E$, $||w|| > \varrho_\epsilon$ and $\lambda \in \Lambda$, the relation

$$|g(x, w(x), \lambda)| < \epsilon\, C_1^{-1}\, ||w|| \tag{3.2}$$

holds.

We have the following relations

$$|l| \leq |u(0)| + |v(0)| \leq ||w|| \ \text{ and } \ |s| \leq |u(\pi)| + |v(\pi)| \leq ||w||,$$

and consequently,

$$||\hat{w}||_0 \leq 3||w||. \tag{3.3}$$

Let

$$\hat{\varrho}_\epsilon = 3\varrho_\epsilon \text{ and } ||\hat{w}||_0 > \hat{\varrho}_\epsilon.$$

Then, by (3.3), we have

$$||w|| > \varrho_\epsilon.$$

Hence it follows from (3.2) that for any $x \in [0,\pi]$, $\hat{w} \in E$, $||\hat{w}||_0 > \hat{\varrho}_\epsilon$ and $\lambda \in \Lambda$,

$$|g(x, w, \lambda)| < \epsilon\, C_1^{-1}\, ||w||. \tag{3.4}$$

Therefore, in view of (3.4), by (3.1) for any $\lambda \in \Lambda$ and $\hat{w} \in \hat{E}$, $||\hat{w}||_0 > \hat{\varrho}_\epsilon$, we get

$$\frac{||\hat{G}(\lambda,\hat{w})||_0}{||w||_0} = \frac{||\hat{A}(G(\lambda,\hat{w}))||_0}{||w||_0} \leq C_1 ||G(\lambda,\hat{w})||_0 = C_1 \max_{x\in[0,\,\pi]} |g(x, w(x), \lambda)| < \epsilon, \tag{3.5}$$

which implies that

$$\hat{G}(\lambda,\hat{w}) = o(||w||_0) \text{ as } ||w||_0 \to \infty, \tag{3.6}$$

uniformly for $\lambda \in \Lambda$.

Step 2. Let $(\lambda,\hat{w}) \in \mathbb{R} \times \hat{E}$ such that $\hat{w} \neq \hat{0}$. We define the continuous operator $\hat{\mathcal{G}} : \hat{E} \to \hat{E}$ as follows:

$$\hat{\mathcal{G}}(\lambda,\hat{w}) = \begin{cases} ||\hat{w}||_0^2\, \hat{G}\left(\lambda, \frac{\hat{w}}{||w||_0^2}\right) & \text{if } \ \hat{w} \neq \hat{0}, \\ \\ 0 & \text{if } \ \hat{w} = \hat{0}. \end{cases} \tag{3.7}$$

Let $\epsilon > 0$ be an arbitrary sufficiently small number, $\hat{\eta}_\epsilon = \frac{1}{\hat{\varrho}_\epsilon}$ and $\hat{w} \in \hat{E}$ such that $||\hat{w}||_0 < \hat{\eta}_\epsilon$. Then we have $\left\|\frac{w}{||\hat{w}||_0^2}\right\|_0 > \hat{\varrho}_\epsilon$. Hence it follows from (3.5) that

$$\left\|\hat{G}\left(\lambda, \frac{\hat{w}}{||w||_0^2}\right)\right\|_0 < \frac{\epsilon}{||w||_0} \text{ for any } \lambda \in \Lambda.$$

In view of (3.7) from the last relation for any $\lambda \in \Lambda$, $\hat{w} \in \hat{E}$, $||\hat{w}||_0 < \hat{\eta}_\epsilon$, we get

$$||\hat{\mathcal{G}}(\lambda, \hat{w})||_0 = ||\hat{w}||_0^2 \left\| \hat{G}\left(\lambda, \frac{\hat{w}}{||w||_0^2}\right) \right\|_0 < \epsilon \, ||\hat{w}||_0. \tag{3.8}$$

The relation (3.18) shows that the set

$$\hat{\mathcal{G}}(\Lambda \times \mathcal{B}_{\hat{\eta}_\epsilon}) = \left\{ \hat{\mathcal{G}}(\lambda, \hat{w}) : (\lambda, \hat{w}) \in \Lambda \times \mathcal{B}_{\hat{\eta}_\epsilon} \right\}$$

is bounded.

By (2.9) and (2.11)-(2.15) the vector-function $\bar{\hat{w}} = \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} = \hat{\mathcal{G}}(\lambda, \hat{w})$ satisfies the following relation

$$B\bar{w}'(x) - P(x)\bar{w}(x) = ||\hat{w}||_0^2 \, g\left(x, ||\hat{w}||_0^{-2} w(x), \lambda\right), \ x \in (0, \pi). \tag{3.9}$$

Then by (3.4) and (3.8) it follows from (3.9) that

$$|\bar{w}'(x)| \leq |p\,(x)||\bar{u}(x)| + |r(x)||\bar{v}(x)| + ||\hat{w}||_0^2 \, |g(x, \tfrac{w(x)}{||\hat{w}||_0^2}, \lambda)| \leq M||\bar{w}|| + \epsilon \, C_1^{-1} \, ||w|| \leq$$

$$\epsilon \, M||\hat{w}||_0 + \epsilon \, C_1^{-1} \, ||\hat{w}||_0 < \epsilon \, (M + C_1) \, \hat{\eta}_\epsilon,$$

where $M = \max\{||p\,(x)||_\infty, \, ||r(x)||_\infty\}$. Hence it follows from Arzelà-Ascoli theorem that the set $\hat{\mathcal{G}}(\Lambda \times \mathcal{B}_{\hat{\eta}_\epsilon})$ is compact in $\mathbb{R} \times \hat{E}$, i.e. operator $\hat{\mathcal{G}}$ is completely continuous.

Step 3. Let $(\lambda, \hat{w})$ be a nontrivial solution of the problem (2.15). Setting $\hat{\mathrm{w}} = \frac{w}{||\hat{w}||_0^2}$ we have $||\hat{\mathrm{w}}||_0 = \frac{1}{||\hat{w}||_0}$ and $w = \frac{\hat{\mathrm{w}}}{||\hat{\mathrm{w}}||_0^2}$. Then dividing (2.15) by $||\hat{w}||_0^2$ and using (3.7), we get the problem

$$\hat{\mathrm{w}} = \lambda \hat{A} \hat{\mathrm{w}} + \hat{\mathcal{G}}(\lambda, \hat{\mathrm{w}}). \tag{3.10}$$

Let $\mathfrak{C} \subset \mathbb{R} \times \hat{E}$ be the set of nontrivial solutions of problem (3.10).

By (3.7) we have

$$\hat{\mathcal{G}}(\lambda, \hat{\mathrm{w}}) = o(||\hat{\mathrm{w}}||_0) \text{ as } ||\hat{\mathrm{w}}||_0 \to 0, \tag{3.11}$$

uniformly in $\lambda \in \Lambda$. Since the characteristics values of problem (2.16) are eigenvalues of problem (1.1)-(1.3) and are simple, by (3.11) it follows from [15, Theorem 1.3] that for each $k \in \mathbb{Z}$ there exists a continuum $\hat{\mathfrak{C}}_k^\nu$ of $\mathfrak{C}$ that meet $(\lambda_k, \hat{0})$ and either (i) $\hat{\mathfrak{C}}_k^\nu$ is unbounded in $\mathbb{R} \times \hat{E}$, or (ii) $\hat{\mathfrak{C}}_k^\nu$ meets $(\lambda'_k, \hat{0})$ for some $k' \neq k$. Note that we can decomposed $\hat{\mathfrak{C}}_k^\nu$ into two subcontinua $\hat{\mathfrak{C}}_k^+$ and $\hat{\mathfrak{C}}_k^-$ using the construction that is presented by E.N. Dancer in [10, pp. 1070-1071]. Consequently, there exists sufficiently small $\varepsilon_k > 0$ such that if $(\lambda, \hat{\mathrm{w}}) \in \hat{\mathfrak{C}}_k^\nu \cap \hat{\mathcal{B}}_{\lambda_k, \epsilon_k}$, then

$$\lambda = \lambda_k + o(|\varsigma|), \ \hat{\mathrm{w}} = \varsigma \hat{w}_k + \hat{\mathbf{w}} \text{ with } \hat{\mathbf{w}} = o(|\varsigma|), \tag{3.12}$$

where $\varsigma \in \mathbb{R}^\nu$ ($\mathbb{R}^+ = (0, +\infty)$ and $\mathbb{R}^+ = (-\infty, 0)$). Since for each $k \in \mathbb{Z}$, $k \geq m_1$ or $k \leq m_{-1}$, the set $\hat{S}_k^\nu$ is open in $\mathbb{R} \times \hat{E}$, by Remark 2.3, it follows from (3.12) that

$$\hat{\mathfrak{C}}_k^+ \cap \hat{\mathcal{B}}_{\lambda_k, \epsilon_k} \subset \mathbb{R} \times \hat{S}_k^+, \ \hat{\mathfrak{C}}_k^- \cap \hat{\mathcal{B}}_{\lambda_k, \epsilon_k} \subset \mathbb{R} \times \hat{S}_k^- \ \text{ for } \ k \geq m_1 \text{ or } k \leq m_{-1}. \tag{3.13}$$

The relations in (3.13) show that for each $k \in \mathbb{Z}$, $k \geq m_1$ or $k \leq m_{-1}$, the subcontinua $\hat{\mathfrak{C}}_k^+$ and $\hat{\mathfrak{C}}_k^-$ meet the bifurcation point $(\lambda_k, \hat{0})$ with respect to the sets $\mathbb{R} \times \hat{S}_k^+$ and $\mathbb{R} \times \hat{S}_k^-$, respectively. Moreover, by [10, Theorem 2] for each $k \in \mathbb{Z}$, $k \geq m_1$ or $k \leq m_{-1}$, and each ν the subcontinuum $\hat{\mathfrak{C}}_k^\nu$ is either unbounded in $\mathbb{R} \times \hat{E}$ or there exists some $(k', \nu') \neq (k, \nu)$ such that $\hat{\mathfrak{C}}_k^\nu$ meets $(\lambda_{k'}, \hat{0})$ with respect to the set $\mathbb{R} \times \hat{S}_{k'}^{\nu'}$. Should be noted that if $\hat{\mathfrak{C}}_k^\nu$ is unbounded in $\mathbb{R} \times \hat{E}$, then the following two cases are possible: (a) the natural projection $P_{\mathcal{R}}(\hat{\mathfrak{C}}_k^\nu)$ of $\hat{\mathfrak{C}}_k^\nu$ onto $\mathcal{R}$ is unbounded; (b) the natural projection $P_{\mathcal{R}}(\hat{\mathfrak{C}}_k^\nu)$ of $\hat{\mathfrak{C}}_k^\nu$ onto $\mathcal{R}$ is bounded.

Now we consider the following inversion (see [16, 17])

$$\mathcal{H} : (\lambda, \hat{w}) \to (\lambda, \hat{\mathrm{w}}) = \left(\lambda, \frac{\hat{w}}{||\hat{w}||_0^2}\right), \; w \in \hat{E}.$$

By (3.6) and (3.11) it follows from the above arguments that the inversion $\mathcal{H}$ turns a bifurcation at infinity problem (2.15) into a bifurcation from zero problem (3.10). In this case, the inversion $\mathcal{H}$ maps the set $\hat{\mathcal{C}}$ into $\hat{\mathfrak{C}}$ and, heuristically, interchanges points $(\lambda, \hat{0})$, $\lambda \in \mathbb{R}$, (respectively, (λ, ∞), $\lambda \in \mathbb{R}$) with points (λ, ∞), $\lambda \in \mathbb{R}$ (respectively, $(\lambda, \hat{0})$, $\lambda \in \mathbb{R}$). By $\hat{\mathcal{C}}_k^\nu$ we denote the inverse image $\mathcal{H}^{-1}(\hat{\mathfrak{C}}_k^\nu)$ of the set $\hat{\mathfrak{C}}_k^\nu$ under transformation $\mathcal{H}$. Then the statements of the theorem about the sets $\hat{\mathcal{C}}_k^\nu$ directly follow from the above-mentioned properties of the set $\hat{\mathfrak{C}}_k^\nu$ under the inversion of $\mathcal{H}$. The proof of this theorem is complete.

Now by (2.13) it follows from Theorem 3.1 the following result.

Theorem 3.1 *For each $k \in \mathbb{Z}$ and each ν there exists a component $\mathcal{C}_k^\nu$ of the set $\mathcal{C}$ that meet (λ_k, ∞) with respect to the set $\mathbb{R} \times S_k^\nu$ and for this set at least one of the following statements holds:*

(i) $\mathcal{C}_k^\nu$ meets (λ'_k, ∞) with respect to the set $\mathbb{R} \times S_{k'}^{\nu'}$ for some $(k', \nu') \neq (k, \nu)$;

(ii) $\mathcal{C}_k^\nu$ meets $\mathcal{R} = \{(\lambda, \tilde{0}) : \lambda \in \mathbb{R}\}$ for some $\lambda \in \mathbb{R}$;

(iii) the natural projection $P_{\mathcal{R}}(\mathcal{C}_k^\nu)$ onto $\mathcal{R}$ is unbounded.

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