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Nodal Solutions of Some Nonlinear Dirac Problems

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Abstract. In this paper we consider the nonlinear boundary value problem for the one-dimensional Dirac operator. We show the existence of pair of nodal solutions of this problem. The proof of our main result based on a method on bifurcation from zero and infinity.

Key Words and Phrases: nonlinear Dirac equation, eigenvalue, eigenvector-function, bifurcation from zero, bifurcation from infinity, nodal solution

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1. Introduction

We consider the following nonlinear Dirac equation

$$(\ell w)(x) \equiv Bw'(x) - P(x)w(x) = sf(w(x)),\ 0 < x < \pi, \tag{1.1}$$

with the boundary conditions

$$U_1(w) := (\sin\alpha,\, \cos\alpha)\, w(0) = v(0)\cos\alpha + u\,(0)\sin\alpha\; = 0, \tag{1.2}$$

$$U_2(w) := (\sin\beta,\, \cos\beta) w(\pi) = v(\pi)\cos\beta + u\,(\pi)\sin\beta = 0, \tag{1.3}$$

where

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ P(x) = \begin{pmatrix} p\,(x) & 0 \\ 0 & r(x) \end{pmatrix},\ w(x) = \begin{pmatrix} u(x) \\ v(x) \end{pmatrix},$$

$\lambda \in \mathbb{C}$ is an eigenvalue parameter, $p\,(x)$, $r(x) \in C([0, \pi]; \mathbb{R})$, α, β, are real constants such that $0 \le \alpha, \beta < \pi$. Here s is a nonzero real number and the function $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in C\left(\mathbb{R}^2 \times \mathbb{R}^2; \mathbb{R}^2\right)$ satisfies the following condition:

$$f(w) = \gamma w + h(w) \text{ and } f(w) = \delta w + \hbar(w), \tag{1.4}$$

where γ, δ $(\gamma \ne \delta)$ are positive constants and

$$h(w) = \begin{pmatrix} h_1(w) \\ h_2(w) \end{pmatrix} = o(|w|) \text{ as } |w| \to 0, \tag{1.5}$$

$$\hbar(w) = \begin{pmatrix} \hbar_1(w) \\ \hbar_2(w) \end{pmatrix} = o(|w|) \text{ as } |w| \to \infty \tag{1.6}$$

(here by $|\cdot|$ we denote the norm in $\mathbb{R}^2$).

The Dirac equations describing spin-1/2 particles, such as electrons and positrons, play the important role in both physics and mathematics. The nonlinear Dirac equations describes the self-action in quantum electrodynamics and used as an effective theory in atomic and gravitational physics (see [14]).

The existence of nodal solutions of boundary value problems for ordinary differential operators of second and fourth orders was study in [5-11]. In these papers, the authors, using the bifurcation technique, show the existence of solutions of problems they consider, contained in classes of functions with a fixed usual oscillation count.

In this paper, we will show the existence of nodal solutions of problem (1.1)-(1.3) using global bifurcation results from zero and infinity that we previously obtained for nonlinear Dirac eigenvalue problems.

2. Preliminary

Let $(b.c.)$ be denote the set of vector-functions that satisfies the boundary conditions (1.2) and (1.3).

We consider the following nonlinear eigenvalue problem

$$\begin{cases} (\ell w)(x) = \lambda w(x) + g(x, w(x), \lambda), \; x \in (0, \pi), \\ w \in (b.c.). \end{cases} \tag{2.1}$$

We will impose one or the other of the following conditions on the nonlinear term $g(x, w, \lambda) \in C([0, \pi] \times \mathbb{R}^2 \times \mathbb{R}; \mathbb{R}^2)$:

$$g(x, w, \lambda) = o(|w|) \;\text{ as }\; |w| \to 0, \tag{2.2}$$

or

$$g(x, w, \lambda) = o(|w|) \;\text{ as }\; |w| \to +\infty, \tag{2.3}$$

uniformly with respect to $(x, \lambda) \in [0, \pi] \times \Lambda$, for any bounded interval $\Lambda \subset \mathbb{R}$.

By conditions (2.2) and (2.3) the nonlinear problem (2.1) is linearizable both at zero and at infinity, and the corresponding linear problem is

$$\begin{cases} (\ell w)(x) = \lambda w(x), \; x \in (0, \pi), \\ w \in (b.c.). \end{cases} \tag{2.4}$$

Let E be the Banach space $C\left([0,\pi];\,\mathbb{R}^2\right)\cap\{w\,:\,U(w)=0\}$ with the norm $||w||_0 = \max\limits_{x\in[0,\,\pi]}|u(x)| + \max\limits_{x\in[0,\,\pi]}|v(x)|$. We denote by S a subset of E, which defined as follows:

$$S = \{w\in E:\ |u(x)+|v(x)|\ > 0,\,\forall x\in[0,\pi]\}\,.$$

As in [2] (see also [4]), for each $w=\begin{pmatrix} u \\ v \end{pmatrix}\in S$ we define a continuous function $\theta(w,x)$ on $[0,\pi]$ by

$$\tan\theta(w,x) = \frac{v(x)}{u(x)},\ \ \theta(w,0) = -\,\alpha. \tag{2.5}$$

Theorem 2.1 [2, Theorem 3.1]. *The eigenvalues $\lambda_k, k\in\mathbb{Z}$, of the problem (2.4) are real and simple, and can be numbered in ascending order on the real axis*

$$...\ <\lambda_{-k}<\ ...\ <\lambda_{-1}<\lambda_0<\lambda_1<\ ...\ <\lambda_k<\ ...\ ,$$

so that for the eigenvector-function $w_k(x)$, $k\in\mathbb{Z}$, corresponding to the eigenvalue λ_k, the angular function $\theta(w_k,x)$ at $x=\pi$ satisfy the condition

$$\theta(w_k,\pi) = -\beta + k\pi. \tag{2.6}$$

The eigenvector-functions $w_k(x)=\begin{pmatrix} u_k(x) \\ v_k(x) \end{pmatrix}$ have, with a suitable interpretation, the following oscillation properties: if $k>0$ and $k=0,\alpha\geq\beta$ (except the cases $\alpha=\beta=0$ and $\alpha=\beta=\pi/2$), then

$$\begin{pmatrix} s(u_k) \\ s(v_k) \end{pmatrix} = \begin{pmatrix} k-1+\chi(\alpha-\pi/2)+\chi(\pi/2-\beta) \\ k-1+\mathrm{sgn}\alpha \end{pmatrix}, \tag{2.7}$$

and if $k<0$ and $k=0$, $\alpha<\beta$, then

$$\begin{pmatrix} s(u_k) \\ s(v_k) \end{pmatrix} = \begin{pmatrix} |\,k|-1+\chi(\pi/2-\alpha)+\chi(\beta-\pi/2) \\ |\,k|-1+\mathrm{sgn}\beta \end{pmatrix}, \tag{2.8}$$

where $s(g)$ the number of zeros of the function $g\in C([0,\pi]\,;\,\mathbb{R})$ in the interval $(0,\pi)$ and

$$\chi\,(x) = \begin{cases} 0, & if\ x\leq 0, \\ 1, & if\ x>0. \end{cases}$$

Moreover, the functions $u_k(x)$ and $v_k(x)$ have only nodal zeros in the interval $(0,\pi)$.

For each $k\in\mathbb{Z}$ and each ν, let S_k^ν denote the set of vector functions $w\in S$ with the following properties:

(i) $\theta(w,\pi) = -\beta+k\pi$;

(ii) if $k>0$ or $k=0,\alpha\geq\beta$ (except the cases $\alpha=\beta=0$ and $\alpha=\beta=\pi/2$), then for fixed w, as x increases from 0 to π, the function θ cannot tend to a multiple of $\pi/2$ from above, and as x decreases, the function θ cannot tend to a multiple of $\pi/2$ from below; if

$k < 0$ or $k = 0, \alpha < \beta$, then for fixed w, as x increases, the function θ cannot tend to a multiple of $\pi/2$ from below, and as x decreases, the function θ cannot tend to a multiple of $\pi/2$ from above;

(iii) the function $\nu u(x)$ is positive in a deleted neighborhood of $x = 0$.

Remark 2.1. By (2.5), Theorem 2.1 and [2, Remark 2.1] we have $w_k \in S_k = S_k^- \cup S_k^+$, in other words the sets S_k^-, S_k^+ and S_k are nonempty. From the definition of the sets S_k^-, S_k^+ and S_k it can be seen that they are disjoint and open in E.

Now for problem (1.1)-(1.3) we can give global bifurcation results obtained earlier in [1, 3].

Theorem 2.2 [3, Theorem 3.1]. *Let condition (2.2) holds. Then for each* $k \in \mathbb{Z}$ *and each* ν*, there exists a continuum* C_k^ν *of solutions of problem (2.1) which contains* $\left(\lambda_k, \tilde{0}\right)$*, is lies in* $(\mathbb{R} \times S_k^\nu) \cup \left\{(\lambda_k, \tilde{0})\right\}$ *and is unbounded in* $\mathbb{R} \times E$.

Theorem 2.3 [1, Theorem 1]. *Let condition (2.3) holds. Then for each* $k \in \mathbb{Z}$ *and each* ν *there exists a continua* D_k^ν *of solutions of problem (2.1) which contains* (λ_k, ∞) *and has the following properties:*

(i) there exists a neighborhood V_k *of* (λ_k, ∞) *in* $\mathbb{R} \times E$ *such that* $V_k \cap D_k^\nu \subset \mathbb{R} \times S_k^\nu$*;*

(ii) either D_k^ν *meets* (λ'_k, ∞) *with respect to the set* $\mathbb{R} \times S_{k'}^{\nu'}$ *for some* $(k', \nu') \neq (k, \nu)$*, or* D_k^ν *meets* $(\lambda, \tilde{0})$ *for some* $\lambda \in \mathbb{R}$*, or* D_k^ν *has an unbounded projection onto* $\mathbb{R} \times \left\{\tilde{0}\right\}$.

3. Existence of nodal solutions to the nonlinear problem (1.1)-(1.3)

In this section, for each $k \in \mathbb{Z}$ and each ν, an interval is determined for s in which there are solutions to the problem (1.1)-(1.3) contained in the set S_k^ν.

Theorem 1. *Let* $\lambda_k \neq 0$ *and*

$$\frac{\lambda_k}{\gamma} < s < \frac{\lambda_k}{\delta} \text{ or } \frac{\lambda_k}{\delta} < s < \frac{\lambda_k}{\gamma}$$

for some $k \in \mathbb{Z}$*. Then problem has two solutions* $w_{k,+}$ *and* $w_{k,-}$ *such that* $w_{k,+} \in S_k^+$ *and* $w_{k,-} \in S_k^-$.

Proof. By the first relation of (1.5) we can rewrite (1.1)-(1.3) it following form

$$\begin{cases} (\ell w)(x) = s\gamma w(x) + sh(w(x)),\, x \in (0, \pi), \\ w \in (b.c.). \end{cases} \tag{3.2}$$

Alongside problem (3.2) we shall consider the following nonlinear eigenvalue problem

$$\begin{cases} (\ell w)(x) = \lambda s\gamma w(x) + sh(w(x)),\, x \in (0, \pi), \\ w \in (b.c.), \end{cases} \tag{3.3}$$

or, equivalently

$$\begin{cases} (\tilde{\ell}w)(x) = \lambda w(x) + \tilde{g}(x, w),\ x \in (0, \pi), \\ w \in (b.c.), \end{cases} \tag{3.4}$$

where

$$\tilde{\ell}w = \frac{1}{s\gamma}\ell w \ \text{ and } \ \tilde{g}(x, w) = \frac{1}{\gamma}\, h(w(x)).$$

In view of condition (1.5) we have

$$\tilde{g}(x, w) = o(|w|) \ \text{ as } \ |w| \to 0, \tag{3.5}$$

uniformly with respect to $x \in [0, \pi]$. Then, in view of (3.5), by Theorem 2.2 for each $k \in \mathbb{Z}$ and each ν, there exists a continuum $\tilde{C}_k^\nu$ of solutions of problem (3.4) (also problem (3.3)) which contains $\left(\tilde{\lambda}_k, \tilde{0}\right)$, is lies in $(\mathbb{R} \times S_k^\nu) \cup \left\{(\tilde{\lambda}_k, \tilde{0})\right\}$ and is unbounded in $\mathbb{R} \times E$, where $\tilde{\lambda}_k$ is a kth eigenvalue of the linear problem

$$\begin{cases} (\tilde{\ell}w)(x) = \lambda w(x),\ x \in (0, \pi), \\ w \in (b.c.), \end{cases}$$

also of the linear problem

$$\begin{cases} (\ell w)(x) = \lambda s\gamma\, w(x),\ x \in (0, \pi), \\ w \in (b.c.). \end{cases} \tag{3.6}$$

In view of (3.6) we get

$$\tilde{\lambda}_k = \frac{\lambda_k}{s\gamma}.$$

By the second representation of (1.4) problem (3.3) will take the form

$$\begin{cases} (\ell w)(x) - s(\delta - \gamma)w(x) = \lambda s\gamma w(x) + s\hbar(w(x)),\ x \in (0, \pi), \\ w \in (b.c.), \end{cases} \tag{3.6}$$

or, equivalently

$$\begin{cases} (\hat{\ell}w)(x) = \lambda w(x) + \hat{g}(x, w),\ x \in (0, \pi), \\ w \in (b.c.), \end{cases} \tag{3.7}$$

where

$$\hat{\ell}w = \frac{1}{s\gamma}\ell w - \left(\frac{\delta}{\gamma} - 1\right) w \ \text{ and } \ \hat{g}(x, w) = \frac{1}{\gamma}\, h(w(x)).$$

In view of condition (1.6) for any bounded intervals $\Lambda \subset \mathbb{R}$

$$\hat{g}(x, w, \lambda) = o(|w|) \ \text{ as } \ |w| \to +\infty, \tag{3.8}$$

uniformly with respect to $(x, \lambda) \in [0, \pi] \times \Lambda$. Then, by (3.8), it follows from Theorem 2.3 that for each $k \in \mathbb{Z}$ and each ν there exists a continua $\hat{C}_k^\nu$ of solutions of problem (3.7) (also of problems (3.7) and (3.2)) which contains $(\hat{\lambda}_k, \infty)$ and has the following properties:

(i) there exists a neighborhood $\hat{V}_k$ of $(\hat{\lambda}_k, \infty)$ in $\mathbb{R} \times E$ such that $\hat{V}_k \cap \hat{C}_k^\nu \subset \mathbb{R} \times S_k^\nu$;

(ii) either $\hat{C}_k^\nu$ meets $(\hat{\lambda}'_k, \infty)$ with respect to the set $\mathbb{R} \times S_{k'}^{\nu'}$ for some $(k', \nu') \neq (k, \nu)$, or $\hat{C}_k^\nu$ meets $(\lambda, \tilde{0})$ for some $\lambda \in \mathbb{R}$, or $\hat{C}_k^\nu$ has an unbounded projection onto $\mathbb{R} \times \{\tilde{0}\}$, where $\hat{\lambda}_k$ is a kth eigenvalue of the linear problem

$$\begin{cases} (\hat{\ell} w)(x) = \lambda w(x),\ x \in (0, \pi), \\ w \in (b.c.), \end{cases}$$

also of the linear problem

$$\begin{cases} (\ell w)(x) = s(\lambda\gamma + \delta - \gamma)w(x) + s\hbar(w(x)),\ x \in (0, \pi), \\ w \in (b.c.). \end{cases} \tag{3.9}$$

By (3.9) we get

$$\hat{\lambda}_k = \frac{\lambda_k}{s\gamma} - \left(\frac{\delta}{\gamma} - 1\right).$$

Since conditions (1.5) and (1.6) are satisfied simultaneously, it follows from the proof of the first part of [13, Theorem 3.3] that $\hat{C}_k^\nu \subset \mathbb{R} \times S_k^\nu$, and consequently, the first alternative of property (ii) of $\hat{C}_k^\nu$ cannot hold, i.e., $\hat{C}_k^\nu$ cannot meets (λ'_k, ∞) for any $k' \neq k$. Then either $\hat{C}_k^\nu$ meets $(\lambda, 0)$ for some $\lambda \in \mathbb{R}$, or the projection $\hat{C}_k^\nu$ onto $\mathbb{R} \times \{\tilde{0}\}$ is unbounded.

Let $\hat{C}_k^\nu$ meets $(\lambda, \tilde{0})$ for some $\lambda \in \mathbb{R}$. Then, using [12, Lemma 1.24], we can show that $\lambda = \tilde{\lambda}_k$. Moreover, if $\tilde{C}_k^\nu$ meets (λ, ∞) for some $\lambda \in \mathbb{R}$, then from property (i) of $\hat{C}_k^\nu$ it follows that $\lambda = \hat{\lambda}_k$.

If the projection $\hat{C}_k^\nu$ onto $\mathbb{R} \times \{\tilde{0}\}$ is unbounded, then there exists $\{(\mu_n, z_n)\}_{n=1}^\infty \subset (\hat{C}_k^\nu \backslash \{(\hat{\lambda}_k, \infty)\}) \subset \mathbb{R} \times S_k^\nu$ $\left(z_n = \begin{pmatrix} z_{1n} \\ z_{2n} \end{pmatrix}\right)$ such that either

$$\lim_{n\to\infty} \mu_n \to +\infty \text{ or } \lim_{n\to\infty} \mu_n \to -\infty.$$

We introduce the notations:

$$\begin{aligned} \varphi_{n,1}(x) &= \tfrac{\hbar_1(z_n(x))z_{n,1}(x)}{z_{n,1}^2(x)+z_{n,2}^2(x)},\ \psi_{n,1}(x) = \tfrac{\hbar_1(z_n(x))z_{n,2}(x)}{z_{n,2}^2(x)+z_{n,2}^2(x)} \\ \varphi_{n,2}(x) &= \tfrac{\hbar_2(z_n(x))z_{n,1}(x)}{z_{n,1}^2(x)+z_{n,2}^2(x)},\ \psi_{n,2}(x) = \tfrac{\hbar_2(z_n(x))z_{n,1}(x)}{z_{n,1}^2(x)+z_{n,2}^2(x)}. \end{aligned} \tag{3.10}$$

Let $\epsilon_0 > 0$ be a fixed small number. Then, by conditions (1.5) and (1.6), there exist a small $\rho_0 > 0$ and a large $\varrho_0 > 0$ such that

$$\frac{|h(w)|}{|w|} < \epsilon_0 \text{ for any } w \in S,\ |w| < \rho_0, \tag{3.11}$$

$$\frac{|\hbar(w)|}{|w|} < \epsilon_0 \text{ for any } w \in S,\ |w| > \varrho_0. \tag{3.12}$$

Since $\hbar \in C\left(\mathbb{R}^2 \times \mathbb{R}^2; \mathbb{R}^2\right)$ there exists $K_0 > 0$ such that

$$\frac{|\hbar(w)|}{|w|} < K_0 \text{ for any } w \in E,\ \rho_0 \leq |w| \leq \varrho_0. \tag{3.13}$$

By (1.4) and (3.11) for any $w \in S$ such that $|w| < \rho_0$ we get

$$|\hbar(w)| = |(\gamma - \delta)w + h(w)| \leq (|\gamma - \delta| + \epsilon_0)|w|. \tag{3.14}$$

Thus, it follows from (3.12)-(3.14) that

$$|\hbar(w)| \leq M|w|,\ w \in S, \tag{3.15}$$

where

$$M = \max\{K_0, |\gamma - \delta| + \epsilon_0\}.$$

In view of (3.15), by (3.10) we get

$$\begin{aligned} |\varphi_{n,1}(x)| &\leq \tfrac{|\hbar_1(z_n(x))|\,|z_{n,1}(x)|}{z_{n,1}^2(x)+z_{n,2}^2(x)} \leq 2M, \quad |\psi_{n,1}(x)| \leq \tfrac{|\hbar_1(z_n(x))|\,|z_{n,2}(x)|}{z_{n,1}^2(x)+z_{n,2}^2(x)} \leq 2M, \\ |\varphi_{n,2}(x)| &\leq \tfrac{|\hbar_2(z_n(x))|\,|z_{n,1}(x)|}{z_{n,1}^2(x)+z_{n,2}^2(x)} \leq 2M, \quad |\psi_{n,2}(x)| \leq \tfrac{|\hbar_2(z_n(x))|\,|z_{n,2}(x)|}{z_{n,1}^2(x)+z_{,2}^2(x)} \leq 2M. \end{aligned} \tag{3.16}$$

By (3.10) it follows from (3.9) that (μ_n, z_n), $n \in \mathbb{N}$, solves the linear problem

$$\begin{cases} (\ell w)(x) - sP_n(x)w(x) = s(\lambda\gamma + \delta - \gamma)w(x),\ x \in (0, \pi), \\ w \in (b.c.), \end{cases} \tag{3.17}$$

where

$$P_n(x) = \begin{pmatrix} \varphi_{n,1}(x) & \psi_{n,1}(x) \\ \varphi_{n,2}(x) & \psi_{n,2}(x) \end{pmatrix}.$$

By [3, theorem 2.4] we have

$$\begin{aligned} &\theta'(z_n, x) = \\ &s(\mu_n\gamma + \delta - \gamma) + p(x)\cos^2\theta(z_n, x) + r(x)\sin^2\theta(z_n, x) + s\varphi_{n,1}(x)\cos^2\theta(z_n, x) + \\ &s\psi_{n,2}(x)\sin^2\theta(z_n, x) + \tfrac{1}{2}s\left\{\psi_{n,1}(x) + \varphi_{n,2}(x)\right\}\sin 2\theta(z_n, x). \end{aligned} \tag{3.18}$$

Integrating (3.18) from 0 to π and using (2.5), (2.6) we get

$$\begin{aligned} s(\mu_n\gamma + \delta - \gamma) = k\pi + \alpha - \beta - \int_0^\pi \left\{p\,(x)\cos^2\theta(z_n, x) + r(x)\sin^2\theta(z_n, x)\right\} dx - \\ \left\{\int_0^\pi \left\{\delta + \varphi_{n,1}(x)\cos^2\theta(z_n, x) + \psi_{n,2}(x)\sin^2\theta(z_n, x)\right\} dx + \right. \\ \left. \tfrac{1}{2}\int_0^\pi \left\{\psi_{n,1}(x) + \varphi_{n,2}(x)\right\}\sin 2\theta(z_n, x)dx\right\}. \end{aligned} \tag{3.19}$$

The left side of (3.19) for sufficiently large n takes on sufficiently large values in absolute value, while by p, $r \in C([0,\pi];\mathbb{R})$ and (3.16) the right side remains bounded, which gives a contradiction. Therefore, the projection of $\hat{C}_k^\nu$ onto $\mathbb{R}\times\{\tilde{0}\}$ is bounded, and consequently, $\hat{C}_k^\nu$ meets $(\tilde{\lambda}_k, 0)$. Similarly, we can show that the projection of $\tilde{C}_k^\nu$ onto $\mathbb{R}\times\{\tilde{0}\}$ is also bounded, which implies that $\tilde{C}_k^\nu$ meets $(\hat{\lambda}_k, \infty)$. Therefore, for $k \in \mathbb{Z}$ and each ν we have

$$\tilde{C}_k^\nu = \hat{C}_k^\nu. \tag{3.20}$$

From (3.2) it is clear that the solution to this problem (λ, w) with $\lambda = 1$ is also solution to problem (1.1)-(1.3). Since $\tilde{C}_k^\nu$ is connected, it follows that for the existence of a solution $w \in E$ of problem (1.1)-(1.3) contained in S_k^ν for some $k \in \mathbb{Z}$, by (3.20), it is sufficient that on the real axis $\mathbb{R}$ the point $\frac{\lambda_k}{s\gamma}$ lies to the left of 1 and the point $\frac{\lambda_k}{s\gamma} - \left(\frac{\delta}{\gamma} - 1\right)$ lies to the right of 1, or the point $\frac{\lambda_k}{s\gamma} - \left(\frac{\delta}{\gamma} - 1\right)$ lies to the right of 1, and the point $\frac{\lambda_k}{s\gamma}$ lies to the left of 1.

If $\lambda_k > 0$, $s > 0$ and $\frac{\lambda_k}{\gamma} < s < \frac{\lambda_k}{\delta}$, then we have

$$\frac{\lambda_k}{s\gamma} < 1 \text{ and } 1 < \frac{\lambda_k}{s\delta}.$$

By conditions $s > 0$, $\delta > 0$ and $\gamma > 0$ we get

$$1 < \tfrac{\lambda_k}{s\delta} \Longrightarrow s\delta < \lambda_k \Longrightarrow s\tfrac{\delta}{\gamma} < \tfrac{\lambda_k}{\gamma} \Longrightarrow s\left(1 + \tfrac{\delta}{\gamma} - 1\right) < \tfrac{\lambda_k}{\gamma} \Longrightarrow$$

$$s < \tfrac{\lambda_k}{\gamma} - s\left(\tfrac{\delta}{\gamma} - 1\right) \Longrightarrow 1 < \tfrac{\lambda_k}{s\gamma} - \left(\tfrac{\delta}{\gamma} - 1\right).$$

Therefore, we have the following relation

$$\frac{\lambda_k}{s\gamma} < 1 < \frac{\lambda_k}{s\gamma} - \left(\frac{\delta}{\gamma} - 1\right).$$

Other cases are considered similarly. The proof of this theorem is complete.

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