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# On basicity of eigenfunctions of one spectral problem with the discontinuity point in Morrey-Lebesgue spaces

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**Abstract.** In this paper is studied the spectral problem for a discontinuous second order differential operator with a spectral parameter in transmission conditions, that arises by solving the problem on vibrations of a loaded string with the free ends. An abstract theorem on the stability of the basis properties of multiple systems in a Banach space with respect to certain transformations is proved. This fact is used in the proof of theorems on the basicity of eigenfunctions of a discontinuous differential operator in Morrey type spaces.

**Key Words and Phrases**: spectral problem, eigenfunctions, basicity, Morrey spaces.

**2010 Mathematics Subject Classifications**: 34B24

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## 1. Introduction

We consider a model eigenvalue problem for the discontinuous second order differential operator

$$-y^{''}(x) = \lambda y(x), \ x \in \left(0, \frac{1}{3}\right) \bigcup \left(\frac{1}{3}, 1\right) \tag{1}$$

with boundary conditions

$$y'(0) = y'(1) = 0 \tag{2}$$

and with the following discontinuity conditions

$$\left.\begin{array}{l} y(\frac{1}{3}-0) = y(\frac{1}{3}+0), \\ y'(\frac{1}{3}-0) - y'(\frac{1}{3}+0) = \lambda m y(\frac{1}{3}), \end{array}\right\} \tag{3}$$

where $\lambda$ is the spectral parameter, $m$ is a non-zero complex number. Such spectral problems arise when the problem of vibrations of a loaded string with free ends is solved by applying the Fourier method [1-3]. The spectral problems with a discontinuity conditions inside the interval play an important role in mathematics, mechanics, physics and other fields of science. The applications of boundary value problems with discontinuity conditions inside the interval are connected with discontinuous material properties. Nowadays

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there is a number of papers dedicated to spectral problems for the ordinary differential operator with eigenparameter dependent boundary conditions. There are many papers concerning problems with discontinuity conditions. One can find the similar works in [4-7].

One of the commonly used methods for solving partial differential equations is the method of separation of variables. This method yields the appropriate spectral problem and in order to justify the method, it is very important the question of the expansion of functions of certain class on eigen and association functions of the discrete differential operators. The study of spectral properties of some discrete differential operators motivates the development of new methods for constructing basis. In this context, much attention has been given to the study of basis properties (completeness, minimality and basicity) of systems of special functions, which are frequently eigen and associated functions of differential operators. Additionally, various methods for examining these properties were proposed. Example, one of the works is [8]. Then various perturbation methods are applied. This direction has been well developed (see [9]). In the case of discontinuous differential operators, there appear systems of eigenfunctions whose basicity cannot be investigated by previously known methods. In the work [10,11] is considered an abstract approach to the problem described above and is proposed a new method for constructing bases, which has wide applications in the spectral theory of discontinuous differential operators.

In the paper [12] the problem of oscillation of a loaded string is investigated in the case when the load is placed in the middle of the string and it is shown that an abstract method proposed in [10, 11] can be used in non-standard spaces such as a Morrey type space. The concept of Morrey space was introduced by C. Morrey in 1938. Since then, various problems related to this space have been intensively studied. Playing an important role in the qualitative theory of elliptic differential equations (see, for example, [13,14]), this space also provides a large class of examples of mild solutions to the Navier-Stokes system [15]. In the context of fluid dynamics, Morrey spaces have been used to model fluid flow when vorticity is a singular measure supported on certain sets in $R^n$ [16]. More details about Morrey spaces can be found in [17,18]. Morrey type space is introduced in [17] (see e.g. [18]). In [19] the basicity of the exponential system, and in [20-23] - the perturbed exponential system in Morrey type spaces are proved. The present paper is continuity of [24] and [25].

## 2. Necessary information and preliminary results

For obtaining the main results we need some notions and facts from the theory of basis in a Banach space.

**Definition 2.1.** *Let $X$- be a Banach space. If there exists a sequence of indexes, such that $\{n_k\} \subset N$ , $n_k < n_{k+1}, n_0 = 0$, and any vector $x \in X$ is uniquely represented in the* *form*

$$x = \sum_{k=0}^{\infty} \sum_{i=n_k+1}^{n_{k+1}} c_i u_i.$$

*then the system* $\{u_n\}_{n\in N} \in X$ *is called a basis with parentheses in* $X$ .

For $n_k = k$ the system $\{u_n\}_{n\in N}$ forms a usual basis for $X$.

We need the following easily proved statements.

**Statement 2.1.** *Let the system* $\{u_n\}_{n\in N}$ *forms a basis with parentheses for a Banach space* $X$. *If the sequence* $\{n_{k+1} - n_k\}_{k\in N}$ *is bounded and the condition*

$$\sup_n \|u_n\| \, \|\vartheta_n\| < \infty,$$

*holds, where* $\{\vartheta_n\}_{n\in N}$*- is a biorthogonal system, then the system* $\{u_n\}_{n\in N}$ *forms a usual basis for* $X$.

**Definition 2.2.** *The basis* $\{u_n\}_{n\in N}$ *of Banach space* $X$ *is called a p-basis, if for any* $x\in X$ *the condition*

$$\left( \sum_{n=1}^{\infty} |\langle x, \vartheta_n \rangle|^p \right)^{\frac{1}{p}} \leq M \|x\|,$$

*holds, where* $\{\vartheta_n\}_{n\in N}$ *- is a biorthogonal system to* $\{u_n\}_{n\in N}$.

**Definition 2.3.** *The sequences* $\{u_n\}_{n\in N}$ *and* $\{\phi_n\}_{n\in N}$ *of Banach space* $X$ *are called a p- close, if the following condition holds:*

$$\sum_{n=1}^{\infty} \|u_n - \phi_n\|^p < \infty.$$

We will also use the following results from [8] (see also [24]).

**Theorem 2.1.** *Let* $\{x_n\}_{n\in N}$ *forms a q-basis for a Banach space* $X$, *and the system* $\{y_n\}_{n\in N}$ *is p-close to* $\{x_n\}_{n\in N}$, *where* $\frac{1}{p} + \frac{1}{q} = 1$. *Then the following properties are equivalent:*

*a)* $\{y_n\}_{n\in N}$*- is complete in* $X$;

*b)* $\{y_n\}_{n\in N}$*- is minimal in* $X$;

*c)* $\{y_n\}_{n\in N}$*- forms an isomorphic basis to* $\{x_n\}_{n\in N}$ *for* $X$.

Let $X_1 = X \oplus C^m$ and $\{\hat{u}_n\}_{n\in N} \subset X_1$ be some minimal system and $\left\{\hat{\vartheta}_n\right\}_{n\in N} \subset X_1^* = X^* \oplus C^m$ be its biorthogonal system:

$$\hat{u}_n = (u_n; \alpha_{n1}, ..., \alpha_{nm}); \hat{\vartheta}_n = (\vartheta_n; \beta_{n1}, ..., \beta_{nm}).$$

Let $J = \{n_1, ..., n_m\}$ some set of $m$ natural numbers. Suppose

$$\delta = det \|\beta_{n_i j}\|_{i,j=\overline{1,m}}.$$

In [27] (see also [28]) has been proved the following theorem :

**Theorem 2.2.** *Let the system $\{\hat{u}_n\}_{n\in N}$ forms a basis for $X_1$.In order to the system $\{u_n\}_{n\in N_J}$, where $N_J = N\backslash J$ forms a basis for $X$ it is necessary and sufficient that the condition $\delta \neq 0$ be satisfied. In this case the biorthogonal system to $\{u_n\}_{n\in N_J}$ is defined by*

$$\vartheta_n^* = \frac{1}{\delta} \begin{vmatrix} \vartheta_n & \vartheta_{n1} & \dots & \vartheta_{nm} \\ \beta_{n1} & \beta_{n_1 1} & \dots & \beta_{n_m 1} \\ \dots & \dots & \dots & \dots \\ \beta_{nm} & \beta_{n_1 m} & \dots & \beta_{n_m m} \end{vmatrix}.$$

*In particular, if $X$ is a Hilbert space and the system $\{u_n\}_{n\in N}$- forms a Riesz basis for $X_1$, then under the condition $\delta \neq 0$, the system $\{u_n\}_{n\in N_J}$ also forms a Riesz basis for $X$. For $\delta = 0$ the system $\{u_n\}_{n\in N_J}$ is not complete and is not minimal in $X$.*

Let $X$ be a Banach space and the system $\{u_{kn}\}_{k=\overline{1,m};n\in N}$ is any system in $X$. Let $a_{ik}^{(n)}, i,k = \overline{1,m}, n \in N$, any complex numbers. Let,

$$A_n = \left(a_{ik}^{(n)}\right)_{i,k=\overline{1,m}} \quad and \ \ \Delta_n = det A_n \ , n \in N.$$

Consider the following system in space $X$ :

$$\hat{u}_{kn} = \sum_{i=1}^{m} a_{ik}^{(n)} u_{in}, k = \overline{1,m}; n \in N. \tag{4}$$

Following theorems have been proved in [8] (also [11])

**Theorem 2.3.** *If the system $\{u_{kn}\}_{k=\overline{1,m};n\in N}$ forms basis for $X$ and*

$$\Delta_n \neq 0, \forall n \in N \tag{5}$$

*then the system $\{\hat{u}_{kn}\}_{k=\overline{1,m};n\in N}$ forms basis with parentheses for $X$. If in addition the following conditions*

$$\sup_n \left\{\|A_n\|, \left\|A_n^{-1}\right\|\right\} < \infty, \qquad \sup_n \left\{\|u_{kn}\|, \|\vartheta_{kn}\|\right\} < \infty, \tag{6}$$

*hold, where $\{\vartheta_{kn}\}_{k=\overline{1,m};n\in N} \subset X^*$- is biorthogonal to $\{u_{kn}\}_{k=\overline{1,m};n\in N}$, then the system $\{\hat{u}_{kn}\}_{k=\overline{1,m};n\in N}$ forms a usual basis for $X$.*

We will also need some facts about the theory of Morrey-type spaces. Let $\Gamma$ be some rectifiable Jordan curve on the complex plane $C$. By $|M|_\Gamma$ we denote the linear Lebesgue measure of the set $M \subset \Gamma$. By the Morrey-Lebesgue space $L^{p,\alpha}(\Gamma)$ , $0 \leq \alpha \leq 1$ , $p \geq 1$, we mean a normed space of all functions $f(\cdot)$ measurable on $\Gamma$ equipped with a finite norm $\|f\|_{L^{p,\alpha}(\Gamma)}$:

$$\|f\|_{L^{p,\alpha}(\Gamma)} = \sup_B \left( \left|B \bigcap \Gamma\right|_\Gamma^{\alpha-1} \int_{B\bigcap\Gamma} |f(\xi)|^p \, |d\xi| \right)^{\frac{1}{p}} < +\infty.$$

$L^{p,\alpha}(\Gamma)$ is a Banach space and $L^{p,1}(\Gamma) = L_p(\Gamma)$, $L^{p,0}(\Gamma) = L_\infty(\Gamma)$ . The embedding $L^{p,\alpha_1}(\Gamma) \subset L^{p,\alpha_2}(\Gamma)$ is valid for $0 \leq \alpha_1 \leq \alpha_2 \leq 1$ . Thus $L^{p,\alpha}(\Gamma) \subset L_p(\Gamma)$ , $\forall \alpha \in [0,1]$ , $\forall p \geq 1$ . The case of $\Gamma \equiv [a,b]$ will be denoted by $L^{p,\alpha}(a,b)$.

Denote by $\tilde{L}^{p,\alpha}(a,b)$ linear subspace of $L^{p,\alpha}(a,b)$ consisting of functions whose shifts are continuous in $L^{p,\alpha}(a,b)$, i.e. $\|f(\cdot+\delta)-f(\cdot)\|_{L^{p,\alpha}(a,b)}\to 0$ as $\delta\to 0$.The closure of $\tilde{L}^{p,\alpha}(a,b)$ in $L^{p,\alpha}(a,b)$ will be denoted by $M^{p,\alpha}(a,b)$ . In [19,21] the following theorem is proved.

**Theorem 2.4.** *The exponential system* $\left\{e^{int}\right\}_{n\in Z}$ *is the bases in* $M^{p,\alpha}(-\pi,\pi)$, $1<p<+\infty$, $0<\alpha\le 1$.

Using this theorem, it is easy to obtain the following

**Statement 2.2.** *The trigonometric systems* $\{\cos nx\}_{n=0}^{\infty}$ *forms a basis for* $M^{p,a}(0,p)$, $1<p<+\infty, 0<a\le 1$.

## 3. Main results

In [24] it was proved that the eigenvalues of the problem (1)-(3) are asymptotically simple and consist of $\lambda_0=0$ and two series: $\lambda_{i,n}=\rho_{i,n}^2, i=1,2; n\in Z^+$, where $Z^+=\{0\}\cup N$ and the numbers $\rho_{i,n}$ hold the following asymptotically formulas:

$$\begin{cases}\rho_{1,n}=3\pi n+\frac{3\pi}{2}+O\left(\frac{1}{n}\right)\\ \rho_{2,n}=\frac{3\pi n}{2}+\frac{3\pi}{4}+O\left(\frac{1}{n}\right).\end{cases}\tag{7}$$

Also the eigenfunctions $y_0(x)$ and $y_{i,n}(x)$ of the problem (1)-(3) corresponding to the eigenvalues $\lambda_{i,n}=(\rho_{1,n})^2, i=\overline{1,2}; n\in Z^+$ are the following form:

$$y_0(x)\equiv 1,\quad y_{i,n}(x)=\begin{cases}cos\frac{2\rho_{i,n}}{3}\ cos\rho_{i,n}\ x, x\in\left[0,\frac{1}{3}\right],\\ cos\frac{\rho_{i,n}}{3}\ cos\rho_{i,n}\ (1-x),\ \ x\in\left[\frac{1}{3},1\right],\end{cases}\quad i=1,2; n\in Z^+.\tag{8}$$

Now consider a problem on basicity of eigen and associated functions of the problem (1)-(3) in spaces $L_p(0,1)\oplus C$ and $L_p(0,1)$. Since the eigenvalues are asymptotically simple, the problem can only have a finite number of associated functions. In paper [24] we were constructed linearizing operator as following form. By $W_p^k\left(0,\frac{1}{3}\right)\oplus W_p^k\left(\frac{1}{3},1\right)$ we denoted a space functions whose contractions on segments $\left[0,\frac{1}{3}\right]$ and $\left[\frac{1}{3},1\right]$ belong correspondingly to Sobolev spaces $W_p^k\left(0,\frac{1}{3}\right)$ and $W_p^k\left(\frac{1}{3},1\right)$. Let us define the operator $L$ in $L_p(0,1)\oplus C$ as follows:

$$D(L)=\left\{\begin{array}{l}\hat{y}\in L_p(0,1)\oplus C:\hat{y}=\left(y,my\left(\frac{1}{3}\right)\right),y\in W_p^2\left(0,\frac{1}{3}\right)\oplus W_p^2\left(\frac{1}{3},1\right),\\ y'(0)=y'(1)=0,y\left(\frac{1}{3}-0\right)=y\left(\frac{1}{3}+0\right),\end{array}\right\}$$

and for $\hat{y}\in D(L)$

$$L\hat{y}=(-y'';y'\left(\frac{1}{3}-0\right)-y'\left(\frac{1}{3}+0\right),\hat{y}\in D(L)).$$

Operator defined by these formula is a linear closed operator with dense definitional domain in $L_p(0,1)\oplus C$. Eigenvalues of the operator $L$ and problem (1)-(3) coincide, and

$\{\hat{y}_0\} \cup \{\hat{y}_{i,n}\}_{i=1,2;\ n\in Z^+}$ are eigenvectors of the operator $L$ , where

$$\hat{y}_0 = (1; m), \quad \hat{y}_{i,n} = \left( y_{i,n}(x); my\left(\frac{1}{3}\right)\right),\ i = 1,2; n \in Z^+.$$

**Theorem 3.1.** *The system* $\{\hat{y}_0\} \cup \{\hat{y}_{i,n}\}_{i=1,2;\ n\in Z^+}$ *of eigen and associated vectors of the operator* $L$*, which linearized the problem* (1)- (3) *forms basis in* $M^{p,a}(0,1) \oplus C, 1 < p < \infty, 0 < a \le 1$.

**Proof.** Considering (7) in (8), we will get the following functional system for the head parts of the asymptotic formulas:

$$\begin{cases} u_{1,n}(x) = \begin{cases} -\cos\left(3\pi n + \frac{3\pi}{2}\right) x\ , & x \in \left[0, \frac{1}{3}\right], \\ 0, & x \in \left[\frac{1}{3}, 1\right], \end{cases} \\ u_{2,n}(x) = \begin{cases} 0, & x \in \left[0, \frac{1}{3}\right], \\ a_n \cos\left(\frac{3\pi n}{2} + \frac{3\pi}{4}\right)(1-x) & x \in \left[\frac{1}{3}, 1\right], \end{cases} \end{cases} \tag{9}$$

where $\alpha_n = \cos\left(\frac{\pi n}{2} + \frac{\pi}{4}\right)$ . If $n = 4k$ and $n = 4k+3$, $\alpha_n = \frac{1}{\sqrt{2}}$ and $n = 4k+1$ and $n = 4k+2$, $\alpha_n = -\frac{1}{\sqrt{2}}$. Then from the formulas (8) and (9) imply that, the following asymptotic relations are true:

$$\begin{cases} y_{1,n}(x) = u_{1,n}(x) + O\left(\frac{1}{n}\right) \\ y_{2,n}(x) = u_{2,n}(x) + O\left(\frac{1}{n}\right). \end{cases} \tag{10}$$

Since the operator $L$ has compact resolvent, the system $\{\hat{y}_0\} \cup \{\hat{y}_{i,n}\}_{i=1,2;\ n\in Z^+}$ of eigenfunctions and associated vectors is minimal in $L_p(0,1) \oplus C$. The conjugate system $\{\hat{v}_0\} \cup \{\hat{v}_{i,n}\}_{i=1,2; n\in Z^+}$ is the eigenvectors and associated vectors of the conjugating operator $L^*$ and is in the form $\hat{v}_0 = c_0(1; \overline{m})$, $\hat{v}_{i,n} = \left(v_{i,n}(x); \overline{m} v_{i,n}\left(\frac{1}{3}\right)\right)$ , where $v_0(x)$ and $v_{i,n}(x), i = 1,2; n \in Z^+$ are the eigen- and associated vectors of the conjugate problem and analogically we obtain the asymptotically formulas:

$$v_{i,n}(x) = \begin{cases} c_{i,n} cos\frac{2\rho_{i,n}}{3}\ cos\rho_{i,n}\ x, & x \in \left[0, \frac{1}{3}\right], \\ c_{i,n} cos\frac{\rho_{i,n}}{3}\ cos\rho_{i,n}\ (1-x), & x \in \left[\frac{1}{3}, 1\right], \end{cases} \quad i = 1,2; n \in Z^+, \tag{11}$$

here $c_0, c_{i,n}$ are the normalized multipliers. We can easily calculate that, the $c_{1n}, c_{2n}$ normalized multipliers hold

$$c_0 = \frac{1}{1+|m|^2}, \quad c_{1n} = 6 + O\left(\frac{1}{n}\right), c_{2n} = 6 + O\left(\frac{1}{n}\right).$$

If we consider these formulas at (11) we will obtain for $v_{i,n}(x)$, $i = 1,2;; n \in Z^+$ the following formulas:

$$v_{1,n} = \begin{cases} -6cos\left(3\pi n + \frac{3\pi}{2}\right) x\ + O\left(\frac{1}{n}\right), & x \in \left[0, \frac{1}{3}\right] \\ O\left(\frac{1}{n}\right), & x \in \left[\frac{1}{3}, 1\right] \end{cases} \tag{12}$$

$$v_{2.n} = \begin{cases} O\left(\frac{1}{n}\right), & x \in \left[0, \frac{1}{3}\right] \\ 6a_n \cos\left(\frac{3\pi n}{2} + \frac{3\pi}{4}\right)(1-x)\ + O\left(\frac{1}{n}\right), & x \in \left[\frac{1}{3}, 1\right]. \end{cases} \tag{13}$$

One can easily seen that the system (10) implies from the following system by the conversion

$$u_{i,n} = a_{i,1}e_{1,n} + a_{i,2}e_{2,n}, i = 1,2; \quad n \in Z^+$$

$$\left\{ \begin{array}{c} e_{1,n}(x) = \left\{ \begin{array}{cc} \cos\left(3\pi n + \frac{3\pi}{2}\right)x\ , & x \in \left[0, \frac{1}{3}\right], \\ 0, & x \in \left[\frac{1}{3}, 1\right], \end{array} \right. \\ e_{2,n}(x) = \left\{ \begin{array}{cc} 0, & x \in \left[0, \frac{1}{3}\right], \\ \cos\left(\frac{3\pi n}{2} + \frac{3\pi}{4}\right)(1-x)\ , & x \in \left[\frac{1}{3}, 1\right] \end{array} \right. \end{array} \right. \tag{14}$$

where the numbers $a_{i,1}$ and $a_{i,2}$, $i = 1,2$ are the elements of the following matrix

$$A = \begin{pmatrix} -1 & 0 \\ 0 & a_n \end{pmatrix}. \tag{15}$$

Note that,

$$detA \ = -a_n \neq 0.$$

On the other hand the system $\{e_{i,n}\}_{i=1,2,n\in Z^+}$ forms a basis for $M^{p,\alpha}(0,1)\,, 1 < p < \infty$. Indeed, according to the decomposition $M^{p,\alpha}(0,1) = M^{p,\alpha}\left(0, \frac{1}{3}\right) \oplus M^{p,\alpha}\left(\frac{1}{3}, 1\right)$, and since the system $\{e_{1,n}\}_{n\in Z^+}$ forms basis in $M^{p,\alpha}\ \left(0, \frac{1}{3}\right)$, and the system $\{e_{1,n}\}_{n\in Z^+}$ forms basis in $M^{p,\alpha}\left(\frac{1}{3}, 1\right)$, therefore, their combination will forms a basis in $M^{p,\alpha}(0,1)$. If we take it into consideration and apply Theorem 2.3, then we obtain that the system $\{u_{i,n}\}_{i=1,2;n\in Z^+}$ forms basis in $M^{p,a}(0,1)$. Consider the system in $\{\hat{u}_0\}\cup\{\hat{u}_{i,n}\}_{i=1,2;n\in Z^+}$ in $M^{p,\alpha}(0,1)\oplus C$, where

$$\hat{u}_0 = (0,1)\,, \quad \hat{u}_{i,n} = (u_{i,n};\ \ 0)\,, i = \overline{1,2}; \quad n \in Z^+. \tag{16}$$

It is clear that, the system $\{\hat{u}_0\} \cup \{\hat{u}_{i,n}\}_{i=1,2;\ n\in Z^+}$ forms basis in $M^{p,\alpha}(0,1) \oplus C$. Let us show that it also forms a $q$-basis, where $q = p/(p-1)$. One can easily check that the system $\{\hat{\upsilon}_0\} \cup \left\{\hat{\vartheta}_{i,n}\right\}_{i=1,2;n\in Z^+}$, which biorthogonal to it is in the following form:

$$\hat{\upsilon}_0 = \frac{1}{1+|m|^2}\,(1;\overline{m})\,, \hat{\vartheta}_{i,n} = (\vartheta_{i,n};0)\,, i = 1,2;\ n \in Z^+, \tag{17}$$

where

$$\vartheta_{1.n} = \left\{ \begin{array}{cc} -6\cos\left(3\pi n + \frac{3\pi}{2}\right)x\ , & x \in \left[0, \frac{1}{3}\right] \\ 0, & x \in \left[\frac{1}{3}, 1\right] \end{array} \right. \tag{18}$$

$$\vartheta_{2.n} = \left\{ \begin{array}{cc} 0, & x \in \left[0, \frac{1}{3}\right] \\ 6\alpha_n\cos\left(\frac{3\pi n}{2} + \frac{3\pi}{4}\right)(1-x)\ \ , & x \in \left[\frac{1}{3}, 1\right]. \end{array} \right. \tag{19}$$

Let $1 < p \leq 2$. Then according to inequality Hausdorf-Young for trigonometric system (see [28], p.153) for each $f \in L_p(0,1)$ the inequality

$$\left(\sum_{i=1}^{2}\sum_{n=0}^{\infty} |< f, e_{i,n} >|^q\right)^{\frac{1}{q}} \leq M\|f\|_{L_p}$$

is fulfilled, where $M > 0$ is a fixed number which does not depend on $f$. Taking into consideration that, the system $\{\vartheta_{i,n}\}_{i=1,2;n\in Z^+}$ implies from the system $\{e_{i,n}\}_{i=1,2,n\in Z^+}$ by conversion

$$\vartheta_{i,n} = b_{i,1}e_{1,n} + b_{i,2}e_{2,n}, i = 1,2;\ n \in Z^+$$

where $b_{i1}$ and $b_{i,2}$, $i = 1,2$ are the elements of the matrix

$$B = \begin{pmatrix} -6 & 0 \\ 0 & 6\alpha_n \end{pmatrix}.$$

We obtain from here that for an arbitrary $\hat{f} \in L_p(0,1) \oplus C$ the following inequality holds:

$$\left( \left| \left\langle \hat{f}, \hat{\vartheta}_0 \right\rangle \right|^q + \sum_{i=1}^{2} \sum_{n=0}^{\infty} \left| \left\langle \hat{f}, \hat{\vartheta}_{i,n} \right\rangle \right|^q \right)^{\frac{1}{q}} \leq M \left\| \hat{f} \right\|_{L_p \oplus C}.$$

Then, taking into account the embedding $M^{p,\alpha}(0,1) \subset L_p(0,1)$, we obtain that for any $\hat{f} \in M^{p,\alpha}(0,1) \oplus C$ the inequality

$$\left( \left| \left\langle \hat{f}, \hat{\vartheta}_0 \right\rangle \right|^q + \sum_{i=1}^{2} \sum_{n=0}^{\infty} \left| \left\langle \hat{f}, \hat{\vartheta}_{i,n} \right\rangle \right|^q \right)^{\frac{1}{q}} \leq M \left\| \hat{f} \right\|_{M^{p,\alpha} \oplus C},$$

holds, i.e. the system $\{\hat{u}_{i,n}\}_{i=\overline{1,2};n\in N}$ forms a $q$-basis for $M^{p,\alpha}(0,1) \oplus C$.

Let's point

$$\hat{y}_{i,n} = \left( y_{i,n}(x); m y_{i,n}\left(\frac{1}{3}\right) \right), i = 1,2; n \in Z^+,$$

According to the formulas (8) since $y_{i,n}\left(\frac{1}{3}\right) = O\left(\frac{1}{n}\right)$, from (10) implies that the systems $\{\hat{y}_0\} \cup \{\hat{y}_{i,n}\}_{i=1,2;\ n\in Z^+}$ and $\{\hat{u}_0\} \cup \{\hat{u}_{i,n}\}_{i=1,2;\ n\in Z^+}$ are $p$-close,

$$\sum_{i=1}^{2} \sum_{n=0}^{\infty} \|\hat{y}_{i,n} - \hat{u}_{i,n}\|^p < \infty.$$

Thus, all the conditions of Theorem 2.1 are fulfilled and according to this theorem the system $\{\hat{y}_0\} \cup \{\hat{y}_{i,n}\}_{i=1,2;\ n\in Z^+}$ also forms an isomorphic basis to the system $\{\hat{u}_0\} \cup \{\hat{u}_{i,n}\}_{i=1,2;\ n\in Z^+}$ in $L_p(0,1) \oplus C$, therefore, it is minimal in this space and in view of the embedding

$$(L_q(0,1) \oplus C) \subset (M^{p,\alpha}(0,1) \oplus C)^*,$$

we find that it is minimal in $M^{p,\alpha}(0,1) \oplus C$. Thus, all the conditions of Theorem 2.3 hold and by this theorem, the system $\{\hat{y}_0\} \cup \{\hat{y}_{i,n}\}_{i=1,2;\ n\in Z^+}$ forms an equivalent basis to the system $\{\hat{u}_0\} \cup \{\hat{u}_{i,n}\}_{i=1,2;\ n\in Z^+}$ for space $M^{p,\alpha}(0,1) \oplus C$ .

Now suppose that $p > 2$, then $1 < q < 2$. Taking into account that in this case the following inclusion is fulfilled:

$$L_p(0,1) \subset L_q(0,1), l_p \subset l_q,$$

then for $\hat{f} \in L_p(0,1) \oplus C$ we obtain:

$$\left( \left| \left\langle \hat{f}, \hat{\vartheta}_0 \right\rangle \right|^p + \sum_{i=1}^{2} \sum_{n=0}^{\infty} \left| \left\langle \hat{f}, \hat{\vartheta}_{i,n} \right\rangle \right|^p \right)^{\frac{1}{p}} \leq M \left\| \hat{f} \right\|_{L_p \oplus C} \leq M \left\| \hat{f} \right\|_{M^{p,a} \oplus C}.$$

This implies that the system $\{\hat{u}_0\} \cup \{\hat{u}_{i,n}\}_{i=1,2;\ n \in Z^+}$ forms a $p$- basis in $M^{p,\alpha}(0,1) \oplus C$. Besides, the systems $\{\hat{y}_0\} \cup \{\hat{y}_{i,n}\}_{i=1,2;\ n \in Z^+}$ and $\{\hat{u}_0\} \cup \{\hat{u}_{i,n}\}_{i=1,2;\ n \in Z^+}$ are $q$-close in $M^{p,\alpha}(0,1) \oplus C$ :

$$\sum_{i=1}^{2} \sum_{n=0}^{\infty} \|\hat{y}_{i,n} - \hat{u}_{i,n}\|^q_{M^{p,\alpha} \oplus C} < \infty.$$

According to the system $\{\hat{y}_0\} \cup \{\hat{y}_{i,n}\}_{i=1,2;\ n \in Z^+}$ is minimal in $M^{p,a}(0,1) \oplus C$ and again applying the Theorem 2.1, we obtain that it is a basis in $M^{p,a}(0,1) \oplus C$ isomorphic to $\{\hat{u}_0\} \cup \{\hat{u}_{i,n}\}_{i=1,2;\ n \in Z^+}$.

Now consider the basicity $\{y_0\} \cup \{y_{i,n}\}_{i=2;n \in Z^+}^{\infty}$ of the system of eigenfunctions and associated functions of the problem (1)-(3) in $L_p(0,1)$. Applying the Theorem 2.3 , we obtain the truth of the following theorem.

**Theorem 3.2.** *In order the system $\{y_{i,n}\}_{i=\overline{1,2};n \in Z^+, n \neq n_0}^{\infty}$ of eigenfunctions and associated functions of the problem* (1)-(3) *forms a basis in $M^{p,\alpha}(0,1), 1 < p < \infty$ after eliminate any function $y_{i,n_0}(x)$ it is necessary and sufficient that the corresponding function $v_{i,n_0}(x)$ of the biorthogonal system satisfy the condition $v_{i,n_0}\left(\frac{1}{3}\right) \neq 0$.If $v_{i,n_0}\left(\frac{1}{3}\right) = 0$, then after the eliminating function $y_{i,n_0}(x)$ from the system, obtaining system does not form basis in $M^{p,\alpha}(0,1)$, moreover in this case it is not complete and not minimal in this space.*

In (7) and (8) the parameter $m$ which included in the problem (1)- (3), generally speaking is a complex number. But in some particular cases it is possible to refine the root subspaces of the operator $L$. So, if $m > 0$, then the operator $L$ linearized of the problem (1)- (3) is a self-adjoint operator in $L_2 \oplus C$ , and in this case all the eigenvalues are simple and for each eigenvalue there corresponds only one eigenvector. If $m < 0$, then the operator $L$ is a $J$ -self-adjoint operator in $L_2 \oplus C$, and in this case applying the results of [29,30], we obtain that all eigenvalues are real and simple, with the exception of, may be either one pair of complex conjugate simple eigenvalues or one non-simple real value. In the case of a complex value m the operator $L$ has an infnite number of complex eigenvalues that are asymptotically simple and, consequently, the operator $L$ can have a finite number of associated vectors. If there are associated vectors, they are determined up to a linear combination with the corresponding eigenvector. Therefore depending on the choice of the coefficients of the linear combination there are associated vectors satisfying the condition $v_{i,n_0}\left(\frac{1}{3}\right) \neq 0$, and there are also associated vectors not satisfying this condition.

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