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On Strong Solvability of One Nonlocal Boundary Value Problem for Laplace Equation in Grand Sobolev Space in Rectangle

T.B. Gasymov, B.Q. Akhmadli*

Abstract. We consider a nonlocal boundary value problem for the Laplace equation in a rectangular domain in Sobolev spaces generated by the norm of the grand Lebesgue space. The concept of strong solvability of this problem is introduced and it's correct solvability is proved. At the same time, the basis property of the eigen and associated functions of one spectral problem in separable grand Lebesgue spaces is proved, and this fact is used to establish correct solvability. Note that earlier this problem in a semi-infinite strip in the classical formulation was considered in the works of E.I. Moiseev [24], M.E. Lerner and O.A. Repin [20].

Key Words and Phrases: Laplace equation, nonlocal problem, grand Lebesgue space, strong solvability.

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1. Introduction

The theory of the strong and weak solvability of linear elliptic equations in the Sobolev spaces is well developed and can be found in the classical monographs. In spite of this, a lot of problems, arising in the mechanics and the mathematical physics do not fit to this theory. An example, of such a problem is the following degenerate elliptic equation, studied by Moiseev in [24].

Consider the following (formal for now) nonlocal boundary value problem for the Laplace equation:

$$u_{xx} + u_{yy} = 0, \quad 0 < x < 2\pi, \quad 0 < y < h, \tag{1}$$

$$u(x,0) = \varphi(x), u(x,h) = \psi(x), \qquad 0 < x < 2\pi, \tag{2}$$

$$u_x(0,y) = 0, u(0,y) = u(2\pi,y), \ 0 < y < h. \tag{3}$$

Such problems have specific peculiarities compared to the ones with local conditions. Earlier, F.I.Frankl [13]; [14, p. 453-456] considered the problem with nonlocal boundary condition for a mixed type equation. Bitsadze-Samarski problem [10] for elliptic equations

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*Corresponding author.

is also nonlocal with supports on a part of the boundary of domain, and these supports are free of other boundary conditions. In [18], N.I.Ionkin and E.I.Moiseev solved the boundary value problem for multi-dimensional parabolic equations with nonlocal conditions, whose supports are the characteristic and the improper parts of the boundary of domain.

We note that recently, interest in nonstandard functions spaces has greatly increased in connection with their applications in mechanics, mathematical physics and pure mathematical problems. Such spaces include Lebesgue spaces with a variable summability exponent, Morrey spaces, grand Lebesgue spaces, Orlicz, Lorents, Martsinkevich, etc.. That's why the number of research works dedicated has been growing in recent years (see, e.g., [2, 3, 5, 6, 7, 8, 9, 11, 12, 21, 25]), and the elaboration of a corresponding theory is far from complete. This article is also devoted to this direction. Let us note, that this problem cannot be treated with the classical methods developed for linear elliptic operators. Our method is based on spectral theory, which is used, for example, in the works [19, 21, 24].

2. Auxiliary concepts and facts

We will use standard notations. N will be the set of positive integers, while $\alpha = (\alpha_1;\alpha_2) \in Z^+ \times Z^+$ will denote a multi-index, where $Z^+ = N \cup \{0\}$. Denote $\partial^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x^{\alpha_1} \partial y^{\alpha_2}}$, where $|\alpha| = \alpha_1 + \alpha_2$. By $|M|$ we will denote the Lebesgue measure of the set M; $\overline{M}$ will be the closure of M. $C^\infty\left(\overline{M}\right)$ will stand for the infinitely differentiable functions on $\overline{M}$, and $C_0^\infty(M)$ will denote the infinitely differentiable and finite functions on M. Throughout this paper we will assume that p' is a conjugate number of $p, 1 < p < +\infty$: $\frac{1}{p'} + \frac{1}{p} = 1$. $d\sigma$ is an area element. We also accept $p_\varepsilon = p - \varepsilon$.

Let us define our weighted grand Sobolev space. Let, $\Pi = (0, 2\pi) \times (0, h)$. Denote by $L_{p)}(\Pi)$ a Banach space of functions on Π with the mixed norm

$$\|f\|_{L_{p)}(\Pi)} = \sup_{0<\varepsilon<p-1} \int_0^h \left(\varepsilon \int_0^{2\pi} |f(x;y)|^{p-\varepsilon} dx\right)^{\frac{1}{p-\varepsilon}} dy, 1 < p < +\infty.$$

Denote by $W^2_{p)}(\Pi)$ a grand Sobolev space generated by the norm

$$\|u\|_{W^2_{p)}(\Pi)} = \sum_{|\alpha|\le 2} \|\partial^\alpha u\|_{L_{p)}(\Pi)}.$$

Now denote by $L_{p)}(I)$, where $I = (0, 2\pi)$, a grand Lebesgue space generated by the norm

$$\|f\|_{L_{p)}(I)} = \sup_{0<\varepsilon<p-1} \left(\varepsilon \int_I |f(x)|^{p-\varepsilon} dx\right)^{\frac{1}{p-\varepsilon}}.$$

We will also consider the grand Sobolev space $W^2_{p)}(I)$, generated by the norm

$$\|f\|_{W^2_{p)}(I)} = \|f\|_{L_{p)}(I)} + \|f'\|_{L_{p)}(I)} + \|f''\|_{L_{p)}(I)}.$$

These spaces are non-separable and therefore the method of biorthogonal expansion (essentially the spectral method) is not applicable for studying the solvability of differential equations with respect to these spaces. In this regard we select the subspace $N_{p)}(\Pi) \subset L_{p)}(\Pi)$ (separable) based on the shift operator T_δ:

$$(T_\delta u)(x;y) = \begin{cases} u(x+\delta : y) & (x+\delta : y) \in \Pi, \\ 0 & (x+\delta : y) \notin \Pi. \end{cases}$$

So let us assume

$$N^2_{p)}(\Pi) = \left\{ W^2_{p)}(\Pi) : \sum_{|\alpha|\leq 2} \|T_\delta(\partial^\alpha u) - \partial^\alpha u\|_{L_{p)}(\Pi)} \to 0, \delta \to 0 \right\}.$$

Definition 1. *A system $\{u_n\}_{n\in N} \subset X$ is called a basis if any element $f \in X$ is uniquely represented as a series*

$$f = \sum_{n=1}^{\infty} c_n u_n$$

convergent in the norm X.

Definition 2. *A system $\{u_n\}_{n\in N} \subset X$ is called complete in X if $\overline{Sp}\{u_n\} = X$ and minimal in X if $u_n \notin \overline{Sp}\{u_k\}_{k\neq n}$.*

It is known that each basis of the space X is a complete and minimal system in X, the converse is not true in general.

Minimum criterion. The system $\{u_N\}_{n\in N}$ is minimal in X if and only if there exists a biorthogonal system i.e. there exists a system $\{\vartheta_n\}_{n\in N} \subset X^*$ such that $\langle u_n, \vartheta_k\rangle = \vartheta_k(u_n) = \delta_{nk}$, where δ_{nk} is the Kronecker symbol.

Basis criterion. The system $\{u_n\}_{n\in N} \subset X$ is a basis of the space X if and only if the following conditions are satisfied.

1. $\{u_n\}_{n\in N}$ is complete and minimal in X;
2. *uniformly bounded projectors*

$$P_n f = \sum_{k=1}^{n} \langle f, \vartheta_k\rangle u_k,$$

where $\{v_k\}_{k\in N}$ is a biorthogonal system.

Definition 3. *A system $\{u_n\}_{n\in N} \subset X$ is called a basis with brackets in X if there exists a sequence of integers $0 = n_0 < n_1 < n_2 < \ldots$, such that each element of $f \in X$ is uniquely represented as a series*

$$f = \sum_{k=0}^{\infty} \sum_{i=n_k+1}^{n_{k+1}} c_i u_i,$$

convergent in the norm X.

To obtain our main results we will extensively use also the following Minkowski's inequality for integrals.

Proposition 1 (Minkowski's inequality). *Let $(M_k;\sigma_{M_k};\mu_k)$, $k=\overline{1,2}$, be measurable spaces with σ-finite measures μ_k and $F(x;y)$ be a $\mu_1\times\mu_2$-measurable function. Then*

$$\left\|\int_{M_1} F(x;y)\,d\mu_1(x)\right\|_{L_{p)}(\mu_2)} \le \int_{M_1}\|F(x;\cdot)\|_{L_{p)}(\mu_2)}d\mu_1(x),$$

where

$$\|f\|_{L_{p)}(\mu)} = \sup_{0<\varepsilon<p-1}\left(\varepsilon\int|f|^{p-\varepsilon}d\mu\right)^{\frac{1}{p-\varepsilon}}.$$

So let's introduce the following

Definition 4. *A function $u\in N^2_{p)}(\Pi)$ is called a strong solution of the problem* (1)-(3) *if the equality* (1) *is satisfied for a.e. $(x;y)\in\Pi$ and its trace $u|_{\partial\Pi}$ satisfies the relations* (2), (3).

Introduce the systems of functions $\{u_n(x)\}_{n\in Z^+}$ and $\{\vartheta_n(x)\}_{n\in Z^+}$, where

$$u_{2n}(x)=cos\,nx\ ,n\in Z^+,\qquad u_{2n-1}(x)=xsin\,nx,n\in N, \tag{4}$$

$$\vartheta_0(x)=\frac{1}{2\pi^2}(2\pi-x)\,,\vartheta_{2n}(x)=\frac{1}{\pi^2}(2\pi-x)\,cos\,nx\ ,\vartheta_{2n-1}(x)=\frac{1}{\pi^2}sin\,nx\ ,n\in N. \tag{5}$$

Note that these systems are biorthogonal conjugate, which can be verified directly. To obtain our main result, we will significantly use the following theorem.

Theorem 1. *The system* (4) *forms a basis for $N_{p)}(I)$.*

Proof. Conjugate space of $L_{p)}(I)$ is $L_{p')}(I)$. It is absolutely clear that the system (4) belongs to $L_{p')}(I)$ and is biorthonormalized to the system (4) (see [1]). It follows that (4) is minimal in $L_{p)}(I)$. On the other hand, from [1] it follows that the system (4) forms a basis with brackets for $N_{p)}(I)$ for every $p\in(1,+\infty)$, and, consequently, it is complete in $L_{p_1)}(I)$. Then from the embedding $N_{p_1)}(I)\subset L_{p)}(I)$ it follows that (4) is complete and, consequently, complete and minimal in $L_{p)}(I)$.

Let's prove the basicity of the system (4) for $N_{p)}(I)$. Consider the projectors

$$P_n(f)=\sum_{k=0}^{n}\langle f,\vartheta_k\rangle u_k,\forall n\in Z^+,\forall f\in N_{p)}(I)$$

where

$$\langle f,g\rangle=\int_0^{2\pi}f(x)\,\overline{g(x)}dx.$$

From the basicity with brackets of the system (4) for $N_{p)}(I)$ it follows that

$$\exists C>0:\|P_{2n}(f)\|_{L_{p)}(I)}\le C\|f\|_{L_{p)}(I)},\forall n\in N. \tag{6}$$

On the other hand, from (4), (6) we have

$$\exists M>0:\|u_n\|_{L_{p)}(I)}\le M,\qquad \|\vartheta_n\|_{L_{p')}}\le M,\qquad \forall n\in N. \tag{7}$$

Considering the relations (6), (7), we obtain

$$\|P_{2n+1}(f)\|_{L_{p)}(I)}=\|P_{2n}(f)\left\langle f,\vartheta_{2n+1}\right\rangle u_{2n+1}\|_{L_{p)}(I)}\le\|P_{2n}\left(f\right)\|_{L_{p)}(I)}+$$

$$+\|\left\langle f,\vartheta_{2n+1}\right\rangle u_{2n+1}\|_{L_{p)}(I)}\le C\|f\|_{L_{p)}(I)}+\|f\|_{L_{p)}(I)}\|u_n\|_{L_{p}(I)}\|\vartheta_n\|_{L_{p')}(I)}\le$$

$$\le\left(C+M^2\right)\|f\|_{L_{p)}(I)}. \tag{8}$$

From (6), (8) it follows that the projectors $\{P_n\}_{n\in Z^+}$ are uniformly bounded, and, according to the criterion for basicity, this means that the system (4) forms a basis for $N_{p)}\left(I\right)$. The theorem is proved.

3. Main Results

In this section, we will study the existence and uniqueness of strong solution of the problem (1)-(3) in the sense of Definition 4. First, denote $\Gamma_0=\{(0;y):0<y<h\}$ and $\Gamma_{2\pi}=\{(2\pi;y)\,,0<y<h\}$. Consider the following nonlocal problem:

$$\Delta u=0,\ (x;y)\in\Pi, \tag{9}$$

$$u|_{I_0}=\varphi,\ u|_{I_h}=\psi,\ u|_{\Gamma_0}=u|_{\Gamma_{2\pi}},\ u_x|_{\Gamma_0}=0. \tag{10}$$

By the solution of this problem, we mean a function $u\in N^2_{p)}\left(\Pi\right)$, which satisfies the equality (9) a.e. in Π and whose traces satisfy the relations (10) on the boundary $\partial\Pi=I_0\cup I_h\cup\Gamma_0\cup\Gamma_{2\pi}$. Let's first prove the uniqueness of the solution. The following theorem is true:

Theorem 2. *The functions $\varphi,\psi\in N^2_{p)}\left(I\right)$ satisfy the conditions $\varphi\left(2\pi\right)-\varphi\left(0\right)=\varphi'\left(0\right)=0,\psi\left(2\pi\right)-\psi\left(0\right)=\psi'\left(0\right)=0$. If the problem (1)-(3) has a solution in $N^2_{p)}\left(\Pi\right)$, then it is unique.*

Proof. Suppose $u\left(x,y\right)\in N^2_{p)}\left(\Pi\right)$ is a solution of the problem (1)-(3). Consider $u_n\left(y\right)=\left\langle u\left(\cdot,y\right),\vartheta_n\left(\cdot\right)\right\rangle$, i.e.

$$\left.\begin{array}{c} U_0\left(y\right)=\frac{1}{2\pi^2}\int_0^{2\pi}u\left(x,y\right)(2\pi-x)dx \\ U_{2n}\left(y\right)=\frac{1}{\pi^2}\int_0^{2\pi}u\left(x,y\right)(2\pi-x)cos\,nx\ dx\quad, \\ U_{2n-1}\left(y\right)=\frac{1}{\pi^2}\int_0^{2\pi}u\left(x,y\right)sin\,nx\ \ dx,n\in N. \end{array}\right\} \tag{11}$$

we obtain the following relations for $U_{2n-1}\left(y\right)$ (respectively, for $U_{2n}\left(y\right)$):

$$U''_{2n-1}\left(y\right)-n^2U_{2n-1}\left(y\right)=0,y\in\left(0.h\right), \tag{12}$$

$$U''_{2n}(y) - n^2 U_{2n}(y) = -2nU_{2n-1}(y), y \in (0.h). \tag{13}$$

By Newton-Leibniz formula we have

$$u(x,\xi) = u(x,0) + \int_0^\xi \frac{\partial u(x,y)}{\partial y} dy = \varphi(x) + \int_0^\xi \frac{\partial u(x,y)}{\partial \xi} d\xi, \quad \text{a.e. } x \in I.$$

Consequently,

$$|u(x,\xi) - \varphi(x)| \le \int_0^\xi \left| \frac{\partial u(x,y)}{\partial y} \right| dy, \quad \text{a.e.} \quad x \in I.$$

Hence it immediately follows that

$$\int_I |u(x,\xi) - \varphi(x)| dx \le \int_I \int_0^\xi \left| \frac{\partial u(x,y)}{\partial y} \right| dy dx. \tag{14}$$

We have $|\Pi_\xi| \to 0$ as $\xi \to +0$. Then from (14) it follows that

$$u_\xi(\cdot) \to \varphi(\cdot), \xi \to +0, \tag{15}$$

in the norm of the space $L_1(I)$.
Similarly we have

$$u(x,\xi) = u(x,h) - \int_\xi^h \frac{\partial u(x,y)}{\partial y} dy = \psi(x) - \int_\xi^h \frac{\partial u(x,y)}{\partial y} dy, \quad \text{a.e. } x \in I.$$

Hence,

$$\int_I |u(x,\xi) - \psi(x)| dx \le \int_I \int_\xi^h \left| \frac{\partial u(x,y)}{\partial y} \right| dy dx. \tag{16}$$

As $|\Pi \backslash \Pi_\xi| \to 0$ when $\xi \to h - 0$, from (16) it follows that

$$u_\xi(\cdot) \to \psi(\cdot), \quad \xi \to h - 0, \tag{17}$$

in the norm of the space $L_1(I)$.

On the other hand, it is clear that $U_n(y) \in W_1^2(0,h)$. Hence it immediately follows that there exist the limits

$$\lim_{y \to +0} U_n(y) = U_n(0), \lim_{y \to h-0} U_n(y) = U_n(h), \forall n \in Z^+.$$

By (15) and (17), from the last two relations it immediately follows that

$$U_n(0) = \varphi_n, \qquad U_n(h) = \psi_n, \qquad \forall n \in Z^+, \tag{18}$$

where

$$\begin{cases} \varphi_0 = \frac{1}{2\pi^2}\int_0^{2\pi}\varphi(x)(2\pi - x)\,dx, \\ \varphi_{2n-1} = \frac{1}{\pi^2}\int_0^{2\pi}\varphi(x)\,sin\,nx\;dx, \\ \varphi_{2n} = \frac{1}{\pi^2}\int_0^{2\pi}\varphi(x)(2\pi - x)\,cos\,nx\;dx, \qquad n \in N; \end{cases} \tag{19}$$

$$\begin{cases} \psi_0 = \frac{1}{2\pi^2}\int_0^{2\pi}\psi(x)(2\pi - x)\,dx, \\ \psi_{2n-1} = \frac{1}{\pi^2}\int_0^{2\pi}\psi(x)\,sin\,nx\;dx, \\ \psi_{2n} = \frac{1}{\pi^2}\int_0^{2\pi}\psi(x)(2\pi - x)\,cos\,nx\;dx, \qquad n \in N. \end{cases} \tag{20}$$

The solution of the problem (12), (18) is

$$U_{2n-1}(y) = \psi_{2n-1}\frac{sinh\,ny}{sinh\,nh} + \varphi_{2n-1}\frac{sinh\,n(h-y)}{sinh\,nh}, \forall n \in N, \tag{21}$$

and the solution of the problem (13), (18) is

$$U_0(y) = \frac{\psi_0 - \varphi_0}{h}y + \varphi_0, \tag{22}$$

$$U_{2n}(y) = \frac{\psi_{2n}sin\,nh + h\psi_{2n-1}cos\,nh - h\varphi_{2n-1}}{sinh\,nh}\frac{sinh\,ny}{sinh\,nh} +$$
$$+\varphi_{2n}\frac{sinh\,n(h-y)}{sinh\,nh} - y(\psi_{2n-1}\frac{cosh\,ny}{sinh\,nh} - \varphi_{2n-1}\frac{cosh\,n(h-y)}{sin\,nh}, \forall n \in N. \tag{23}$$

Now we can proceed to the proof of the uniqueness of solution. For this, it suffices to prove that the corresponding homogeneous problem has only trivial solution. In fact, if $\varphi(x) = \psi(x) \equiv 0$, then $\varphi_n = \psi_n = 0, \forall n \in Z^+$, and from the formulas (21)-(23) it follows that $U_n(y) = 0$, $\forall y \in (0,h), \forall n \in Z^+$. As $u_y \in N_{p)}(I)$, $\forall y \in (0,h)$, the basicity of the system (4) for $N_{p)}(I)$ implies $u_y(x) = 0$ a.e. $x \in I$ and $\forall y \in (0,h)$. Hence it follows that $u(x;y) = 0$ a.e. $(x;y) \in \Pi$. Consequently, the homogeneous problem has only trivial solution, and this completes the proof of uniqueness.

Now let's prove the existence of solution. The following theorem is true:

Theorem 3. *Let the boundary functions $\varphi(x)$ and $\psi(x)$ belong to the space $N^2_{p)}(I)$ and satisfy the conditions*

$$\varphi(0) - \varphi(2\pi) = \varphi'(0) = 0, \psi(0) - \psi(2\pi) = \psi'(0) = 0.$$

Then the problem (1)-(3) *has a (unique) solution in* $N^2_{p)}(\Pi)$.

Proof. Consider the function

$$u(x,y) = U_0(y) + \sum_{n=1}^{\infty} U_n(y)\,u_n(x) = U_0(y) +$$

$$+\sum_{k=1}^{\infty}\left(U_{2k}\left(y\right)cos\,kx+U_{2k-1}\left(y\right)xsin\,kx\right),(x,y)\in\Pi, \qquad (24)$$

where the coefficients $U_0\left(y\right),U_{2k}\left(\cdot\right),U_{2k-1}\left(\cdot\right),k\in N$, are defined by (21)-(23). Let's show that the function $u\left(x,y\right)$ belongs to $N^2_{p)}\left(\Pi\right)$. Denote by $u_{\alpha_1,\alpha_2}\left(x,y\right)$ the sum of the series obtained by the formal differentiation of the series (24), i.e.

$$u_{\alpha_1,\alpha_2}\left(x,y\right)=U_0^{(\alpha_2)}\left(y\right)+\sum_{\rm n=1}^{\infty}U_n^{(\alpha_2)}\left(y\right)u_n^{(\alpha_1)}\left(x\right), \qquad (25)$$

where $\alpha_1,\alpha_2\in Z^+,\alpha_1+\alpha_2=0,1,2;$ $u_{0,0}\left(x,y\right)=u\left(x,y\right)$ and $U_n^{(\alpha_2)}\left(y\right)=\frac{d^{\alpha_2}U_n}{dy^{\alpha_2}}$; $U_n^{(\alpha_1)}\left(x\right)=\frac{d^{\alpha_1}U_n}{dx^{\alpha_1}}$.

Let us first consider the following member of series (24).

$$u_1\left(x,y\right)=\sum_{k=1}^{\infty}U_{2k-1}\left(y\right)\ xsin\,kx\ .$$

So, differentiating this series formally term-by-term, we have

$$\frac{\partial^2u_1}{\partial y^2}=\sum_{k=1}^{\infty}U''_{2k-1}\left(y\right)xsin\,kx=\sum_{k=1}^{\infty}k^2U_{2k-1}\left(y\right)xsin\,kx, \qquad (26)$$

$$\frac{\partial u_1}{\partial x}=\sum_{k=1}^{\infty}U_{2k-1}\left(y\right)sin\,kx\ +\sum_{k=1}^{\infty}kU_{2k-1}\left(y\right)xcos\,kx, \qquad (27)$$

$$\frac{\partial^2u_1}{\partial x^2}=2\sum_{k=1}^{\infty}kU_{2k-1}\left(y\right)cos\,kx-\sum_{k=1}^{\infty}k^2U_{2k-1}\left(y\right)xsin\,kx. \qquad (28)$$

Denote

$$w\left(x,y\right)=\sum_{k=1}^{\infty}k^2U_{2k-1}\left(y\right)x\,sin\,kx.$$

Let's show that the function $w\left(x,y\right)$ belongs to $N_{p)}\left(\Pi\right)$. Let

$$\varphi''_{2k-1}=\frac{1}{\pi^2}\int_0^{2\pi}\varphi''\left(x\right)sin\,kxdx,\ \ \psi''_{2k-1}=\frac{1}{\pi^2}\int_0^{2\pi}\psi''\left(x\right)sin\,kxdx.$$

From (19), integrating by parts, we obtain

$$\varphi_{2k-1}=-\frac{1}{\pi^2k}\int_0^{2\pi}\varphi\left(x\right)dcos\,kx\ =-\frac{1}{\pi^2k}\left(\varphi\left(2\pi\right)-\varphi\left(0\right)-\int_0^{2\pi}\varphi'\left(x\right)cos\,kx\ dx\right)=$$

$$=\frac{1}{\pi^2k}\int_0^{2\pi}\varphi'\left(x\right)cos\,kx\ dx\ =\frac{1}{\pi^2k^2}\int_0^{2\pi}\varphi''\left(x\right)sin\,kx\ dx\ =\frac{1}{k^2}\varphi''_{2k-1}.$$

Similarly, from (20) we have

$$\psi_{2k-1} = -\frac{1}{k^2}\psi''_{2k-1}.$$

Thus,

$$w(x,y) = \sum_{k=1}^{\infty}\left(\psi''_{2k-1}\frac{sinh\,ky}{sinh\,kh} + \varphi''_{2k-1}\frac{sinh\,k(h-y)}{sinh\,kh}\right)x sin\,kx\ .$$

Let $\varepsilon \in (0, p-1)$ be an arbitrary number.We have the following continuous embeddings $L_p(I) \subset L_{p-\varepsilon}(I) \subset L_1(I)$. Let us suppose that $\beta = \frac{p}{p-\varepsilon} \Longrightarrow \frac{1}{\beta'} = 1 - \frac{p-\varepsilon}{p} = \frac{\varepsilon}{p}$. Applying the Holder's inequality, we have

$$\int_0^{2\pi}|f|^{p-\varepsilon}dx = \int_0^{2\pi}|f|^{p-\varepsilon}dx \le \left(\int_0^{2\pi}|f|^p dx\right)^{\frac{1}{\beta}}\left(\int_0^{2\pi}dx\right)^{\frac{1}{\beta'}} \Longrightarrow$$

$$\Longrightarrow \left(\varepsilon\int_0^{2\pi}|f|^{p-\varepsilon}dx\right)^{\frac{1}{p-\varepsilon}} \le \left(\int_0^{2\pi}|f|^p dx\right)^{\frac{1}{p}}\left(\int_0^{2\pi}dx\right)^{\frac{\varepsilon}{p-\varepsilon}\frac{1}{p}}\varepsilon^{\frac{1}{p-\varepsilon}} \le c\left(\int_0^{2\pi}|f|^p dx\right)^{\frac{1}{p}},$$

where $c > 0$ is a constant which is independent of f and ε. This immediately follows

$$\|f\|_{L_{p)}(I)} \le c\|f\|_{L_p(I)}, \forall f \in L_{p)}(I).$$

Let $\exists\ \alpha > 1$ and $\frac{1}{\alpha} + \frac{1}{\alpha'} = 1$.
Applying Holder's inequality again, we obtain

$$\left(\int_0^{2\pi}|w(x,y)|^p dx\right)^{\frac{1}{p}} \le \left(\int_0^{2\pi}1dx\right)^{\frac{1}{\alpha}}\left(\int_0^{2\pi}|w(x,y)|^{p\alpha'}dx\right)^{\frac{1}{p\alpha'}} = c\left(\int_0^{2\pi}|w(x,y)|^{p_1}dx\right)^{\frac{1}{p_1}},$$

where $c = (2\pi)^{\frac{1}{\alpha}}$ (consequently, does not depend on $w(x,y)$) and $p_1 = p\alpha'$. Let's consider the cases $p \ge 2$ and $1 < p < 2$. Consider the following separate cases regarding p.

I. $p \ge 2$. From the previous inequality we have

$$\|w(.;y)\|_{L_{p)}} \le c\left(\int_0^{2\pi}|w(x,y)|^p dx\right)^{\frac{1}{p}} \le c\sum_{k=1}^{\infty}|U_{2k-1}(y)| \le c_1\sum_{k=1}^{\infty}|U_{2k-1}(y)| \le$$

$$\le c_1\sum_{k=1}^{\infty}\left|\psi''_{2k-1}\frac{sinh\,ky}{sinh\,kh} + \varphi''_{2k-1}\frac{sinh\,k(h-y)}{sinh\,kh}\right| \le$$

$$\le c_1\sum_{k=1}^{\infty}\left(\left|\psi''_{2k-1}\right|\frac{sinh\,ky}{sinh\,kh} + \left|\varphi''_{2k-1}\right|\frac{sin\,k(h-y)}{sinh\,kh}\right).$$

Hence, first integrating with respect to $y \in (0,h)$ and then applying Holder's inequality for any $\beta \in (1,\infty)$, we obtain

$$\|w\|_{L_{p)}(\Pi)} \le c_1\sum_{k=1}^{\infty}\left(\frac{\left|\psi''_{2k-1}\right|}{sinh\,kh}\int_0^h sinh\,kydy + \frac{\left|\varphi''_{2k-1}\right|}{sinh\,kh}\int_0^h sinh\,k(h-y)\,dy\right) \le$$

$$\leq c_1 \sum_{k=1}^{\infty} \frac{\left|\psi''_{2k-1}\right| + \left|\varphi''_{2k-1}\right|}{sin\, kh} \int_0^h sinh\, kydy \leq$$

$$\leq c_1 \sum_{k=1}^{\infty} \frac{cosh\, kh - 1}{ksinh\, kh} \left(\left|\psi''_{2k-1}\right| + \left|\varphi''_{2k-1}\right|\right) \leq$$

$$\leq c_2 \sum_{k=1}^{\infty} \frac{1}{k} \left(\left|\psi''_{2k-1}\right| + \left|\varphi''_{2k-1}\right|\right) \leq c_2 \left(\sum_{k=1}^{\infty} \frac{1}{k^{\beta'}}\right)^{\frac{1}{\beta'}} \left(\left(\sum_{n=1}^{\infty} \left|\varphi''_n\right|^{\beta}\right)^{\frac{1}{\beta}} + \left(\sum_{n=1}^{\infty} \left|\psi''_n\right|^{\beta}\right)^{\frac{1}{\beta}}\right).$$

Now, assuming $\beta \geq 2$ and applying classical Hausdorff-Young inequality (see, e.g. [27, p. 154]. We have

$$\|w\|_{L_{p)}(\Pi)} \leq c_3 \left(\left\|\psi''\right\|_{L_{\beta'}(I)} + \left\|\varphi''\right\|_{L_{\beta'}(I)}\right). \tag{29}$$

Let us suppose that $r = \frac{p_\varepsilon}{q}$ and $g \in L_{p)}(I)$. Then $1 < r < p_\varepsilon$ and we have

$$\left(\int_I |g|^r dx\right)^{\frac{1}{r}} = \left(\int_I |g|^{\frac{p_\varepsilon}{q}} dx\right)^{\frac{1}{r}} \leq \left(\int_I |g|^{p_\varepsilon} dx\right)^{\frac{1}{qr}} \left(\left(\int_I dx\right)\right)^{\frac{1}{q'r}} =$$

$$= c \left(\int_I |g|^{p_\varepsilon} dx\right)^{\frac{1}{p_\varepsilon}}.$$

Then, the last inequality means $g \in L_r(I)$ and

$$\|g\|_{L_r(I)} \leq c\|g\|_{L_{p_\varepsilon}(I)}, \tag{30}$$

where $c > 0$ is a constant independent of g. Also note that the continuous embedding $L_{p)}(I) \subset L_\alpha(I)$ is true for every $\alpha \in (1, r)$. Let us choose β big enough to satisfy the condition

$$1 < \beta' < r \Rightarrow \|g\|_{L_{\beta'}(I)} \leq c\|g\|_{L_r(I)}$$

is satisfied. Then from inequalities (29),(30) we obtain

$$\|w\|_{L_{p)}(\Pi)} \leq c \left(\left\|\psi''\right\|_{L_{p)}(I)} + \left\|\varphi''\right\|_{L_{p)}(I)}\right).$$

II. $p \in (1, 2)$. Therefore, choosing $\alpha > 1$ close enough to 1, we can provide that $p_1 = p\alpha' > 2$ (this is possible, because $\alpha' \to +\infty$ as $\alpha \to 1 + 0$). With this, further considerations are carried out similar to the previous case.

Other series from (26)-(28), and, consequently, all series from (25) are estimated in a similar way. So, as a result, we obtain

$$\|u\|_{W^2_{p)}(\Pi)} \leq c \left(\|\varphi\|_{W^2_{p)}(I)} + \|\psi\|_{W^2_{p)}(I)}\right),$$

where $c > 0$ is a constant independent of φ and ψ. The fulfillment of equation (9) by $u(\cdot;\cdot)$ can be verified directly. Let's verify the fulfillment of boundary conditions. Denote the trace operators on $\Gamma_0, \Gamma_{2\pi}$, I_0 and I_h by $\theta_0, \theta_{2\pi}$, T_0 and T_h, respectively. Let's show that $T_0 u = \varphi$. It is clear that, $T_0 u \in L_1(I)$ *and* $\varphi \in L_1(I)$. From the boundedness of the operator $T_0 \in \left[W_p^2(\Pi); L_p(I)\right]$, $\forall\ p \geq 1$, it follows that if $u_m \to u$ in $W_{p)}^2(\Pi)$, then $u_m/_I \to u/_I$ in $L_{p)}(I)$.

Now, let's consider the following functions:

$$u_m(x,y) = U_0(y) + \sum_{n=1}^{m} \left(U_{2n}(y)\, cos\, nx + U_{2n-1}(y)\, x sin\, nx\right), (x;y) \in \Pi, m \in N.$$

We have

$$T_0 u_m = u_m(x,0) = U_0(0) + \sum_{n=1}^{m} \left(U_{2n}(0)\, cos\, nx + U_{2n-1}(0)\, x sin\, nx\right) =$$

$$= \frac{1}{2\pi^2} \int_0^{2\pi} \varphi(\tau)(2\pi - \tau)\, d\tau +$$

$$+ \sum_{n=1}^{m} \left(\frac{1}{\pi^2} \int_0^{2\pi} \varphi(\tau)(2\pi - \tau)\, cos\, n\tau d\tau cos\, nx + \frac{1}{\pi^2} \int_0^{2\pi} \varphi(\tau)\, sin\, n\tau d\tau sin\, nx \right). \quad (31)$$

It is clear that, $T_0 u_m \to T_0 u$. On the other hand, the basicity of the system (4) for $N_{p)}(I)$ implies $T_0 u_m \to \varphi, m \to \infty$, in $L_{p)}(I)$. Consequently, $T_0 u = \varphi$, a.e. on I.

Absolutely similar we can show that $T_h u_m \to \psi, m \to \infty$, in $L_{p)}(I)$. Consequently, $T_h u = \psi$, a.e. on I.

Consider the operators θ_0 and $\theta_{2\pi}$. It is clear that $\theta_0 u_m = \theta_{2\pi} u_m$, $\forall m \in N$. Obviously, $\theta_0 u_m \to \theta_0 u$ and $\theta_{2\pi} u_m = \theta_{2\pi} u \Rightarrow \theta_0 u = \theta_{2\pi} u$. Thus, the boundary conditions (10) are fulfilled.

The theorem is proved.

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Telman B. Gasymov
Ministry of Science and Education of Azerbaijan, Institute of Mathematics and Mechanics, Baku, Azerbaijan
Baku State University, Baku, Azerbaijan
E-mail: telmankasumov@rambler.ru

Beherchin Q. Akhmadli
Institute of Mathematics and Mechanics, The Ministry of Science and Education
E-mail: a_beherchin@mail.ru

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