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Approximation of the Hilbert transform in Orlicz spaces

L.Sh. Alizade

Abstract. The Hilbert transform is the main part of the singular integral equations on the real line. Therefore, approximations of the Hilbert transform are of great interest. This article is devoted to the approximation of the Hilbert transform in Orlicz spaces by operators which introduced by V.R.Kress and E.Mortensen to approximate the Hilbert transform of analytic functions in a strip.

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1. Introduction

The Hilbert transform of a function $u \in L_p(\mathbb{R})$, $1 \le p < \infty$ is defined as the Cauchy principle value integral [23]

$$(Hu)(t) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{u(\tau)}{t - \tau} d\tau, \quad t \in \mathbb{R},$$

where the integral is understood in the Cauchy principal value sense. It is well known (see [18, 23]) that the Hilbert transform of the function $u \in L_p(\mathbb{R})$, $1 \le p < \infty$, exists for almost all values of $t \in \mathbb{R}$. In case $1 < p < \infty$, the Hilbert transform is a bounded map in the space $L_p(\mathbb{R})$ and satisfies the equation:

$$H^2 = -I.$$

The Hilbert transform plays an important role in the theory and practice of signal processing operations in continuous system theory because of its relevance to such problems as envelope detection and demodulation, as well as its use in relating the real and imaginary components, and the magnitude and phase components of spectra. The Hilbert transform is the main part of the singular integral equations on the real line (see [32]). Therefore, approximations of the Hilbert transform are of great interest.

Many papers have dealt with the numerical approximation of the Hilbert Transform in the case of bounded intervals and the reader can refer to [2, 7**?** , 12, 13, 15, 16, 19, 20, 21, 28, 30, 33, 34, 36, 37, 38, 42, 43] and the references given there. On the other hand, the literature concerning the numerical integration on unbounded intervals is by far poorer than the one on bounded intervals. The case of the Hilbert Transform has been considered very little and the reader can consult [5, 11, 14, 15, 25, 27, 31, 39, 40, 41, 44]. In particular, in [25] the authors assume that the function u is analytic in the strip $\{z \in \mathbb{C} : |\Im z| < d\}$, in which case they show that the series $\frac{2}{\pi}\sum_{k\in\mathbb{Z}, k\neq even} \frac{u(t+k\delta)}{-k}$ uniformly converges to $(Hu)(t)$ as $\delta \to 0$. In [9] the author replaces the above series with the following one $\frac{1}{\pi}\sum_{k\in\mathbb{Z}} \frac{u(t+(k+1/2)\delta)}{-k-1/2}$ for a suitable choice of the step $\delta \to 0$.

This article is devoted to the approximation of the Hilbert transform of functions from Orlicz spaces by operators of the form

$$(H_\delta u)(t) = \frac{1}{\pi}\sum_{k\in\mathbb{Z}} \frac{u(t+(k+1/2)\delta)}{-k-1/2}, \quad \delta > 0$$

which were introduced in [25]. It is proved that the operators are uniformly bounded maps in the Orlicz spaces, satisfies the equality

$$H_\delta^2 = -I,$$

and for any $\delta > 0$ the sequence of operators H_δ strongly converges to the operator H in these spaces. Note that for Lebesgue and Hölder spaces these approximations were considered respectively in the works [3, 4].

2. Orlicz spaces

Definition 1. *A convex left continuous function $\Phi : [0,\infty) \to [0,\infty]$ satisfying the conditions*

$$\lim_{r\to 0+} \Phi(r) = \Phi(0) = 0, \quad \lim_{r\to\infty} \Phi(r) = \infty$$

is called a Young function.

The class of Young functions satisfying the condition $\Phi(r) \in (0,\infty)$ for any $r \in (0,\infty)$ is denoted by $\mathcal{Y}$. If $\Phi \in \mathcal{Y}$, then Φ is absolutely continuous on each bounded closed interval and bijective from $[0,\infty)$ to $[0,\infty)$.

For the Young function Φ we will write

$$\Phi^{-1}(s) = \inf\{r \geq 0 :\ \Phi(r) > s\}, \quad 0 \leq s \leq \infty.$$

We note that if $\Phi \in \mathcal{Y}$, then Φ^{-1} is the usual inverse of the function Φ. And for each Young function, we have the inequalities

$$\Phi\left(\Phi^{-1}(r)\right) \leq r \leq \Phi^{-1}\left(\Phi(r)\right), \quad r \geq 0.$$

Definition 2. *If there exists a number $C > 1$ such that the inequality*

$$\Phi(2r) \leq C\Phi(r)$$

holds for any $r > 0$, then the Young function Φ is said to satisfy the Δ_2-condition (the notation: $\Phi \in \Delta_2$).

Definition 3. *If there exists a number $C > 1$ such that the inequality*

$$\Phi(r) \leq \frac{1}{2C}\Phi(Cr)$$

holds for any $r > 0$, then the Young function Φ is said to satisfy the ∇_2-condition (the notation: $\Phi \in \nabla_2$).

For the Young function Φ, the function $\tilde{\Phi}$ defined by the relation

$$\tilde{\Phi}(r) = \sup\{rs - \Phi(s) : s \in [0.\infty)\}, \quad r \geq 0,$$

is called the complementary function. The complementary function $\tilde{\Phi}$ also is a Young function and $\tilde{\tilde{\Phi}} = \Phi$. We note that $\Phi \in \Delta_2$ (accordingly, $\Phi \in \nabla_2$) if and only if $\tilde{\Phi} \in \nabla_2$ (accordingly, $\tilde{\Phi} \in \Delta_2$). For any $r > 0$, the following inequalities hold:

$$r \leq \Phi^{-1}(r)\,\tilde{\Phi}^{-1}(r) \leq 2r.$$

Definition 4. *Let Φ be a Young function. The set of measurable functions $f : \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{R}$ satisfying the condition*

$$\exists k > 0 : \int\limits_{\Omega} \Phi(k|f(x)|)\,dx < \infty$$

is called the Orlicz space (the notation: $L_\Phi(\Omega)$).

The space $L_\Phi(\Omega)$ with norm

$$\|f\|_{L_\Phi} = \inf\left\{\lambda > 0 : \int\limits_{\Omega} \Phi\left(\frac{|f(x)|}{\lambda}\right) dx \leq 1\right\}$$

is a Banach space.

We note that if $\Phi(r) = r^p$, $1 \leq p < \infty$, then $L_\Phi(\Omega) = L_p(\Omega)$. The properties of Orlicz spaces are investigated in the works [1, 22, 24].

The following analogue of the Hölder's inequality is well known (see: [22]).

Theorem 1. *Let $\Omega \subset \mathbb{R}$ be a measurable set and functions f and g measurable on Ω. For a Young function Φ and its complementary function $\widetilde{\Phi}$, the following inequality is valid*

$$\int_{\Omega} |f(x)g(x)|dx \leq 2\|f\|_{L_\Phi(\Omega)}\|g\|_{L_{\widetilde{\Phi}(\Omega)}}.$$

Definition 5. *Let Φ be a Young function. The set of sequences $b = \{b_n\}_{n\in\mathbb{Z}}$ satisfying the condition*

$$\exists k > 0 : \sum_{n\in\mathbb{Z}} \Phi(k|b_n|) < \infty$$

is called the Orlicz sequence space (the notation: l_Φ).

The space l_Φ with norm

$$\|b\|_{l_\Phi} = \inf\left\{\lambda > 0 : \sum_{n\in\mathbb{Z}} \Phi\left(\frac{|b_n|}{\lambda}\right) \le 1\right\}$$

is a Banach space.

We note that if $\Phi(r) = r^p$, $1 \le p < \infty$, then $l_\Phi = l_p$. The properties of Orlicz sequence spaces are investigated in the works [17, 26, 29].

3. Properties of the approximating operators H_δ

The sequence $h(b) = \{(h(b))_n\}_{n\in\mathbb{Z}}$ is called the discrete Hilbert transform of the sequence $b = \{b_n\}_{n\in\mathbb{Z}}$, where

$$(h(b))_n = \sum_{m\neq n} \frac{b_m}{n-m}, \quad n \in \mathbb{Z}.$$

M.Riesz (see [35]) proved that if $b \in l_p$, $1 < p < \infty$, then $h(b) \in l_p$ and the inequality

$$\|h(b)\|_{l_p} \le C_p \|b\|_{l_p} \tag{1}$$

holds, where C_p is constant depending only on p.

We will use a modified version of the discrete Hilbert transform:

$$(\tilde{h}(b))_n = \sum_{m\in\mathbb{Z}} \frac{b_m}{n-m-1/2}, \quad n \in \mathbb{Z}.$$

K. Andersen [8] proved that the inequality (1) is also valid for the transform $\tilde{h}$, that is, there exist $\widetilde{C}_p > 0$ such that the inequality

$$\|\tilde{h}(b)\|_{l_p} \le \widetilde{C}_p \|b\|_{l_p} \tag{2}$$

holds for any $b \in l_p$, $1 < p < \infty$. Then it follows from Marcinkiewicz interpolation theorem in Orlicz spaces (see: [22]) that if $\Phi \in \Delta_2 \cap \nabla_2$, then the inequality (2) is also valid for the space l_Φ, that is, there exist $\widetilde{C}_\Phi > 0$ such that the inequality

$$\|\tilde{h}(b)\|_{l_\Phi} \le \widetilde{C}_p \|b\|_{l_\Phi}$$

holds for any $b \in l_\Phi$.

In [3], the authors prove that the operators H_δ are uniformly bounded maps in the spaces $L_p(\mathbb{R})$, $1 < p < \infty$ and

$$\|H_\delta\|_{L_p(\mathbb{R})\to L_p(\mathbb{R})} \le \|\tilde{h}\|_{l_p\to l_p}.$$

Then it follows from Marcinkiewicz interpolation theorem in Orlicz spaces (see: [22]) that if $\Phi \in \Delta_2 \cap \nabla_2$, then the operators H_δ are also uniformly bounded maps in the space $L_\Phi(\mathbb{R})$, that is there exist $C_\Phi > 0$ such that for any $\delta > 0$

$$\|H_\delta\|_{L_\Phi(\mathbb{R})\to L_\Phi(\mathbb{R})} \le C_\Phi. \tag{3}$$

Theorem 2. *For any $\delta > 0$ and $u \in L_\Phi(\mathbb{R})$ the following inequality holds:*

$$H_\delta(H_\delta u)(t) = -u(t). \tag{4}$$

Proof. For any $u \in L_\Phi(\mathbb{R})$ we have

$$H_\delta(H_\delta u)(t) = -\tfrac{1}{\pi}\sum_{k\in Z}\tfrac{(H_\delta u)(t+(k+1/2)\delta)}{k+1/2} = \tfrac{1}{\pi}\sum_{k\in Z}\tfrac{1}{k+1/2}\cdot\tfrac{1}{\pi}\sum_{m\in Z}\tfrac{u(t+(k+m+1)\delta)}{m+1/2}$$

$$= \tfrac{1}{\pi^2}\sum_{k\in Z}\sum_{m\in Z}\tfrac{u(t+(k+m+1)\delta)}{(k+1/2)(m+1/2)} = \tfrac{1}{\pi^2}\sum_{k\in Z}\sum_{n\in Z}\tfrac{u(t+n\delta)}{(k+1/2)(n-k-1/2)}$$

$$= \tfrac{1}{\pi^2}\sum_{n\in Z}\left(\sum_{k\in Z}\tfrac{1}{(k+1/2)(n-k-1/2)}\right)u(t+n\delta). \tag{5}$$

Since for $n = 0$

$$\sum_{k\in Z}\frac{1}{(k+1/2)(n-k-1/2)} = -4\sum_{k\in Z}\frac{1}{(2k+1)^2} = -\pi^2,$$

and for $n \neq 0$

$$\sum_{k\in Z}\frac{1}{(k+1/2)(n-k-1/2)} = \sum_{k\in Z}\frac{1}{n}\left[\frac{1}{k+1/2}+\tfrac{1}{n-k-1/2}\right]$$

$$= \frac{1}{n}\lim_{N\to\infty}\sum_{|k|\le N}\left[\frac{1}{k+1/2}+\tfrac{1}{n-k-1/2}\right] = 0,$$

then equality (4) follows from (5).

4. Approximation of the singular integral with Hilbert kernel in Orlicz spaces

Let $\Phi \in \Delta_2 \cap \nabla_2$. Denote by $L_\Phi(T)$, the space of all measurable, 2π-periodic functions with finite norm $\|\varphi\|_{L_\Phi(T)} = \|\varphi\|_{L_\Phi([-\pi,\pi])}$.

It is well known that the singular integral with Hilbert kernel

$$(S\varphi)(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\cot\frac{t-\tau}{2}\varphi(\tau)d\tau,$$

is a bounded map in the space $L_\Phi(T)$ (see [45]).

Consider in $L_\Phi(T)$ the sequence of operators

$$(S_n\varphi)(t) = \frac{1}{n}\sum_{k=0}^{n-1}\cot\left(-\tfrac{\pi(2k+1)}{2n}\right)\varphi\left(t+\tfrac{\pi(2k+1)}{n}\right), \quad n\in N.$$

It is easy to verify that if

$$\varphi(t)=\frac{a_0}{2}+\sum_{m=1}^{\infty}(a_m\cos mt+b_m\sin mt),$$

then

$$(S_n\varphi)(t)=\sum_{m=1}^{\infty}\lambda_m^{(n)}(a_m\cos mt+b_m\sin mt),$$

where $\lambda_m^{(n)}=1$ for $m=\overline{1,n-1}$, $\lambda_n^{(n)}=\lambda_{2n}^{(n)}=0$, $\lambda_m^{(n)}=-1$ for $m=\overline{n+1,2n-1}$ and $\lambda_{m+2n}^{(n)}=\lambda_m^{(n)}$ for $m\in Z$. It follows from here that for any trigonometric polynomial $P(t)$ of order at most $n-1$

$$(S_nP)(t)=(SP)(t).$$

In [3], the authors prove that the operators S_n are uniformly bounded in $L_p(T)$, $1<p<\infty$, and for any $n\in N$ the inequality

$$\|S_n\|_{L_p(T)\to L_p(T)}\le 4+2\|\tilde{h}\|_{l_p\to lp}$$

holds. Then it follows from Marcinkiewicz interpolation theorem in Orlicz spaces (see: [22]) that if $\Phi\in\Delta_2\cap\nabla_2$, then the operators S_n are also uniformly bounded maps in the space $L_\Phi(T)$, that is there exist $\widetilde{C_\Phi}>0$ such that for any $n\in\mathbb{N}$ and for any $\varphi\in L_\Phi(T)$

$$\|S_n\varphi\|_{L_\Phi(T)}\le\widetilde{C_\Phi}\|\varphi\|_{L_\Phi(T)}.$$

In [6], it is proved that if $\Phi\in\Delta_2\cap\nabla_2$, then the sequence of operators S_n strongly converges to the operator S in $L_\Phi(T)$ and for any $\varphi\in L_\Phi(T)$ the following estimate holds:

$$\|S\varphi-S_n\varphi\|_{L_\phi(T)}\le\left(\|S\|_{L_\Phi(T)\to L_\Phi(T)}+\widetilde{C_\Phi}\right)\cdot E_{n-1}^{\Phi}(\varphi),\ n\in\mathbb{N},\tag{6}$$

where $E_{n-1}^{\Phi}(\varphi)$ – is the best approximation of the function φ in the metric $L_\Phi(T)$ by trigonometric polynomials of order at most $n-1$.

5. Approximation of the Hilbert transform in Orlicz spaces

Consider the regular integral operator

$$(\mathrm{K}\varphi)(t)=\frac{1}{2\pi}\int_{-\pi}^{\pi}K(t,\tau)\varphi(\tau)d\tau,\quad t\in[-\pi,\pi],$$

and the sequence of operators

$$(\mathrm{K}_n\varphi)(t)=\frac{1}{n}\sum_{k=0}^{n-1}K\left(t,t+\tfrac{\pi(2k+1)}{n}\right)\varphi\left(t+\tfrac{\pi(2k+1)}{n}\right),\quad t\in[-\pi,\pi],\quad n\in\mathbb{N},$$

where $K(t,\tau)$ is a continuous function on $[-\pi,\pi]^2$ and $K(t,\tau)=K(t,\tau-2\pi)$ for $(t,\tau)\in[-\pi,\pi]\times(\pi,3\pi)$.

Lemma 1. *Let* $\Phi \in \Delta_2 \cap \nabla_2$. *Then the sequence of operators* $\{\mathrm{K}_n\}$ *strongly converges to the operator* K *in* $L_\Phi(T)$.

Proof. First assume that $K(t,\tau)$ is a 2π - periodic function by τ . Denote

$$\|K\|_\infty = \max_{t,\tau\in[-\pi,\pi]} |K(t,\tau)|, \quad E_n(K) = \inf \|K - \Phi_n\|_\infty,$$

where $\Phi_n(t,\tau) = \frac{\alpha_0(t)}{2} + \sum_{m=1}^{n}(\alpha_m(t)\cos m\tau + \beta_m(t)\sin m\tau)$, and infimum is taken over all trigonometric polynomials $\alpha_m(t)$, $m = \overline{0,n}$, $\beta_m(t)$, $m = \overline{1,n}$ of order at most n.

Denote $n_0 = \left[\frac{n-1}{2}\right]$. Suppose that

$$q_{n_0}(t) = \frac{a_0}{2} + \sum_{m=1}^{n_0}(a_m \cos mt + b_m \sin mt)$$

and

$$\Phi_{n_0}^{(0)}(t,\tau) = \frac{\alpha_0^{(0)}(t)}{2} + \sum_{m=1}^{n_0}(\alpha_m^{(0)}(t)\cos m\tau + \beta_m^{(0)}(t)\sin m\tau)$$

are the best approximations of the functions φ and K by trigonometric polynomials of order at most n_0, that is

$$\|\varphi - q_{n_0}\|_{L_\Phi} = \inf \|\varphi - P_{n_0}\|_{L_\Phi},$$

where infimum is taken over all trigonometric polynomials P_{n_0} of order at most n_0 and

$$\|K - \Phi_{n_0}^{(0)}\|_\infty = E_n(K).$$

For any trigonometric polynomial $r_{n-1}(t)$ of order at most $n-1$, the equality

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} r_{n-1}(\tau)d\tau = \frac{1}{n}\sum_{k=0}^{n-1} r_{n-1}\left(t + \tfrac{\pi(2k+1)}{n}\right)$$

holds. Therefore

$$(\mathrm{K}\varphi)(t) - (\mathrm{K}_n\varphi)(t) = (\mathrm{K} - \mathrm{K}_n)(\varphi - q_{n_0})(t) + \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[K(t,\tau) - \Phi_{n_0}^{(0)}(t,\tau)\right] q_{n_0}(\tau)d\tau$$

$$+\frac{1}{n}\sum_{k=0}^{n-1}\left[K(t,t+\tau_k^{(n)}) - \Phi_{n_0}^{(0)}(t,t+\tau_k^{(n)})\right] q_{n_o}(t+\tau_k^{(n)}),$$

where $\tau_k^{(n)} = \frac{\pi(2k+1)}{n}$, $k \in Z$. It follows from here and from inequalities

$$\|\mathrm{K}\|_{L_\Phi(T)\to L_\Phi(T)} \le \|K\|_\infty, \quad \|\mathrm{K}_n\|_{L_\Phi(T)\to L_\Phi(T)} \le \|K\|_\infty$$

that

$$\|\mathrm{K}\varphi - \mathrm{K}_n\varphi\|_{L_\Phi(T)} \le 2\|K\|_\infty E_{n_0}^\Phi(\varphi) + 2E_{n_0}(K)\left[\|\varphi\|_{L_\Phi(T)} + E_{n_0}^\Phi(\varphi)\right].$$

This completes the proof of the lemma in this case. Now consider the general case.

Let $\varphi \in L_\Phi(T)$ and $\varepsilon > 0$. Since $\varphi \in L_\Phi(T) \subset L_1(T)$, then it follows from Lebesgue theorem that there exist $\delta_\varepsilon^{(0)} > 0$ such that for any $0 < \delta < \delta_\varepsilon^{(0)}$

$$\int_{\pi-\delta}^{\pi} |\varphi(\tau)| d\tau < \frac{\pi\varepsilon}{4(\|K\|_\infty + 1)} \cdot \Phi^{-1}\left(\frac{1}{2\pi}\right).$$

Denote

$$K^*(t,\tau) = K(t,\tau) \quad \text{for} \quad (t,\tau) \in [-\pi,\pi] \times [-\pi, \pi - \delta_\varepsilon],$$

$$K^*(t,\tau) = K(t,\pi-\delta_\varepsilon) + \frac{\tau - \pi + \delta_\varepsilon}{\delta_\varepsilon}[K(t,-\pi) - K(t,\pi-\delta_\varepsilon)]$$

$$\text{for} \quad (t,\tau) \in [-\pi,\pi] \times [\pi-\delta_\varepsilon, \pi],$$

$$K^*(t,\tau+2\pi) = K^*(t,\tau) \quad \text{for any } (t,\tau) \in [-\pi,\pi] \times \mathbb{R},$$

where $\delta_\varepsilon = \min\left\{\delta_\varepsilon^{(0)}, \frac{\pi\varepsilon}{8\|K\|_\infty \|\varphi\|_{L_\Phi(T)}}, 1\right\}$.

Since the function $K^*(t,\tau)$ is continuous and 2π-periodic by τ, the sequence of operators

$$(\mathrm{K}_n^*\varphi)(t) = \frac{1}{n}\sum_{k=0}^{n-1} K^*(t, t+\tau_k^{(n)})\varphi(t+\tau_k^{(n)}), \quad t \in [-\pi,\pi], \quad n \in \mathbb{N}$$

strongly converges to the operator

$$(\mathrm{K}^*\varphi)(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} K^*(t,\tau)\varphi(\tau) d\tau$$

in $L_\Phi(T)$.Therefore, the inequality

$$\|\mathrm{K}_n^*\varphi - \mathrm{K}^*\varphi\|_{L_\Phi(T)} < \varepsilon/2$$

is satisfied for large values of n. Moreover, since for any $t \in [-\pi,\pi]$

$$|(\mathrm{K}\varphi)(t) - (\mathrm{K}^*\varphi)(t)| \le \frac{1}{2\pi}\int_{\pi-\delta_\varepsilon}^{\pi} |K(t,\tau) - K^*(t,\tau)||\varphi(\tau)| d\tau$$

$$\le \frac{\|K\|_\infty}{\pi}\int_{\pi-\delta_\varepsilon}^{\pi} |\varphi(\tau)| d\tau < \frac{\varepsilon}{4} \cdot \Phi^{-1}\left(\frac{1}{2\pi}\right),$$

then

$$\|\mathrm{K}\varphi - \mathrm{K}^*\varphi\|_{L_\Phi(T)} \le \frac{\varepsilon}{4}.$$

For $n \ge \frac{16\|K\|_\infty \|\varphi\|_{L_p(T)}}{\varepsilon}$ we have

$$\|\mathrm{K}_n\varphi - \mathrm{K}_n^*\varphi\|_{L_\Phi(T)} \le \frac{1}{n} \cdot \left(\frac{n}{2\pi} \cdot \delta_\varepsilon + 1\right) \cdot 2\|K\|_\infty \|\varphi\|_{L_\Phi(T)} \le \frac{\varepsilon}{4}.$$

Therefore for sufficiently large values n

$$\|\mathrm{K}_n\varphi - \mathrm{K}\varphi\|_{L_\Phi(T)}$$

$$\le \|\mathrm{K}_n\varphi - \mathrm{K}_n^*\varphi\|_{L_\Phi(T)} + \|\mathrm{K}_n^*\varphi - \mathrm{K}^*\varphi\|_{L_\Phi(T)} + \|\mathrm{K}^*\varphi - \mathrm{K}\varphi\|_{L_\Phi(T)} < \varepsilon.$$

Corollary 1. *The sequence of operators*

$$(\widetilde{\mathrm{K}}_n\varphi)(t)=\frac{1}{n}\sum_{\{k\in\mathbb{Z}:t+\tau_k^{(n)}\in[-\pi,\pi]\}}K(t,t+\tau_k^{(n)})\varphi(t+\tau_k^{(n)}),\ t\in[-\pi,\pi],\ n\in\mathbb{N}$$

strongly converges to the operator K *in* $L_\Phi([-\pi,\pi])$.

Corollary 2. *If the function $K(t,\tau)$ is continuous on $[\pi m,\pi m+2\pi q]\times[-\pi,\pi]$, then for any $\varphi\in L_\Phi(T)$ the sequence of functions*

$$(\widetilde{\mathrm{K}}_n\varphi)(t)=\frac{1}{n}\sum_{\{k\in\mathbb{Z}:t+\tau_k^{(n)}\in[-\pi,\pi]\}}K(t,t+\tau_k^{(n)})\varphi(t+\tau_k^{(n)}),\ t\in[\pi m,\pi m+2\pi q],$$

converges to the function

$$(\mathrm{K}\varphi)(t)=\frac{1}{2\pi}\int_{-\pi}^{\pi}K(t,\tau)\varphi(\tau)d\tau,\quad t\in[\pi m,\pi m+2\pi q]$$

in $L_\Phi([\pi m,\pi m+2\pi q])$, where $m\in\mathbb{Z}$, $q\in\mathbb{N}$.

Corollary 3. *If the function $K_0(t)$ is continuous on $[-\pi,\pi]$, then the sequence of operators*

$$(\mathrm{K}_n^0\varphi)(t)=\frac{1}{2n}\sum_{k=-n}^{n-1}K_0\left(\tfrac{\pi(2k+1)}{2n}\right)\varphi\left(t+\tfrac{\pi(2k+1)}{2n}\right),\ t\in[-\pi,\pi],\ n\in\mathbb{N}$$

strongly converges to the operator

$$(\mathrm{K}^0\varphi)(t)=\frac{1}{2\pi}\int_{-\pi}^{\pi}K_0(\tau)\varphi(t+\tau)d\tau,\quad t\in[-\pi,\pi]$$

in $L_\Phi(T)$.

In the following theorem we prove that for any $\delta>0$ the sequence of operators $\{H_{\delta/n}\}_{n\in\mathbb{N}}$ strongly converges to the operator H in $L_\Phi(\mathbb{R})$.

Theorem 3. *Let $\Phi\in\Delta_2\cap\nabla_2$. Then for any $\delta>0$ the sequence of the operators $\{H_{\delta/n}\}_{n\in\mathbb{N}}$ strongly converges to the operator H in $L_\Phi(\mathbb{R})$, that is for any $u\in L_\Phi(\mathbb{R})$ the following inequality holds:*

$$\lim_{n\to\infty}\|H_{\delta/n}u-Hu\|_{L_\Phi(\mathbb{R})}=0.$$

Proof. We have divided the proof into three steps.

Step 1. Let us first prove that the operator

$$(H^*\varphi)(t)=\frac{1}{\pi}\int_{t-\pi}^{t+\pi}\frac{\varphi(\tau)}{t-\tau}d\tau$$

is a bounded operator in $L_\Phi(T)$. Indeed, for any $\varphi \in L_\Phi(T)$ we have

$$(H^*\varphi)(t) = \frac{1}{\pi}\int_{t-\pi}^{t+\pi} \tfrac{\varphi(\tau)}{t-\tau}d\tau = \frac{1}{\pi}\int_{t-\pi}^{t+\pi}\left[\tfrac{1}{t-\tau} - \tfrac{1}{2}\cot\tfrac{t-\tau}{2}\right]\varphi(\tau)d\tau + (S\varphi)(t)$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\cot\tfrac{\tau}{2} - \tfrac{2}{\tau}\right]\varphi(t+\tau)d\tau + (S\varphi)(t). \tag{7}$$

Since the function

$$K_0(\tau) = \cot\frac{\tau}{2} - \frac{2}{\tau} \quad \text{for} \quad \tau \neq 0, \quad K_0 = 0$$

is continuous on $[-\pi,\pi]$, then it follows from (7) that the operator H^* is bounded in $L_\Phi(T)$.

Consider the sequence of operators

$$(H_n^*\varphi)(t) = \frac{1}{\pi}\sum_{k=-n}^{n-1} \tfrac{1}{-k-1/2}\varphi\left(t + \tfrac{\pi(2k+1)}{2n}\right), \; t \in [-\pi,\pi], \; n \in \mathbb{N}.$$

Since for any $\varphi \in L_\Phi(T)$

$$(H_n^*\varphi)(t) = \frac{1}{2n}\sum_{k=-n}^{n-1}\left[\cot\left(\tfrac{\pi(2k+1)}{4n}\right) - \tfrac{4n}{\pi(2k+1)}\right]\varphi\left(t + \tfrac{\pi(2k+1)}{2n}\right) + (S_{2n}\varphi)(t)$$

$$= \frac{1}{2n}\sum_{k=-n}^{n-1} K_0\left(\tfrac{\pi(2k+1)}{2n}\right)\varphi\left(t + \tfrac{\pi(2k+1)}{2n}\right) + (S_{2n}\varphi)(t),$$

then it follows from (6) and from Corollary 3 that the sequence of operators H_n^* strongly converges to the operator H^* in $L_\Phi(T)$.

Step 2. Let us prove that the sequence of operators

$$(H_{\pi/(4n)}u)(t) = \frac{1}{\pi}\sum_{k\in Z} \tfrac{1}{-k-1/2}u\left(t + \tfrac{\pi(k+1/2)}{4n}\right), \; t \in \mathbb{R}, \; n \in \mathbb{N}$$

strongly converges to the operator H in $L_\Phi(\mathbb{R})$. At first assume that *supp* $u \subset [-\pi/4, \pi/4]$. Denote by φ 2π-periodic function, coinciding with the function u on $[-\pi/4,\pi/4]$ and equal to zero in $[-\pi,\pi] \setminus [-\pi/4,\pi/4]$. Since for any $t \in [-\pi/2,\pi/2]$

$$(Hu)(t) = \frac{1}{\pi}\int_{-\pi/4}^{\pi/4} \tfrac{u(\tau)}{t-\tau}d\tau = (H^*\varphi)(t), \tag{8}$$

$$(H_{\pi/n}u)(t) = \frac{1}{\pi}\sum_{k=-n}^{n-1} \tfrac{1}{-k-1/2}u\left(t + \tfrac{\pi(k+1/2)}{n}\right)$$

$$= \frac{1}{\pi}\sum_{k=-n}^{n-1} \tfrac{1}{-k-1/2}\varphi\left(t + \tfrac{\pi(k+1/2)}{n}\right) = (H_n^*\varphi)(t), \tag{9}$$

and the sequence of operators H_n^* strongly converges to the operator H^* in $L_\Phi(T)$, then it follows from (8) and (9) that for any $\varepsilon > 0$ for large values of n

$$\|H_{\pi/n}u - Hu\|_{L_\Phi([-\pi/2,\pi/2])} = \|H_n^*\varphi - H^*\varphi\|_{L_\Phi([-\pi/2,\pi/2])} \\ \le \|H_n^*\varphi - H^*\varphi\|_{L_\Phi(T)} < \varepsilon. \quad (10)$$

Due to the inequalities

$$|(Hu)(t)| \le \frac{1}{\pi}\int_{-\pi/4}^{\pi/4}\left|\frac{u(\tau)}{t-\tau}\right| d\tau \le \frac{\|u\|_{L_1([-\pi/4,\pi/4])}}{\pi(|t|-\pi/4)},\ |t| > \pi/4,$$

$$|(H_{\pi/n}u)(t)| \le \frac{1}{\pi}\sum_{k\in\mathbb{Z}_{(n)}^{(t)}} \frac{1}{|k+1/2|}\left|u\left(t+\tfrac{\pi(k+1/2)}{n}\right)\right| \\ \le \frac{1}{n(|t|-\pi/4)}\sum_{k\in\mathbb{Z}_{(n)}^{(t)}}\left|u\left(t+\tfrac{\pi(k+1/2)}{n}\right)\right|,\quad |t| > \pi/4,$$

where $\mathbb{Z}_{(n)}^{(t)} = \{k \in \mathbb{Z} : t + \frac{\pi(k+1/2)}{n} \in [-\pi/4, \pi/4]\}$, we get that for any $M > 2\pi$

$$\|Hu\|_{L_\Phi([M,\infty))} \le \frac{\|u\|_{L_1([-\pi/4,\pi/4])}}{\pi}\cdot\|f_0\|_{L_\Phi([M,\infty))},$$

where $f_0(t) = \frac{1}{t-\pi/4} \in L_\Phi([M,\infty))$,

$$\|H_{\pi/n}u\|_{L_\Phi([M,\infty))} \le \frac{1}{n(M-\pi/4)}\left\|\sum_{k\in\mathbb{Z}_{(n)}^{(t)}}\left|u\left(\cdot+\tfrac{\pi(k+1/2)}{n}\right)\right|\right\|_{L_\Phi([M,\infty))}$$

$$\le \frac{1}{n(M-\pi/4)}\cdot n\cdot\|u\|_{L_\Phi([-\pi/4,\pi/4])} = \frac{\|u\|_{L_\Phi([-\pi/4,\pi/4])}}{M-\pi/4}$$

Similar inequalities holds for $\|Hu\|_{L_\Phi((-\infty,-M])}$ and for $\|H_{\pi/n}u\|_{L_\Phi((-\infty,-M])}$. Therefore, for any $\varepsilon > 0$ there exist $m_0 \ge 4$ such that

$$\|Hu\|_{L_\Phi(R\backslash[-\frac{\pi m_0}{2},\frac{\pi m_0}{2}])} < \varepsilon,\quad \|H_{\pi/n}u\|_{L_\Phi(R\backslash[-\frac{\pi m_0}{2},\frac{\pi m_0}{2}])} < \varepsilon. \quad (11)$$

Since the function $\frac{1}{t-\tau}$ is continuous on a rectanhle $[2\pi, 2\pi m_0]\times[-\pi,\pi]$, then it follows from Corollary 2 that the sequence of functions

$$(\mathrm{W}_n\varphi)(t) = \frac{2}{n}\sum_{\{k\in Z:t+\frac{\pi(2k+1)}{n}\in[-\pi,\pi]\}}\frac{\varphi(t+\pi(2k+1)/n)}{-\pi(2k+1)/n} \\ = \frac{1}{\pi}\sum_{\{k\in Z:t+\frac{\pi(2k+1)}{n}\in[-\pi,\pi]\}}\frac{\varphi(t+\pi(2k+1)/n)}{-k-1/2},\quad n\in N$$

converges to the function

$$(\mathrm{W}\varphi)(t) = \int_{-\pi}^{\pi} \frac{\varphi(\tau)}{t-\tau} d\tau$$

in $L_\Phi([2\pi, 2\pi m_0])$. Denote by ψ the function, defined on $[-\pi, \pi]$ by the equality $\psi(\tau) = u(\tau/4)$. Then it follows from the equations

$$(Hu)(t) = \frac{1}{\pi}\int_{-\pi/4}^{\pi/4} \frac{u(\tau)}{t-\tau} d\tau = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{u(\tau/4)}{4t-\tau} d\tau = (\mathrm{W}\psi)(4t),\ t \in [\pi/2, \pi m_0/2],$$

$$(H_{\pi/(4n)}u)(t) = \frac{1}{\pi}\sum_{k\in Z_{(4n)}^{(t)}} \tfrac{u(t+\pi(k+1/2)/4n)}{-k-1/2}$$

$$= \frac{1}{\pi}\sum_{k\in Z_{(4n)}^{(t)}} \tfrac{\psi(4t+\pi(k+1/2)/n)}{-k-1/2} = (\mathrm{W}_n\psi)(4t),\ t \in [\pi/2, \pi m_0/2],$$

that the sequence of functions $H_{\pi/(4n)}u$ converges to the function Hu in the space $L_\Phi([\pi/2, \pi m_0/2])$. Therefore, for large values of n

$$\|H_{\pi/(4n)}u - Hu\|_{L_\Phi([\pi/2, \pi m_0/2])} < \varepsilon. \tag{12}$$

Similarly, for large values on n

$$\|H_{\pi/(4n)}u - Hu\|_{L_\Phi([-\pi m_0/2, -\pi/2])} < \varepsilon. \tag{13}$$

It follows from (10)-(13) that in the case *supp* $u \subset [-\pi/4, \pi/4]$

$$\lim_{n\to\infty} \|H_{\pi/(4n)}u - Hu\|_{L_\Phi(R)} = 0. \tag{14}$$

Now suppose that *supp* $u \subset [-\pi m/4, \pi m/4]$ for some $m \in N$. Denote by u_0 the function, defined on $[-\pi/4, \pi/4]$ by the equatin $u_0(t) = u(mt)$. Then for any $t \in R$

$$(Hu)(t) = \frac{1}{\pi}\int_{-\pi m/4}^{\pi m/4} \frac{u(\tau)}{t-\tau} d\tau = \frac{1}{\pi}\int_{-\pi/4}^{\pi/4} \frac{u(m\tau)}{t-\tau} m d\tau = (Hu_0)(t/m),$$

$$(H_{\pi/(4n)}u)(t) = \frac{1}{\pi}\sum_{\{k\in Z: t+\frac{\pi(k+1/2)}{4n}\in[-\frac{\pi m}{4}, \frac{\pi m}{4}]\}} \tfrac{u(t+\pi(k+1/2)/4n)}{-k-1/2}$$

$$= \frac{1}{\pi}\sum_{k\in Z_{(4mn)}^{(t/m)}} \tfrac{u_0(t/m+\pi(k+1/2)/(4mn))}{-k-1/2} = (H_{\pi/(4mn)}u_0)(t/m).$$

Since equation (14) holds for u_0, we obtain that

$$\lim_{n\to\infty} \|H_{\pi/(4n)}u - Hu\|_{L_\Phi(R)} = m^{1/p} \lim_{n\to\infty} \|H_{\pi/(4mn)}u_0 - Hu_0\|_{L_\Phi(R)} = 0.$$

Now consider the general case. Let us prove that equation (14) holds for any $u \in L_\Phi(R)$. For any $u \in L_\Phi(R)$ and $\varepsilon > 0$ there exist $m \in N$ such that

$$\|u - u_m\|_{L_\Phi(R)} < \varepsilon, \tag{15}$$

where $u_m(t) = u(t) \cdot \chi_{[-\pi m/4, \pi m/4](t)}$. Since equation (14) holds for u_m, and it follows from (3), (15) that

$$\|H_{\pi/(4n)}(u - u_m) - H(u - u_m)\|_{L_\Phi(R)}$$
$$\leq \left[\|H_{\pi/(4n)}\|_{L_\Phi(R)\to L_\Phi(R)} + \|H\|_{L_\Phi(R)\to L_\Phi(R)}\right] \cdot \|u - u_m\|_{L_\Phi(R)}$$
$$\leq \varepsilon \cdot \left[C_\Phi + \|H\|_{L_\Phi(R)\to L_\Phi(R)}\right],$$

then we get that the equation (14) also holds for the function u.

Step 3. Let us prove that for any $\delta > 0$ the secuence of the operators $\{H_{\delta/n}\}_{n\in N}$ strongly converges to the operator H in $L_\Phi(R)$. Let $u \in L_\Phi(R)$. Denote $w(t) = u(4\delta t/\pi)$, $t \in R$. Then for any $t \in R$

$$(Hu)(t) = \frac{1}{\pi}\int_R \tfrac{u(\tau)}{t-\tau} d\tau = \frac{1}{\pi}\int_R \tfrac{w(\pi\tau/(4\delta))}{t-\tau} d\tau$$
$$= \frac{1}{\pi}\int_R \tfrac{w(\tau)}{\pi t/(4\delta)-\tau} d\tau = (Hw)(\pi t/(4\delta)), \tag{16}$$

$$(H_{\delta/n}u)(t) = \frac{1}{\pi}\sum_{k\in Z} \tfrac{u(t+(k+1/2)\delta/n)}{-k-1/2}$$
$$= \frac{1}{\pi}\sum_{k\in Z} \tfrac{w(\pi t/(4\delta)+\pi(k+1/2)/(4n))}{-k-1/2} = (H_{\pi/(4n)}w)(\pi t/(4\delta)). \tag{17}$$

Since $\lim_{n\to\infty} \|H_{\pi/(4n)}w - Hw\|_{L_\Phi(R)} = 0$, then it follows from (16), (17) that

$$\lim_{n\to\infty} \|H_{\delta/n}u - Hu\|_{L_\Phi(R)} = 0.$$

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Lale Sh. Alizade
Baku State University, AZ1148, Baku, Azerbaijan
E-mail: lale-alizade-98@mail.ru

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