Caspian Journal of Applied Mathematics, Ecology and Economics
V. 12, No 1, 2024, July
ISSN 1560-4055

Asymptotic Behavior of Eigenvalues of a Boundary Value Problem for Laplace Equation

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Abstract. In the square $\Omega = (0, 2\pi) \times (0, 2\pi)$ we consider a spectral boundary value problem for the Laplace equation in the case when one of the boundary conditions contain mixed derivatives on the line $x = 2\pi$, when $y \in [0, 2\pi]$.
It is shown that the considered spectral problem has two series of eigenvalues one of which is finite, the another one asymptotically behaves as $\lambda_{n,k} \sim k^2 + \frac{n^2}{4} \quad (k, n \to \infty)$.

Key Words and Phrases: eigenvalues, Hilbert space, Laplace equations, asymptotic formula.

2010 Mathematics Subject Classifications: 35G25, 35P20, 47A75

1. Introduction

It is known that in the bounded domain there exist spectral problems for the Laplace equation (for example, first, second, third) in which eigenvalues are nonnegative, discrete and have a unique limit point $+\infty$. It this case, it is customary to say that eigenvalues of spectral boundary value problems for the Laplace equation behave as classic. These exist spectral boundary value problems for the Laplace equation in which the classic nature of eigenvalues is violated. Non-classic nature of eigenvalues of boundary value problems for the Laplace equation appear mainly in two moments:

I. Boundary conditions contain a linear operator (differential or non-differential)

II. One and the same spectral parameter is involved in the equation and in the boundary conditions.

We cite some papers where the classic nature of the eigenvalues of boundary value problems for the Laplace equation is violated.

In [1] V.A. Il'in and A.F. Filippov consider such a spectral problem for the Laplace equation where any point on the positive axis is a condensation point, i.e. classic nature of eigenvalues is violated. In this paper, one of the boundary conditions contains a non-differential operator.

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In S.Ya. Yakubov is work [2], asymptotic behavior of eigenvalues of the following boundary value problems is studied for the Laplace equation in the square $\Omega = (0, 2\pi) \times (0, 2\pi)$, containing in one of the boundary conditions the differential operator.

$$-\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = \lambda u\,(x, y)\,, \tag{1}$$

$$u(0, y) = 0, \ \ \frac{\partial u\,(2\pi, y)}{\partial x} + i\frac{\partial u\,(2\pi, y)}{\partial y} = 0, \ \ y \in [0, 2\pi]\,, \tag{2}$$

$$u(x,\,0) = u(x,\,2\pi), \quad \frac{\partial u\,(x, 0)}{\partial y} = \frac{\partial u\,(x, 2\pi)}{\partial y}, \ \ x \in [0, 2\pi] \tag{3}$$

It is proved that for boundary value problems (1)- (3) there exists a sequence of eigenvalues $\{\lambda_k\}$ convergent to zero. More exactly, the classic case on the behavior of eigenvalues for the boundary value problems (1)-(3) is violated .

In A.N. Kozhevnikov's work [3], in the boundary domain. $.\Omega \in R^n$ with a rather smooth boundary Γ for the Laplace equation the following spectral problem is studied:

$$-\Delta u = \lambda u \ \text{ in } \ \Omega, \tag{4}$$

$$-u = \lambda\frac{\partial u}{\partial \nu} \ \text{ in } \ \Gamma, \tag{5}$$

where λ is a spectral parameter, ν is an interior normal to the boundary Γ. It is proved that the spectrum of the boundary value problem (4), (5) is discrete and consists of two series of eigenvalues convergent to zero and to $+\infty$, respectively. More exactly, the eigenvalues of spectral boundary value problem (4), (5) behave as non- classic .

It is known that many boundary value problems for the Laplace equation given in a rectangle are reduced to spectral boundary value problems for second order elliptic differential operator equations in some Hilbert space.

Usually, in spectral problems for elliptic differential operator equations, the operator appearing in the equation is an operator with a discrete spectrum, the eigenevalues of which form a complete orthonormal basis in the same Hilbert space. After spectral expansion in eigenelements of the operator appearing in the equation, the spectral problem stated for elliptic differential operator equations is reduced to a spectral boundary value problem for ordinary differential equations with respect to Fourier coefficients.

In the present paper, using the above method we study asymptotic behavior of eigen values of the following boundary value problems for a two-dimensional Laplace equation in the square $\Omega = (0, 2\pi) \times (0, 2\pi)$:

$$-\frac{\partial^2 u(x, y)}{\partial x^2} - \frac{\partial^2 u(x, y)}{\partial y^2} = \lambda u\,(x, y)\,, \tag{6}$$

$$u(0, y) = 0, \ \ u\,(2\pi, y) + i\frac{\partial^2 u(2\pi, y)}{\partial y \partial x} = 0, \ \ y \in [0, 2\pi]\,, \tag{7}$$

$$u(x,\,0) = \,u\,(x, 2\pi)\,, \quad \frac{\partial u(x,\,0)}{\partial y} = \frac{\partial u(x,\;2\pi)}{\partial y}, \;\; x \in [0, 2\pi]\,. \tag{8}$$

where i is an imaginary number.

It is proved that the eigenvalues of boundary value problem (6)-(8) are real. Then it is shown that problem (6)-(8) has two series of eigenvalues one of which consist of a finitely many numbers, more exactly of six eigenvalues satisfying the inequality : $k^2 - 1 < \lambda_k < k^2, \;\; k = 1, ..., 6$; the another one behaves as $\lambda_{n,k} \sim k^2 + \frac{n^2}{4}; \;\; k, n \in N$. In other works, the classicsum of eigenvalues of or boundary value problems (6)-(8) is violeted.

By the above [4-7] studies asymptotic behavior of eigenvalues of boundary values problems for second order elliptic differential operator equations in the case one and the same spectral parameter is involved in the equation and in the boundary condition. It is proved that in [4-7] the eigenvalues of the problem under consideration behave with non-classic asymptotics.

2. Asymptotic formulas for eigenvalues

Definition 1. *Am eigenfunctions of the problem* (6)-(8) *is a identically non-zero function* $u(x, y) \in C^{(2)}\left(\overline{\Omega}\right)$ *that on the boundary* Ω *satisfy boundary conditions* (7), (8) *and for some* λ *satisfies on* Ω *the equation* (6). *The values of* λ *for which there exist eigenfunctions, are called eigenvalues of problem* (6)-(8).

Lemma 1. *The eigenvalues of the boundary value of problem* (6)-(8) *are real.*

Proof. In Hilbert space $L_2\,(0, 2\pi)$ we consider the operators A and B, determined by the following equalities:

$$D(A) := W_2^2\,\left((0.2\pi)\,, u\,(0) = u\,(2\pi)\,, u'\,(0) = u'\,(2\pi)\right)\,,\;\; Au = -\frac{d^2u}{dy^2};$$

$$D(B) := W_2^1\,((0, 2\pi)\,;\;\, u(0) = u\,(2\pi))\,,\;\;\, Bu = i\frac{du}{dy}.$$

The eigenvalues of the operator A are the numbers $\mu_k(A) = k^2, \;\; k = 0, 1, ...\infty$ that for $k > 0$ correspond to the pair of eigenfunctions

$$u_{k1}(y) = \frac{1}{\sqrt{2\pi}}e^{iky}, \;\; u_{k2}(y) = \frac{1}{\sqrt{2\pi}}e^{-iky}, \;\; k = 1, 2, ...\infty.$$

The eigenvalues $\mu_k(B) = -k, \;\; k = 0, 1, ..., \infty$ correspond to the pair of eigenfunction $u_{k1}(y)$, the eigenvalues $\mu_{-k}(B) = k, \;\; k = 1, 2, ..., \infty$, correspond to the eigenfunctions $u_{k2}(y)$.

The problem (6)-(8) is equivalent to the problem

$$-u''(x) + Au(x) = \lambda u(x), \;\; x \in (0, 2\pi)\,, \tag{9}$$

$$u(0) = 0, \quad u\,(2\pi) + Bu'\,(2\pi) = 0, \tag{10}$$

where $u(x)$ is a vector function with the values from $L_2(0, 2\pi)$. It is known that the system $\{u_{k1}(y), u_{k2}(y)\}$ $k = \overline{1, \infty}$ forms a complete orthonormed basis in $L_2(0, 2\pi)$. Then almost everywhere on $(0, 1)$ any element $u(x) \in L_2(0, 2\pi)$ expands in Fourier series:

$$u(x) = \sum_{k=1}^{\infty} \left[(u(x), u_{k1}) u_{k1} + (u(x), u_{k2}) u_{k2} \right].$$

If $Au(x) \in L_2(0, 2\pi)$, $u''(x) \in L_2(0, 2\pi)$, then almost everywhere on $(0, 2\pi)$ we have the expansion

$$Au(x) = \sum_{k=1}^{\infty} k^2 \left[(u(x), u_{k1}) u_{k1} + (u(x), u_{k2}) u_{k2} \right],$$

$$u''(x) = \sum_{k=1}^{\infty} \left[\left(u''(x), u_{k1} \right) u_{k1} + \left(u''(x), u_{k2} \right) u_{k2} \right].$$

If $u(x) \in D(B)$, then almost everywhere on $(0, 2\pi)$ we have the expansion $u(x) = \sum\limits_{k=1}^{\infty} \left[-k (u(x), u_{k1}) u_{k1} + k (u(x), u_{k2}) u_{k2} \right]$.

Taking into account these spectral expansions in problem (9), (10) for the Fourier coefficients $\tilde{u}_{k1}(x) = (u(x), u_{k1})$ we obtain the problem

$$-\tilde{u}''_{k1} + k^2 \tilde{u}_{k1}(x) = \lambda \tilde{u}_{k1}(x), \quad x \in (0, 2\pi), \tag{11}$$

$$\tilde{u}_{k1}(0) = 0, \quad \tilde{u}_{k1}(2\pi) - k\tilde{u}'_{k1}(2\pi) = 0, \tag{12}$$

and for the coefficients $\tilde{u}_{k2}(x) = (u(x), u_{k2})$ we obtain the boundary value problem

$$-\tilde{u}''_{k2}(x) + k^2 \tilde{u}_{k2}(x) = \lambda \tilde{u}_{k2}(x), \quad x \in (0, 2\pi), \tag{13}$$

$$\tilde{u}_{k2}(0) = 0, \quad \tilde{u}_{k2}(2\pi) + k\tilde{u}'_{k2}(2\pi) = 0. \tag{14}$$

Thus, the study of eigenvalues of boundary value problems (6)-(8) is reduced to the study of eigenvalues of boundary value problems (11), (12) and (13), (14).

We show that the eigenvalues of problem (11), (12) are real.

Muitiplying the equation (11) by the function $\overline{\tilde{u}}_{k1}(x)$ and integrating on the interval $(0, 2\pi)$, we obtain the equality

$$-\int_0^{2\pi} \tilde{u}''_{k1}(x) \overline{\tilde{u}}_{k1}(x) \, dx + k^2 \int_0^{2\pi} |\tilde{u}_{k1}(x)|^2 \, dx = \lambda \int_0^{2\pi} |\tilde{u}_{k1}(x)|^2 \, dx.$$

Integrating by parts, we obtain :

$$-\overline{\tilde{u}}_{k1}(2\pi) \tilde{u}'_{k1}(2\pi) + \overline{\tilde{u}}_{k1}(0) \tilde{u}'_{k1}(0) + \int_0^{2\pi} \left| \tilde{u}'_{k1}(x) \right|^2 dx +$$

$$+k^2 \int_0^{2\pi} |\tilde{u}_{k1}(x)|^2 \, dx = \lambda \int_0^{2\pi} |\tilde{u}_{k1}(x)|^2 \, dx. \tag{15}$$

Using boundary conditions (12) from equality (15) we obtain

$$-\frac{1}{k}\left|\tilde{u}_{k1}\left(2\pi\right)\right|^{2}+\int_{0}^{2\pi}\left|\tilde{u}_{k1}^{\prime}\left(x\right)\right|^{2}dx+k^{2}\int_{0}^{2\pi}\left|\tilde{u}_{k1}\left(x\right)\right|^{2}dx=\lambda\int_{0}^{2\pi}\left|\tilde{u}_{k1}\left(x\right)\right|^{2}dx.$$

Hence it follows that the eigenvalues of problem (11), (12) are real. ◄

In a similar way it is shown that the egienvalues of boundary value problem (13), (14) are also real.

Lemma 1 is proved.

Obviously, there must be $\lambda \neq k^2$. Since for $\lambda = k^2$ problem (11), (12) has only a trivial solution .

Theorem 1. *Boundary value problem* (6)-(8) *has two series of eigenvalues one of which is a finite sequence satisfying the inequality* $k^2 - 1 < \lambda_k < k^2,\ k = \overline{1,6}$, *and the another one behaves asymptotically as* $\lambda_{n,k} \sim k^2 + \frac{n^2}{4}$ *for* $k, n \to \infty$.

Proof. The general solution of ordinary differential equation (11) is in the form

$$\tilde{u}_{k1}(x)=c_1e^{-x\sqrt{k^2-\lambda}}+c_2e^{-(2\pi-x)\sqrt{k^2-\lambda}}, \tag{16}$$

where c_i $(i = 1, 2)$ are arbitrary constants .

Having substituted (16) in (12), we obtain a system with respect to c_i $(i = 1, 2)$, whose determinant is of the form :

$$D_k\left(\lambda\right)=\left(k\sqrt{k^2-\lambda}+1\right)e^{-4\pi\sqrt{k^2-\lambda}}-\left(k\sqrt{k^2-\lambda}-1\right).$$

The eigenvalues of boundary value problem (11), (12) consist of real $\lambda \neq k^2$, that even if for one k satisfy the transcendental equation:

$$\left(k\sqrt{k^2-\lambda}+1\right)e^{-4\pi\sqrt{k^2-\lambda}}-\left(k\sqrt{k^2-\lambda}-1\right)=0. \tag{17}$$

Equation (17) is equivalent to the equation

$$k\sqrt{k^2-\lambda}ch\left(2\pi\sqrt{k^2-\lambda}\right)-sh\left(2\pi\sqrt{k^2-\lambda}\right)=0. \tag{18}$$

Let us write equation (18) in the form

$$k\sqrt{k^2-\lambda}-th\left(2\pi\sqrt{k^2-\lambda}\right)=0. \tag{19}$$

We find the eigenvalues of boundary value problem (11), (12) for which $\lambda < k^2,\ k \in N$. Let us put to equation (19), $2\pi\sqrt{k^2-\lambda} = y$ $(0 < y < 2\pi k)$. Then equation (19) takes the form

$$thy-\frac{k}{2\pi}y=0,\ 0<y<2\pi k,\ \ k=1,2,...\infty. \tag{20}$$

We note that the roots of equation (20) are the abscissa of intersection point of the graphs of the function $f_k(y) = thy,\ y \in (0, 2\pi k)$ and $g_k(y) = \frac{k}{2\pi} y,\ y \in (0, 2\pi k)$, $k = 1, 2, ...\infty$.

Show that for each fixed $k = \overline{1,6}$ the equation (20) has a unique solution. Indeed, since the angular coefficient of the tangent drawn to the curve $f_k(y) = thy,\ y \in (0, 2\pi k)$, $k = \overline{1.\infty}$ at the point $y = 0$ equals 1, and the angular coefficient of the straightline $g_k(y) = \frac{k}{2\pi} y,\ y \in (0, 2\pi k)$, is less than 1, only for $k = \overline{1,\, 6}$. Consequently, the graphs of these functions may intersect for $k = \overline{1,\, 6}$. For $k = 7, 8, ..., \infty$ the angular coefficient of the straightline $g_k(y) = \frac{k}{2\pi} y,\ y \in (0, 2\pi k)$ is greater than a unit, therefore we confirm that the graph of the functions $f_k(y) = thy,\ \ y \in (0, 2\pi k)$ and $g_k(y) = \frac{k}{2\pi} y,\ y \in (0, 2\pi k)$ can not intersect for $k = 7, 8, ..., \infty$. Obviously, the intersection points of these functions are located in the open rectangle $(0, 2\pi) \times (0, 1)$.

Consequently, the abscissa of the intersection points of the curve $f_k(y) = thy,\ y \in (0, 2\pi k)$, and of the straightline $g_k(y) = \frac{k}{2\pi} y,\ y \in (0, 2\pi k)$ for $k = \overline{1,\, 6}$ are located in the interval $(0, 2\pi)$.

Let us consider the functions

$$\varphi_k(y) = thy - \frac{k}{2\pi} y, 0 < y < 2\pi\,, k = \overline{1,\, 6}. \tag{21}$$

For a rather small $\varepsilon > 0$ we have

$$\varphi_k(\varepsilon) = th\varepsilon - \frac{k}{2\pi}\varepsilon > 0,$$

since for small $\varepsilon > 0$, $th\varepsilon \sim \varepsilon$.

$$\varphi_k(2\pi) = \lim_{y \to 2\pi - 0} \left(thy - \frac{k}{2\pi} y \right) = th2\pi - k < 0,$$

since for any $y > 0$, $0 < thy < 1$. In what follows, by virtue of the Cauchy theorem on the zeros of a continuous function, for each $k = \overline{1,\, 6}$ the function $\varphi_k(y)$, determined by formula (21) has a unique zero belonging to the interval $(0, 2\pi)$. The uniquesness of zeros of the function $\varphi_k(y)$ for each $k = \overline{1,\, 6}$ follows from the fact that the function $f_k(y)$, $y \in (0, 2\pi)$ and the staightline $g_k(y) = \frac{k}{2\pi} y,\ y \in (0, 2\pi)$ are monotonically increasing. We denote the zeros of the function $\varphi_k(y)$ by y_k: $0 < y_k < 2\pi,\ k = \overline{1,\, 6}$. Consequently, for each $k = \overline{1,\, 6}$, $0 < 2\pi\sqrt{k^2 - \lambda} < 2\pi$. Hence, for each $k = \overline{1,\, 6}$ we have

$$\lambda_k > k^2 - 1.$$

On the other hand, we consider such eigenvalues λ, for which $\lambda_k < k^2$. As a result, for the eigenvalues of boundary value problem (11), (12) satisfying the inequality $\lambda < k^2$, we have

$$k^2 - 1 < \lambda_k < k^2, k = \overline{1,\, 6}.$$

We now find the eigenvalues of boundary value problems (11), (12) for which $\lambda > k^2$. We put in equation (18) $2\pi\sqrt{\lambda - k^2} = z$, $0 < z < +\infty$. Then equation (18) takes the form

$$\frac{k}{2\pi} z \cos z - \sin z = 0, z \in (0, +\infty)\,. \tag{22}$$

Let $z \neq n\pi, \ n \in N$. In this case equation (22) is equivalent to the equation

$$zctgz - \frac{2\pi}{k} = 0, \ \ z \in (0, +\infty), \ z \neq n\pi, \ n \in N. \tag{23}$$

Let us consider the function

$$\psi_k(z) = zctgz - \frac{2\pi}{k}, \ z \in (0, +\infty), \ z \neq n\pi, \ \ n \in N.$$

Since at each interval $(n\pi, (n+1)\pi)$, $n \in N$ the function $\psi_k(z)$ accepts the values from $-\infty$ to $+\infty$, and its derivative

$$\psi_k'(z) = \frac{\sin 2z - 2z}{2\sin^2 z} < 0,$$

then n the given interval for each k,the function $\psi_k(z)$ has only one zero z_{nk}: $n\pi < z_{n,k} < (n+1)\pi, \ n \in N$. For each $k \in N$ we find asymptotic formulas for $z_{n,k}$, as $n \to \infty$. From equation (23) we have

$$ctgz = \frac{2\pi}{kz}, \ \ z \in (0, +\infty), \ z \neq n\pi, \ \ n \in N. \tag{24}$$

Obviously, the points $z_{n,k}$ are the abscissa of the interaction of the graphs of the function $q_k(z) = \frac{2\pi}{kz}$, $z \in (0, +\infty)$, $k \in N$ and branches of the function $ctgz$. More exactly , $z_{n,k}$ are approximate solutions of equation (23) . From the location of graphs of these functions it is clear that for each k, with increasing n, the points $z_{n,k}$ will approach the points $n\pi$, i.e., $z_{n,k} \sim n\pi$.

Hence and from the equality $2\pi\sqrt{\lambda - k^2} = z$ for the eigenvalues of boundary value problem (11), (12) satisfying the condition $\lambda > k^2$, we have the formula: $\lambda_{n,k} \sim k^2 + \frac{n^2}{4}$.

We now study boundary value problem (13), (14). In the same way we show that the eigenvalues of boundary value problems (13), (14) are also real, and $\lambda = k^2$, $k \in N$ is not an eigenvalue.

The general solution of ordinary differential equation (13) is of the form

$$\tilde{u}_{k2}(x) = c_1 e^{-x\sqrt{k^2-\lambda}} + c_2 e^{-(2\pi - x)\sqrt{k^2-\lambda}}, \tag{25}$$

where $c_i \, (i = 1, 2)$ are arbitrary constants .

Having substituted (25) in (14), we obtain a system with respect to $c_i \, (i = 1, 2)$, whose determinant is of the form

$$F_k(\lambda) = \left(1 - k\sqrt{k^2 - \lambda}\right) e^{-4\pi\sqrt{k^2-\lambda}} - \left(1 + k\sqrt{k^2 - \lambda}\right).$$

The eigenvalues of boundary value problem (13), (14) consist of real $\lambda \neq k^2$, $k \in N$, that for even if one k satisfy the equation

$$F_k(\lambda) = 0. \tag{26}$$

Equation (26) is equivalent to the equation

$$k\sqrt{k^2-\lambda}\,ch\left(2\pi\sqrt{k^2-\lambda}\right)+sh\left(2\pi\sqrt{k^2-\lambda}\right)=0, \tag{27}$$

that for $\lambda < k^2$, $k \in N$ has no solutions. Consequently, boundary value problem (13), (14) has no eigenvalues satisfying the condition $\lambda < k^2$, $k \in N$.

We now look for the solutions of equation (27) , that are greater than k^2. Assume $2\pi\sqrt{\lambda-k^2}=z,\ 0<z<+\infty$. Then we can write equation (27) in the form

$$-\frac{k}{2\pi}z\sin z+\cos z=0,\ \ z\in(0,+\infty)\,. \tag{28}$$

Let $z \neq \frac{\pi}{2}+n\pi,\quad n=0,1,...\infty$. Then equation (28) is equivalent to the equation

$$tgz-\frac{2\pi}{kz}=0,\quad z\in(0,+\infty)\,,z\neq\frac{\pi}{2}+n\pi,\quad n=0,1,...\infty. \tag{29}$$

Let us consider the function

$$\eta_k\,(z)=tgz-\frac{2\pi}{kz},\quad z\in(0,+\infty)\,,z\neq\frac{\pi}{2}+n\pi,\quad n=0,1,...\infty.$$

Since at each interval $\left(\pi\left(\frac{1}{2}+n\right),\ \pi\left(\frac{3}{2}+n\right)\right),\quad n=0,1,...\infty$ the function $\eta_k\,(z)$ accepts the values from $-\infty$ to $+\infty$, and its derivative

$$\eta_k'\,(z)=\frac{1}{\cos^2 z}+\frac{2\pi}{k}>0,\quad z\in(0,+\infty)\,,$$

then in it $\eta_k\,(z)$ for each k has only one zero z_{nk}: $\pi\left(\frac{1}{2}+n\right)<z_{n,k}<\pi\left(\frac{3}{2}+n\right)$, $n=0,1,2,...\infty$. For each $k\in N$ we find asymptotic formulas for z_{nk}, as $n\to+\infty$.

Obviously, the points $z_{n,k}$ are the abscissa of the intersection of the function $q_k\,(z)=\frac{2\pi}{kz},\quad z\in(0,+\infty)\,,k\in N$ and branches of the function tgz, i.e. $z_{n,k}$ is approximate solution of the equation (29). From the location of the graps of these functions it is clear that for each k, with increasing n, the point z_{nk} will approach the point $n\pi$, i.e., $z_{nk}\sim n\pi$. Hence and from the equality $2\pi\sqrt{\lambda-k^2}=z$ for the eigenvalues of the boundary value problem (13), (14) satisfying the condition $\lambda>k^2$ we have the asymptotic formula $\lambda_{n,k}\sim k^2+\frac{n^2}{4}$. ◀

The Theorem 1 is proved .

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Received 12 February 2024
Accepted 23 March 2024